M-estimation

and Complexity Regularization

Sara van de Geer
Seminar für Statistik, ETH Zürich

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Program

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1. Notation and motivation

Let $Z_1, \ldots, Z_n$ be independent random variables in $\mathcal{Z}$.

Notation: for $\gamma : \mathcal{Z} \rightarrow \mathbf{R}$,

the theoretical measure is

$$P\gamma := \frac{1}{n}\sum_{i=1}^{n} E\gamma(Z_i),$$

and the empirical measure is

$$P_n\gamma := \frac{1}{n}\sum_{i=1}^{n} \gamma(Z_i).$$

We moreover write

$$\nu_n(\gamma) := \sqrt{n}(P_n - P)\gamma.$$

Consider a class Γ of functions γ on $\mathcal{Z}$.

EPT (Empirical Process Theory) is about the study of

$$\nu_n := \{\sqrt{n}(P_n - P)\gamma : \ \gamma \in \Gamma\}$$

as process indexed by Γ.

In particular, the study of probability and moment inequalities for

$$\mathbf{V}_n := \sup_{\gamma \in \Gamma} |\nu_n(\gamma)|.$$

Statistical motivation:

We will consider empirical risk minimization (M-estimation).

Let Γ be a class of loss functions, indexed by a parameter.

Parametric:

$$\Gamma = \{\gamma_\theta : \theta \in \Theta\}, \ \Theta \subset \mathbf{R}^r.$$

Nonparametric:

$$\Gamma = \{\gamma_f : f \in \mathcal{F}\},$$

with $\mathcal{F}$ some collection of functions.

Empirical risk minimizer

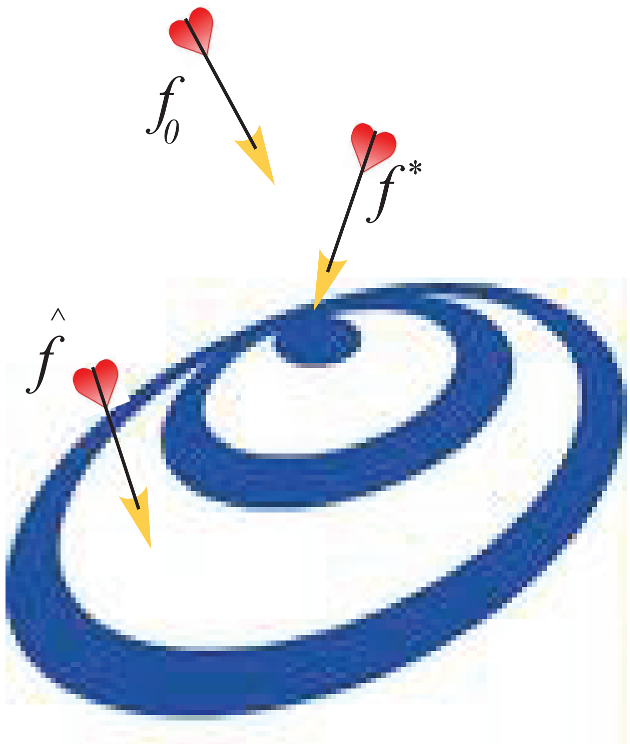
$$\hat{f} := \arg\min_{f \in \mathcal{F}} P_n \gamma_f, \ \hat{\gamma} := \gamma_{\hat{f}}.$$

Let $\mathbf{F} \supset \mathcal{F}$. The target is

$$f^0 := \arg\min_{f \in \mathbf{F}} P \gamma_f, \ \gamma^0 := \gamma_{f^0}.$$

Best approximation in the class

$$f^* := \arg\min_{f \in \mathcal{F}} P \gamma_f, \ \gamma^* := \gamma_{f^*}.$$


f_0
f^*
$\hat{f}$

Excess risk

$$\mathcal{E}(\gamma) := P(\gamma - \gamma^0).$$

Approximation error

$$\mathcal{E}^* := \mathcal{E}(\gamma^*).$$

Basic inequality: For $\hat{\mathcal{E}} = \mathcal{E}(\hat{\gamma})$,

$$\hat{\mathcal{E}} \leq -(P_n - P)(\hat{\gamma} - \gamma^*) + \mathcal{E}^*.$$

Proof. $\ldots$

Define

$$\sigma^2(\gamma) := \frac{1}{n}\sum_{i=1}^{n} E[\gamma(Z_i) - E\gamma(Z_i)]^2.$$

Let $\hat{\sigma} := \sigma(\hat{\gamma} - \gamma^0)$, $\sigma^* := \sigma(\gamma^* - \gamma^0)$.
More generally, let d be some metric on $\mathbf{F}$, and $\hat{d} := d(\hat{f}, f^0)$, $d^* := d(f^*, f^0)$.

