Empirical log-optimal portfolio selections
Abstract

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Consider a market consisting of d assets. The evolution of the market in time is represented by a sequence of price vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots \in \mathbb{R}_+^d$, where

$$\mathbf{s}_n = (s_n^{(1)}, \ldots, s_n^{(d)})$$

such that the j-th component $s_n^{(j)}$ of $\mathbf{s}_n$ denotes the price of the j-th asset on the n-th trading period. In order to normalize, put $s_0^{(j)} = 1$. $\{\mathbf{s}_n\}$ has exponential trend:

$$s_n^{(j)} = e^{nW_n^{(j)}} \approx e^{nW^{(j)}},$$

with average growth rate (average yield)

$$W_n^{(j)} := \frac{1}{n} \ln s_n^{(j)}$$

and with asymptotic average growth rate

$$W^{(j)} := \lim_{n \to \infty} \frac{1}{n} \ln s_n^{(j)}.$$

In order to apply the usual prediction techniques for time series analysis one has to transform the sequence price vectors $\{\mathbf{s}_n\}$ into a more or less stationary sequence of return vectors $\{\mathbf{x}_n\}$ as follows:

$$\mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(d)})$$

such that

$$x_n^{(j)} = \frac{s_n^{(j)}}{s_{n-1}^{(j)}}.$$

Thus, the j-th component $x_n^{(j)}$ of the return vector $\mathbf{x}_n$ denotes the amount obtained after investing a unit capital in the j-th asset on the n-th trading period.

The *static portfolio selection* is a single period investment strategy. A portfolio vector is denoted by $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$. The j-th component $b^{(j)}$ of $\mathbf{b}$ denotes the proportion of the investor's capital invested in asset j. We assume that the portfolio vector $\mathbf{b}$ has nonnegative components sum up to 1, that means that short selling is not permitted. The set of portfolio vectors is denoted by

$$\Delta_d = \left\{ \mathbf{b} = (b^{(1)}, \dots, b^{(d)});\ b^{(j)} \geq 0,\ \sum_{j=1}^{d} b^{(j)} = 1 \right\}.$$

For static portfolio selection, at time $n = 0$ we distribute the initial capital S_0 according to a fix portfolio vector $\mathbf{b}$, i.e., if S_n denotes the wealth at the trading period n, then

$$S_n = S_0 \sum_{j=1}^{d} b^{(j)} s_n^{(j)}.$$

One can show that

$$W := \lim_{n\to\infty} \frac{1}{n} \ln S_n = \lim_{n\to\infty} \max_j \frac{1}{n} \ln s_n^{(j)} = \max_j W^{(j)}.$$

Thus, any static portfolio selection achieves the maximal growth rate $\max_j W^{(j)}$.

One can achieve even higher growth rate for long run investments, if the tuning of the portfolio is allowed dynamically trading period after trading period. The *dynamic portfolio selection* is a multi-period investment strategy, where at the beginning of each trading period we rearrange the wealth among the assets. A representative example of the dynamic portfolio selection is the *constantly rebalanced portfolio (CRP)*, where we fix a portfolio vector $\mathbf{b} \in \Delta_d$, i.e., we are concerned with a hypothetical investor who neither consumes nor deposits new cash into his portfolio, but reinvest his portfolio each trading period. Note that in this case the investor has to rebalance his portfolio after each trading day to "corrigate" the daily price shifts of the invested stocks.

Let S_0 denote the investor's initial capital. Then at the beginning of the first trading period $S_0 b^{(j)}$ is invested into asset j, and it results in return

$S_0 b^{(j)} x_1^{(j)}$, therefore at the end of the first trading period the investor's wealth becomes

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b}\,, \mathbf{x}_1 \rangle\,,$$

where $\langle \cdot\,, \cdot \rangle$ denotes inner product. For the second trading period, S_1 is the new initial capital

$$S_2 = S_1 \cdot \langle \mathbf{b}\,, \mathbf{x}_2 \rangle = S_0 \cdot \langle \mathbf{b}\,, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}\,, \mathbf{x}_2 \rangle\,.$$

By induction, for the trading period n the initial capital is S_{n-1}, therefore

$$S_n = S_{n-1} \langle \mathbf{b}\,, \mathbf{x}_n \rangle = S_0 \prod_{i=1}^{n} \langle \mathbf{b}\,, \mathbf{x}_i \rangle\,.$$

The asymptotic average growth rate of this portfolio selection is

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \ln S_n &= \lim_{n\to\infty} \left(\frac{1}{n} \ln S_0 + \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b}\,, \mathbf{x}_i \rangle \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b}\,, \mathbf{x}_i \rangle\,, \end{aligned}$$

therefore without loss of generality one can assume in the sequel that the initial capital $S_0 = 1$.

