1. Introduction

2. Maurey's empirical method

We denote by $\mathcal{U}_d$ the set of all the unit vectors $x \in \mathcal{S}^{N-1}$ such that $|\mathrm{Supp}(x)| \leq d$, where $\mathrm{Supp}(x)$ stands for the support of x and $\mathcal{S}^{d-1}$ is the unit euclidean ball of $\mathbb{R}^N$. Let Γ be an orthogonal matrix with size N. We assume that all the entries of Γ are such that

$$|\Gamma_{ij}| \leq \frac{c}{\sqrt{N}}, \forall i, j \in \{1, \dots, N\}, \tag{0.1}$$

where $c > 0$ is an absolute constant.

We denote by $\Gamma_1, \dots, \Gamma_N$ the row vectors of Γ and we define for some $p \in \{1, \dots, N\}$ the norm

$$\|x\|_{\infty,p} := \max_{1 \leq j \leq p} | \langle \Gamma_j, x \rangle |, \forall x \in \mathbb{R}^N.$$

We want to compute the entropy numbers $N(\epsilon, \mathcal{U}_d, \|\cdot\|_{\infty,p})$. For that we will use the following ℓ_1^N approximation of the set $\mathcal{U}_d$:

$$\mathcal{U}_d \subset \sqrt{d}\mathcal{B}_1^N,$$

where $\mathcal{B}_1^N$ denotes the unit ℓ_1^N ball of $\mathbb{R}^N$. Remark that for every $\epsilon > 0$, we have $N(\epsilon, \sqrt{d}\mathcal{B}_1^N, \|\cdot\|_{\infty,p}) = N(\epsilon/\sqrt{d}, \mathcal{B}_1^N, \|\cdot\|_{\infty,p})$.

Theorem 0.1. *There exists an absolute constant $c > 0$ such that the following holds.*

$$N(\epsilon, \mathcal{B}_1^N, \|\cdot\|_{\infty,p}) \leq \left\{ \begin{array}{ll} \exp\left(\frac{c \log p}{N\epsilon^2} \log(2N+1)\right), & \forall \epsilon > 0; \\ \exp\left(N \log\left(1 + \frac{2c}{\epsilon\sqrt{N}}\right)\right) & \forall \epsilon > 0. \end{array} \right.$$

Proof.

The proof is splited in two cases: for small "scale" ($\epsilon \leq n^{-1/2}$), we will use the volumetric estimate. For larger scale ($\epsilon \geq n^{-1/2}$), we will use the empirical method of Maurey.

Let $x \in \mathcal{B}_1^N$. We define the random variable Z with values in $\{\pm e_1, \dots, \pm e_N\} \cup \{0\}$ (where $(e_1, \dots, e_N)$ is the canonical basis of $\mathbb{R}^N$) by

$$\mathbb{P}[Z = \mathrm{Sign}(x_i)e_i] = |x_i|, \forall i = 1, \dots, N \text{ and } \mathbb{P}[Z = 0] = 1 - \|x\|_1.$$

Note that $\mathbb{E}Z = x$.

Take $Z_1, \dots, Z_m$ be m i.i.d. random variables having the same probability distribution as Z. By the Giné-Zinn symmetrization argument and the classical Gaussian bound on Rademacher processes, we obtain

$$(\star) := \mathbb{E}\left\| x - \frac{1}{m}\sum_{i=1}^m Z_i \right\|_{\infty,p} \leq \frac{2}{m}\mathbb{E}_Z\mathbb{E}_\epsilon \left\| \sum_{i=1}^m \epsilon_i Z_i \right\|_{\infty,p} \leq \frac{c}{m}\mathbb{E}_Z\mathbb{E}_g \left\| \sum_{i=1}^m g_i Z_i \right\|_{\infty,p}$$

where $\epsilon_1, \dots, \epsilon_m$ are m i.i.d. Rademacher r.v. and $g_1, \dots, g_m$ are m i.i.d. standard Gaussian r.v..