Let $\psi(\cdot)$ (some concave function) be the "modulus of continuity" of the empirical process, that is, ψ is such that

$$\mathbf{V}_n := \sup_{\gamma \in \Gamma} \frac{|\sqrt{n}(P_n - P)(\gamma - \gamma^*)|}{\psi(d(f, f^0) \vee d^*)}$$

is a "bounded" random variable.

Let $G(\cdot)$ (some convex function) be the margin, i.e.,

$$\mathcal{E}(\gamma_f) \geq G(d(f, f^0)), \; \forall \; f \in \mathcal{F}.$$

Example(Classification)
Suppose that $Z_i = (X_i, Y_i)$, with $Y_i \in \mathcal{Y} := \{0, 1\}$ a label, $i = 1, \ldots, n$. Let **F** be a class of functions $f : \mathcal{X} \to [0, 1]$. We consider 0/1-loss

$$\gamma_f(x, y) = \gamma(f(x), y) := (1-y)f(x)+y(1-f(x)).$$

For $a \in [0, 1]$, write

$$l(a, \cdot) := \mathbb{E}(\gamma(a, Y_i)|X_i = \cdot)$$

$$= (1 - \eta)a + \eta(1 - a) = a(1 - 2\eta) + \eta \ ,$$

where $\eta = \mathbb{E}(Y_i|X_i = \cdot)$.

The target is the overall minimizer

$$f^0 := \arg \min_{a \in [0,1]} l(a, \cdot) \ .$$

It is clear that f^0 is the Bayes rule

$$f^0 = \mathrm{l}\{1 - 2\eta < 0\} + q\{1 - 2\eta = 0\} \ ,$$

with q an arbitrary value in $[0, 1]$.
We moreover have

$$P(\gamma_f - \gamma_{f^0}) = P|(f - f^0)(1 - 2\eta)| \ .$$

Consider the functions

$$H_1(v) = vP\mathrm{l}\{|1-2\eta| < v\},\ v \in [0,1]\ ,$$

and

$$G_1(u) = \max_v\{uv - H_1(v)\},\ u \in [0,1]$$

(assuming the maximum exists).

Lemma *The inequality*

$$P(\gamma_f - \gamma_{f^0}) \geq G(\sigma(\gamma_f - \gamma_{f^0}))$$

holds with $G(u) = G_1(u^2)$, $u \in [0,1]$.

Suppose that $G_\psi := G \circ \psi^{-1}$ is strictly convex.

Definition The convex conjugate of G_ψ is

$$H_\psi(v) := \sup_u [uv - G_\psi(u)].$$

Lemma For all $0 < \lambda_n^2 < 1$,

$$(1-\delta)\hat{\mathcal{E}} \le \delta H_\psi\left(\frac{\mathbf{V}_n}{\sqrt{n}\delta}\right) + (1+\delta)\mathcal{E}^*.$$

Proof. $\ldots$

Bernstein's inequality

Let $\gamma_j : \mathcal{Z} \to \mathbf{R}$, $j = 1, \ldots, p$.

Assume that for all j, i *and* $m \geq 2$,

$$E\gamma_j(Z_i) = 0, \; P|\gamma_j - \gamma^*|^m \leq \frac{m!}{2}(2K)^{m-2} d^2(f_j, f^*).$$

Then for all $m \leq 1 + \log p$,

$$\mathbb{E}^{1/m}\left(\max_{1 \leq j \leq p} \frac{|P_n(\gamma_j - \gamma^*)|}{d(f_j, f^*) \vee \tau} \right)^m \leq \lambda_n + \frac{K\lambda_n^2}{\tau},$$

where $\lambda_n^2 := \frac{2\log(2p)}{n}$.