If the market process $\{\mathbf{X}_i\}$ is memoryless, i.e., it is a sequence of independent and identically distributed (i.i.d.) random return vectors then we show that the best constantly rebalanced portfolio (BCRP) is the log-optimal portfolio:

$$\mathbf{b}^* := \arg\max_{\mathbf{b}\in\Delta_d} \mathbb{E}\{\ln \langle \mathbf{b}\,, \mathbf{X}_1 \rangle\}.$$

This optimality means that if $S_n^* = S_n(\mathbf{b}^*)$ denotes the capital after day n achieved by a log-optimum portfolio strategy $\mathbf{b}^*$, then for any portfolio strategy $\mathbf{b}$ with finite $\mathbb{E}\{(\ln \langle \mathbf{b}\,, \mathbf{X}_1 \rangle)^2\}$ and with capital $S_n = S_n(\mathbf{b})$ and for any memoryless market process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n \le \lim_{n\to\infty} \frac{1}{n} \ln S_n^* \quad \text{almost surely}$$

and maximal asymptotic average growth rate is

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = W^* := \mathbb{E}\{\ln \langle \mathbf{b}^*\,, \mathbf{X}_1 \rangle\} \quad \text{almost surely.}$$

We show several *examples* for constantly rebalanced portfolio.

In order to decrease the computational complexity of log-optimal portfolio we introduce the *semi-log-optimal portfolio*, where the function $\ln z$ is replaced by its second order Taylor expansion.

For a *general dynamic portfolio selection*, the portfolio vector may depend on the past data. Let $\mathbf{b} = \mathbf{b}_1$ be the portfolio vector for the first trading period. For initial capital S_0, we get that

$$S_1 = S_0 \cdot \langle \mathbf{b}_1 , \mathbf{x}_1 \rangle .$$

For the second trading period, S_1 is new initial capital, the portfolio vector is $\mathbf{b}_2 = \mathbf{b}(\mathbf{x}_1)$, and

$$S_2 = S_0 \cdot \langle \mathbf{b}_1 , \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}(\mathbf{x}_1) , \mathbf{x}_2 \rangle .$$

For the nth trading period, a portfolio vector is $\mathbf{b}_n = \mathbf{b}(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) = \mathbf{b}(\mathbf{x}_1^{n-1})$ and

$$S_n = S_0 \prod_{i=1}^{n} \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) , \mathbf{x}_i \right\rangle = S_0 e^{n W_n(\mathbf{B})}$$

with the average growth rate

$$W_n(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) , \mathbf{x}_i \right\rangle .$$

The fundamental limits reveal that the so-called *log-optimum portfolio* $\mathbf{B}^* = \{\mathbf{b}^*(\cdot)\}$ is the best possible choice. More precisely, on trading period n let $\mathbf{b}^*(\cdot)$ be such that

$$\mathbb{E}\left\{ \ln \left\langle \mathbf{b}^*(\mathbf{X}_1^{n-1}) , \mathbf{X}_n \right\rangle \Big| \mathbf{X}_1^{n-1} \right\} = \max_{\mathbf{b}(\cdot)} \mathbb{E}\left\{ \ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}) , \mathbf{X}_n \right\rangle \Big| \mathbf{X}_1^{n-1} \right\} .$$

If $S_n^* = S_n(\mathbf{B}^*)$ denotes the capital achieved by a log-optimum portfolio strategy $\mathbf{B}^*$, after n trading periods, then for any other investment strategy $\mathbf{B}$ with capital $S_n = S_n(\mathbf{B})$ and with

$$\sup_n \mathbb{E}\left\{ (\ln \left\langle \mathbf{b}_n(\mathbf{X}_1^{n-1}) , \mathbf{X}_n \right\rangle)^2 \right\} < \infty,$$

and for any stationary and ergodic process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{S_n}{S_n^*} \le 0 \quad \text{almost surely}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\ln S_n^* = W^* \quad \text{almost surely,}$$

where

$$W^* := \mathbb{E}\left\{\max_{\mathbf{b}(\cdot)}\mathbb{E}\left\{\ln\left\langle \mathbf{b}(\mathbf{X}_{-\infty}^{-1}),\, \mathbf{X}_0\right\rangle \Big| \mathbf{X}_{-\infty}^{-1}\right\}\right\}$$

is the maximal possible growth rate of any investment strategy.