Set $\gamma_j := \sum_{i=1}^m g_i \langle \Gamma_j, Z_i \rangle$ for all $j = 1, \dots, p$. For every $1 \leq j \leq p$, γ_j is a centered Gaussian r.v. with variance $\sigma_j^2 = \sum_{i=1}^m \langle \Gamma_j, Z_i \rangle^2 \leq cm/N$ (where we use the upper bound (0.1). Thus, by using the Gaussian maximal inequality,

$$\mathbb{E}_g \left\| \sum_{i=1}^m g_i Z_i \right\|_{\infty,p} \leq \sqrt{\log p}\sqrt{\frac{cm}{N}}.$$

This yields $(\star) \leq c\sqrt{(\log p)/(mN)}$.

Now, we choose the minimal m such that

$$c\sqrt{\frac{\log p}{mN}} \leq \epsilon \text{ i.e. } m \sim \frac{\log p}{\epsilon^2 N}.$$

For this choice of m we have

$$\mathbb{E}\left\| x - \frac{1}{m}\sum_{i=1}^{m} Z_i \right\|_{\infty,p} \leq \epsilon.$$

In particular, there exists $\omega \in \Omega$ such that

$$\left\| x - \frac{1}{m}\sum_{i=1}^{m} Z_i(\omega) \right\|_{\infty,p} \leq \epsilon$$

and so for $z := \frac{1}{m}\sum_{i=1}^{m} Z_i(\omega)$, we have $\|x - z\|_{\infty,p} \leq \epsilon$.

We finish the proof for the large scale by noting that there exists at most $(2N+1)^m$ different values of $z := \frac{1}{m}\sum_{i=1}^{m} Z_i(\omega)$.

For the small scale, remark that $\|x\|_{\infty,p} \leq cN^{-1/2}$ because $|\langle \Gamma_j, x\rangle| \leq |\Gamma_j|_\infty |x|_1 \leq cN^{-1/2} \forall j = 1, \dots, N$. Thus, $\mathcal{B}_1^N \subset cN^{-1/2}\mathcal{B}_{\infty,p}$, where $\mathcal{B}_{\infty,p}$ denotes the unit ball w.r.t. $\|\cdot\|_{\infty,p}$.Now, we want to compute $N(\epsilon, cN^{-1/2}\mathcal{B}_{\infty,p}, \|\cdot\|_{\infty,p}) = N(c^{-1}\sqrt{N}\epsilon, \mathcal{B}_{\infty,p}, \|\cdot\|_{\infty,p})$. Denote by Λ a maximal set of $c^{-1}\sqrt{N}\epsilon$-separated points of $\mathcal{B}_{\infty,p}$ w.r.t. $\|\cdot\|_{\infty,p}$. We know that $N(c^{-1}\sqrt{N}\epsilon, \mathcal{B}_{\infty,p}, \|\cdot\|_{\infty,p}) \leq |\Lambda|$. We have

$$\cup_{x\in\Lambda}\Big(x + \frac{\sqrt{N}\epsilon}{2c}\mathcal{B}_{\infty,p}\Big) \subset \Big(1 + \frac{\sqrt{N}\epsilon}{2c}\Big)\mathcal{B}_{\infty,p},$$

where, by defintion of Λ, the balls $\Big(x + \frac{\sqrt{N}\epsilon}{2c}\mathcal{B}_{\infty,p}\Big), \forall x \in \Lambda$ are all disjoint. Thus, by taking the volume, we get

$$|\Lambda|\Big(\frac{\sqrt{N}\epsilon}{2c}\Big)^N |\mathcal{B}_{\infty,p}| \leq \Big(1 + \frac{\sqrt{N}\epsilon}{2c}\Big)^N |\mathcal{B}_{\infty,p}|,$$

which yields

$$N(c^{-1}\sqrt{N}\epsilon, \mathcal{B}_{\infty,p}, \|\cdot\|_{\infty,p}) \leq |\Lambda| \leq \Big(1 + \frac{2c}{\sqrt{N}\epsilon}\Big)^N.$$

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3. Combinatorial argument and Theorem of the majorizing measure

We denote by $\mathcal{U}_d$ the set of all the unit vectors $x \in \mathcal{S}^{N-1}$ such that $|\mathrm{Supp}(x)| \leq d$, where $\mathrm{Supp}(x)$ stands for the support of x and $\mathcal{S}^{d-1}$ is the unit euclidean ball of $\mathbb{R}^N$.

Theorem 0.2. *Let T be a subset of $\mathbb{R}^N$ and $g_1, \dots, g_n$ be n i.i.d. standard gaussian random variables. We have*