Moreover, for all $t > 0$,

$$\mathbf{P}\left[\max_{1\le j\le p}\frac{|P_n(\gamma_j-\gamma^*)|}{d(f_j,f^*)\vee\tau}\ge\sqrt{\lambda_n^2+2t}+\frac{K(\lambda_n^2+2t)}{\tau}\right]$$

$$\le\exp[-nt].$$

Bousquet's inequality. *Let* $\gamma : \mathcal{Z} \to \mathbf{R}$, $\gamma \in \Gamma$. *Assume that for all* γ, i,

$$E\gamma(Z_i) = 0,\ |\gamma - \gamma^*| \leq 2K.$$

Let

$$\mathbf{Z} := \sup_{\gamma \in \Gamma} \frac{|P_n(\gamma - \gamma^*)|}{\sigma(\gamma - \gamma^*) \vee \tau}.$$

Then $\forall\, t > 0$,

$$\mathbb{P}\left(\mathbf{Z} \geq \mathbb{E}\mathbf{Z} + \sqrt{2t}\sqrt{1 + 4K\mathbb{E}\mathbf{Z}} + \frac{2tK}{3} \right) \leq \mathrm{e}^{-nt}.$$

Hoeffding's inequality

Suppose for $1 \leq j \leq p$,

$$E\gamma_j(Z_i) = 0,\ |(\gamma_j - \gamma^*)(Z_i)| \leq c_{i,j}\ \forall\ i.$$

Let

$$d^2(\gamma_j, \gamma^*) := \frac{1}{n}\sum_{i=1}^{n} c_{i,j}^2.$$

Then

$$\mathbb{E}\left(\max_{1\leq j\leq p} \frac{|P_n(\gamma_j - \gamma^*)|}{d(\gamma_j, \gamma^*)}\right) \leq \lambda_n.$$

Moreover, for all $t > 0$*,*

$$\mathbf{P}\left(\max_{1\leq j\leq p} \frac{|P_n(\gamma_j - \gamma^*)|}{d(\gamma_j, \gamma^*)} \geq \ \lambda_n + \sqrt{2t}\right)$$

$$\leq \exp\left[-nt\right].$$

3. Empirical risk minimization over a finite class

Let $\gamma_j : \mathcal{Z} \rightarrow \mathbf{R}$, $j = 1, \ldots, p$ be given loss functions in a class $\Gamma \subset \mathbf{\Gamma}$. We define the model selection estimator

$$P_n \hat{\gamma} := \min_{1 \leq j \leq p} P_n \gamma_j .$$

The target is

$$P \gamma^0 := \arg \min_{\gamma \in \mathbf{\Gamma}} P \gamma .$$

The best approximation is

$$P \gamma^* := \min_{1 \leq j \leq p} P \gamma_j .$$

We define the excess risks

$$\hat{\mathcal{E}} := P(\hat{\gamma} - \gamma^0),$$

and

$$\mathcal{E}^* := P(\gamma^* - \gamma^0).$$

Moreover, we let

$$\sigma^2(\gamma) := \frac{1}{n} \sum_{i=1}^{n} E[\gamma(Z_i) - E\gamma(Z_i)]^2.$$

3.1 Bounded loss, standard margin condition

Lemma. *Suppose*

$$\mathbf{P}|\gamma_j - \gamma^*|^m \leq \frac{m!}{2} K^{m-2} d^2(f_j, f^*),$$

and the standard margin condition

$$\mathcal{E}(\gamma_j) \geq d^2(f_j, f^*)/C.$$

Then for $\mathcal{E}^* \geq \lambda_n^2$*,*

$$\mathbb{E}^{1/m} \left(\sqrt{\frac{\hat{\mathcal{E}}}{\mathcal{E}^*}} \right)^m \leq 1 + \sqrt{\frac{C\lambda_n^2}{\mathcal{E}^*}} + \frac{K\lambda_n^2}{\mathcal{E}^*}.$$

Remark. The result with $m = 2$ reads

$$\mathbb{E}\left[\frac{\hat{\mathcal{E}}}{\mathcal{E}^*}\right] \leq \left[1 + \sqrt{\frac{C\lambda_n^2}{\mathcal{E}^*}} + \frac{K\lambda_n^2}{\mathcal{E}^*}\right]^2 .$$

Corollary. *When*

$$\mathcal{E}^* \gg (K + C)\,\lambda_n^2,$$

it holds that

$$\mathbb{E}\left(\sqrt{\frac{\hat{\mathcal{E}}}{\mathcal{E}^*}}\right)^m \to 1.$$

Example: density estimation.