An empirical (data driven) portfolio strategy $\mathbf{B}$ is called *universally consistent* with respect to a class $\mathcal{C}$ of stationary and ergodic processes $\{\mathbf{X}_n\}_{-\infty}^{\infty}$, if for each process in the class,

$$\lim_{n\to\infty}\frac{1}{n}\ln S_n(\mathbf{B}) = W^* \quad \text{almost surely.}$$

For a fixed integer $k > 0$ large enough, let's apply the following approximation:

$$\mathbb{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{n-1}),\, \mathbf{X}_n\right\rangle \mid \mathbf{X}_1^{n-1}\} \approx \mathbb{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}),\, \mathbf{X}_n\right\rangle \mid \mathbf{X}_{n-k}^{n-1}\}$$

and

$$\mathbf{b}^*(\mathbf{X}_1^{n-1}) \approx \mathbf{b}_k(\mathbf{X}_{n-k}^{n-1}) = \arg\max_{\mathbf{b}(\cdot)}\mathbb{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}),\, \mathbf{X}_n\right\rangle \mid \mathbf{X}_{n-k}^{n-1}\}.$$

Because of stationarity

$$\mathbf{b}_k(\mathbf{x}_1^k) \quad = \quad \arg\max_{\mathbf{b}}\mathbb{E}\{\ln\langle \mathbf{b},\, \mathbf{X}_{k+1}\rangle \mid \mathbf{X}_1^k = \mathbf{x}_1^k\},$$

which is the maximization of the regression function

$$m_{\mathbf{b}}(\mathbf{x}_1^k) = \mathbb{E}\{\ln\langle \mathbf{b},\, \mathbf{X}_{k+1}\rangle \mid \mathbf{X}_1^k = \mathbf{x}_1^k\}.$$

Thus, a possible way for asymptotically optimal empirical portfolio selection is that, based on the past data, sequentially estimate the regression function $m_{\mathbf{b}}(\mathbf{x}_1^k)$, and choose the portfolio vector, which maximizes the regression function estimate.

Next briefly summarize the basics of *nonparametric regression function estimation.*

Introduce the *kernel-based portfolio selection* strategies. Define an infinite array of portfolio selections $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}^{(k,\ell)}(\cdot)\}$, where k, ℓ are positive integers. For fixed positive integers k, ℓ, choose the radius $r_{k,\ell} > 0$ such that for any fixed k,

$$\lim_{\ell \to \infty} r_{k,\ell} = 0.$$

Then, for $n > k+1$, define the expert $\mathbf{b}^{(k,\ell)}$ by

$$\mathbf{b}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg\max_{\mathbf{b} \in \Delta_d} \sum_{\left\{k<i<n: \|\mathbf{x}_{i-k}^{i-1} - \mathbf{x}_{n-k}^{n-1}\| \le r_{k,\ell}\right\}} \ln \langle \mathbf{b}\,,\, \mathbf{x}_i \rangle \ ,$$

if the sum is non-void, and $\mathbf{b}_0 = (1/d, \ldots, 1/d)$ otherwise.

The good, data dependent choice of k and ℓ is doable borrowing current techniques from *machine learning*. In machine learning setup k and ℓ are considered as parameters of the estimates, called experts. The basic idea of machine learning is the combination of the experts, where an expert has large weight if its past performance is good. Combine the elementary portfolio strategies $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}_n^{(k,\ell)}\}$ as follows: let $\{q_{k,\ell}\}$ be a probability distribution on the set of all pairs (k,ℓ) such that for all k, ℓ, $q_{k,\ell} > 0$. The combined strategy $\mathbf{B}$ arises from weighting the elementary portfolio strategies $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}_n^{(k,\ell)}\}$ such that the investor's capital becomes

$$S_n(\mathbf{B}) = \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}).$$

We prove that the portfolio scheme $\mathbf{B}$ is *universally consistent* with respect to the class of all ergodic processes such that $\mathbb{E}\{|\ln X^{(j)}|\} < \infty$, for $j = 1, 2, \ldots d$.

We present some *numerical results* obtained by applying the kernel based log-optimal algorithm to a *NYSE data set* from www.szit.bme.hu/˜oti/portfolio .