Define

$$\hat{\mathcal{K}} := P(\gamma_{(\hat{f}+f^*)/2} - \gamma_{f^0}),$$

and

$$\mathcal{K}^* := P(\gamma_{f^*} - \gamma_{f^0}).$$

Lemma *Suppose that*

$$\sqrt{\frac{f^0}{f^*}} \le \frac{C}{8} .$$

Then

$$\mathbb{E}^{1/m}\left(\frac{\hat{\mathcal{K}}}{\mathcal{K}^*}\right)^{m/2} \le 1 + C\sqrt{\frac{\lambda_n^2}{\mathcal{K}^*}} + \frac{\lambda_n^2}{\mathcal{K}^*}.$$

3.2 Bounded loss, general margin condition

Lemma *Suppose that the margin condition holds, with strictly convex margin function* G. *Let* H *be the convex conjugate of* G. *Assume that for some* $r \leq 1 + \log p$, *the function* $H(v^{\frac{1}{r}}),\ v > 0$, *is concave. Assume moreover that the exponential moment condition holds for some* $K > 0$.

Then for all $0 < \delta < 1$*, and* $\varepsilon > 0$*, we have*

$$(1-\delta)\mathbb{E}\hat{\mathcal{E}} \leq 2\delta H \left(\sqrt{\frac{\lambda_n^2}{\delta}} + \frac{K\lambda_n^2}{2\delta G^{-1}(\mathcal{E}^* \vee \varepsilon)} \right) + (1+\delta)\mathcal{E}^* \, .$$

Lemma.

Suppose

$$\mathbf{P}|\gamma_j - \gamma^*|^m \leq \frac{m!}{2} K^{m-2} d^2(f_j, f^*),$$

and the general margin condition

$$\mathcal{E}(\gamma_j) \geq d^{2\kappa}(f_j, f^0)/C.$$

Then,

$$\mathbb{E}^{1/m}\left(\left(\frac{\hat{\mathcal{E}}}{\mathcal{E}^*}\right)^{\frac{1}{2\kappa}}\right)^m \leq 1$$

$$+\bar{c}_\kappa \left(\sqrt{\frac{C^{\frac{1}{\kappa}} \lambda_n^2}{(\mathcal{E}^*)^{\frac{2\kappa-1}{\kappa}}}} + \frac{K\lambda_n^2}{\mathcal{E}^*} \right)^{\frac{1}{2\kappa-1}},$$

where

$$\bar{c}_\kappa := \left(\frac{1 + (2\kappa - 1)^{\frac{1}{2\kappa-1}}}{(2\kappa)^{\frac{1}{2\kappa-1}}} \right).$$

3.3 Unbounded loss, under standard margin condition

Lemma. *Suppose*

$$|\gamma_j(\cdot) - \gamma_l(\cdot)| \le \mathbf{K}(\cdot),\ \forall (j, l),$$

and that for some $s > 1$,

$$\|\mathbf{K}\|_s^s := P\mathbf{K}^s < \infty.$$

Assume the standard margin condition

$$\mathcal{E}(\gamma_j) \ge \sigma^2(\gamma_j - \gamma^0)/C,\ \forall j.$$

Then for $\log p \geq 1$,

$$\mathbb{E}^{1/2}\left(\frac{\hat{\mathcal{E}}}{\mathcal{E}_*}\right) \leq 1 + \sqrt{\frac{C\lambda_n^2}{\mathcal{E}^*}}$$

$$+c_s(2\|\mathbf{K}\|_s)^{\frac{s}{s+1}}\left(\lambda_n^2\right)^{\frac{s-1}{s+1}}(\mathcal{E}^*)^{-\frac{s}{s+1}},$$

where

$$c_{\mathcal{S}} := \left(\frac{s-1}{2}\right)^{\frac{2}{s+1}} + \left(\frac{s-1}{2}\right)^{-\frac{s-1}{s+1}}.$$

Corollary. *When*

$$\mathcal{E}^* \gg 2c_{\mathcal{S}}^{\frac{s+1}{s}} \|\mathbf{K}\|_s \left(\lambda_n^2\right)^{\frac{s-1}{s}} + C\lambda_n^2,$$

it holds that

$$\mathbb{E}\frac{\hat{\mathcal{E}}}{\mathcal{E}^*} \to 1.$$

4. Empirical risk minimization over an infinite class

Estimator

$$\hat{\gamma} := \arg\min_{\gamma \in \Gamma} P_n \gamma$$

Target

$$\gamma^0 := \arg\min_{\gamma \in \mathbf{\Gamma}} P\gamma, \ \mathbf{\Gamma} \supset \Gamma.$$

Best approximation

$$\gamma^* := \arg\min_{\gamma \in \Gamma} P\gamma.$$

We assume $\forall \ \gamma \in \Gamma$,

1. boundedness: $\|\gamma - \gamma^*\| \leq K$,

2. margin condition: $\mathcal{E}(\gamma) \geq G(\sigma(\gamma - \gamma^0))$.

Recall: we sketched a result involving the weighted empirical process

$$\mathbf{V}_n := \sup_{\gamma \in \Gamma} \frac{\sqrt{n}|(P_n - P)(\gamma - \gamma^*)|}{\psi(\sigma(\gamma - \gamma^0) \vee \sigma^*)}.$$

We will now show how to obtain this from the un-weighted process, using the peeling device.

Let for $\sigma > 0$,

$$\mathbf{Z}_n(\sigma) := \sup_{\sigma(\gamma - \gamma^0) \le \sigma} |(P_n - P)(\gamma - \gamma^*)|.$$

Assume that (for some σ_0 and) all $\sigma \geq \sigma_0$, we have the
3. increments of the empirical process:

$$\mathbb{E}\mathbf{Z}_n(\sigma) \leq \psi(\sigma),$$

where $\psi(\sigma) \geq \sigma$.

Let (as before)

$$G_\psi := G \circ \psi^{-1},$$

so that

$$G_\psi^{-1} = \psi \circ G^{-1}.$$

Assume the
4. decay condition: there is an $0 < \alpha < 1$ such that,

$$\frac{G_\psi^{-1}(\epsilon)}{\epsilon^\alpha} \downarrow \text{ in } \epsilon.$$

Peeling device. *Let* $\epsilon \geq G(\sigma_0)$. *Then*

$$\mathbb{E}\left(\sup_{\mathcal{E}(\gamma) > \epsilon} \frac{\sqrt{n}|(P_n - P)(\gamma - \gamma^*)|}{\mathcal{E}(\gamma)} \right) \leq C_\alpha \frac{G_\psi^{-1}(\epsilon)}{\epsilon},$$

where $C_\alpha := \alpha^{-\frac{\alpha}{1-\alpha}}/(1-\alpha)$.

Theorem *Let* $\epsilon_t \geq G(\sigma_0)$ *and*

$$(1-\delta)\epsilon_t > \delta H_\psi \left(\frac{4C_\alpha + 2\sqrt{2t}}{\delta\sqrt{n}} \right) + \frac{2Kt}{3n} + \mathcal{E}^*,$$

where H_ψ *is the convex conjugate of* G_ψ. *Then*

$$\mathbf{P}(\hat{\mathcal{E}} > \epsilon_t) \leq \exp[-t].$$

5. Symmetrization

Definition A Rademacher sequence is a sequence of independent random variables $\{\xi_i\}_{i=1}^n$ with

$$\mathbf{P}(\xi_i = +1) = \mathbf{P}(\xi_i = -1) = \frac{1}{2}.$$

Notation

Let $\{Z_i'\}$ be an independent copy of $\{Z_i\}$, and let $\{\xi_i\}$ be a Rademacher sequence, independent of $\{Z_i\}$ and $\{Z_i'\}$. Define

$$P_n' := \frac{1}{n}\sum_{i=1}^n \gamma(Z_i'),$$

$$P_n^{\xi} := \frac{1}{n} \sum_{i=1}^{n} \xi_i \gamma(Z_i),$$

and write

$$\|P_n - P\|_{\Gamma} := \sup_{\gamma \in \Gamma} |(P_n - P)\gamma|,$$

$$\|P_n' - P\|_{\Gamma} := \sup_{\gamma \in \Gamma} |(P_n' - P)\gamma|,$$

$$\|P_n^{\xi}\|_{\Gamma} := \sup_{\gamma \in \Gamma} |P_n^{\xi}\gamma|.$$

Lemma *We have*

$$\mathbb{E}\|P_n - P\|_{\Gamma} \leq 2\mathbb{E}\|P_n^{\xi}\|_{\Gamma}.$$

6. Entropy

Let (Γ, d) be a subset of a metric space.

Definition

For $u > 0$, a u-covering of Γ is defined as a collection $\{\gamma_j\}_{j=1}^N$ such that
$\forall \gamma$ there is a γ_j with $d(\gamma, \gamma_j) \leq u$.

The covering number $N(\cdot, \Gamma, d)$ is defined for all $u > 0$ as
$N(u, \Gamma, d) := \min\{N :$ there is a u-covering $\{\gamma_j\}_{j=1}^N\}$.

The entropy is $\mathcal{H}(\cdot, \Gamma, d) := \log(1 + N(\cdot, \Gamma, d))$.

Example. Let

$$\Gamma = \{\gamma : [0,1] \to [0,1] :\ \|\gamma^{(m)}\|_\infty \leq 1\}.$$

Then

$$\mathcal{H}(u, \Gamma, \|\cdot\|_\infty) \leq A_m u^{-\frac{1}{m}},\ u > 0.$$

Example. Let

$$\Gamma = \{\gamma : \mathbf{R} \to [0,1] :\ \gamma \uparrow\}.$$

Let Q be some probability measure and $\|\cdot\|_Q$ be the $L_2(Q)$-norm. Then

$$\mathcal{H}(u, \Gamma, \|\cdot\|_Q) \leq A u^{-1},\ u > 0.$$

7. Moment inequalities for an infinite class

Let Γ be some class of functions.

We assume

$$\sup_{\text{prob. measures } Q} \mathcal{H}(\cdot, \Gamma, \|\cdot\|_Q) \le \mathcal{H}(\cdot),$$

and write

$$\psi(\cdot) := 24 \int_0^{\cdot} \sqrt{\mathcal{H}(u)} du.$$

Let $\|\cdot\|$ be the $L_2(P)$-norm and $\|\cdot\|_n$ be the $L_2(P_n)$-norm.

Define

$$\sigma := \sup_{\gamma \in \Gamma} \|\gamma\|, \ \hat{\sigma} := \sup_{\gamma \in \Gamma} \|\gamma\|_n.$$

Lemma *We have*

$$\mathbb{E}\|P_n^\xi\|_\Gamma \leq \frac{\mathbb{E}\psi(\hat{\sigma})}{2\sqrt{n}}.$$

Proof. ...

Contraction principle *(Ledoux and Talagrand (1991)) (It holds more generally for Lipschitz functions.) Suppose* $\|\gamma\|_\infty \leq K$ *for all* $\gamma \in \Gamma$. *Then*

$$\mathbb{E}\left(\sup_{\gamma\in\Gamma} |P_n^\xi \gamma^2|\right) \leq 2K\mathbb{E}\left(\sup_{\gamma\in\Gamma} |P_n^\xi \gamma|\right).$$

Lemma. Suppose $\|\gamma\|_\infty \leq K$ for all $\gamma \in \Gamma$. Let H be the convex conjugate of $G_\psi = G \circ \psi^{-1}$, with $G(u) = u^2$, $u > 0$. Then for all

$$\sigma^2 \geq \frac{\delta}{1-\delta} H\left(\frac{2K}{\sqrt{n}\delta}\right),$$

we have

$$\mathbb{E}\psi(\hat{\sigma}) \leq \psi\left(\sigma\sqrt{2/1-\delta}\right).$$

Proof. $\cdots$

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