About the spectral analysis of large random Markov kernels

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Abstract

This tiny note concerns the spectral analysis of models of random Markov kernels, centered around joint works in collaboration with Charles Bordenave and Pietro Caputo, with a "large random matrices" point of view. Some open problems are given.

A (rather old now) survey about the spectral analysis of large random Markov kernels can be found in [9]. An up to date review can be found in the Habilitation thesis of Charles Bordenave under preparation (gives a complete bibliography). These models can be seen as random walks or Markov chains in random environment. From the random matrix theory point of view, these random matrices have dependent entries, and what is know concerns mostly first order asymptotics for empirical spectral distributions, spectral edge, and principal eigenvector. In particular, the study of the asymptotic fluctuations is completely open.

Irreversible models

We say that $M \in \mathcal{M}_n([0,1])$ is Markov when each row sums up to 1. This set is a convex and compact polytope, and also a compact semigroup. If X is $n \times n$ with i.i.d. real non negative entries of mean m and variance σ^2 and if D is the diagonal matrix with $D_{ii} = X_{i1} + \cdots + X_{in}$ then $P = D^{-1}X$ is a random Markov (transition) matrix. When X_{11} is exponential, then we recover the uniform probability measure on Markov matrices (Dirichlet Markov Ensemble [10]). The asymptotic analysis of the singular values and the eigenvalues of such matrices is done in [7]. Some aspects can be improved using the technology developed in [8] and [3]. For instance, it is conjectured that the invariant vector is asymptotically uniform, and this can be proved probably by imitating the argument used in [3]. It is also conjectured that the second largest eigenvalue in module has Gumbel fluctuations as for Ginibre Ensembles. The study of the case where X_{11} has a heavy tail is open, and may be done by reusing part of [6]. In another direction, the study of random matrices with i.i.d. log-concave rows is done in [1]. The study of fully log-concave matrices (on the n^2 entries) is open.

We say that a Markov matrix M is doubly stochastic if its transpose is also Markov. The set of $n \times n$ doubly stochastic matrices, known as the Birkhoff polytope, is convex and compact, and is a semigroup. It can be equipped with the uniform probability measure.

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The asymptotic behavior of the singular values was studied in [11]. The behavior of the eigenvalues is open and a circular law is conjectured[1]. One of the most fascinating question regarding doubly stochastic matrices is the construction of the uniform measure from an i.i.d. structural recipe, without using rejection, as for the Markov polytope. Recall that the Birkhoff polytope is the convex envelope of permutation matrices (Birkhoff-von Neumann theorem). The spectral analysis of random permutation matrices is quite fun and can be linked with the Ewens distribution (Chinese restaurant process), see [9].

If X is as above, then $L = X - D$ is a random Markov generator, studied in [3]. Here again, the heavy tailed case is open, and a possible solution is to exploit part of [6].

Reversible models

If X is symmetric, then P is a reversible Markov kernel and its reversible measure is given by the diagonal of D. The spectrum is real. A Wigner theorem is available and the first order edge behavior is well understood [4]. It is conjectured that the second largest eigenvalue has Tracy-Widom fluctuations. The case where X_{11} has a heavy tail is particularly interesting and gives rise to new limiting spectral distributions [5], while the behavior of the invariant vector gives rise to Poisson or Poisson-Dirichlet statistics.

The study of random reversible generators instead of random reversible transition matrices (Markov matrices) is discussed in [4] and may be revisited with [3] in mind.

Beyond the complete graph

The Markov transition matrix P can be seen as a random walk in random environment (i.i.d. conductances) on the complete graph. One may consider other sequences of graphs. The study of chain graphs corresponds to random birth and death processes, and to tridiagonal matrices, giving rise to non universal limiting spectral distributions, see [4] and [12].

Other models

If we allow the law X_{11} to depend on n with an atom at 0 then we give rise to an Erdős-Rényi random graph skeleton as in [3]. However, the complete graph model with i.i.d. conductances does not allow to consider preferential attachment graphs, ubiquitous in nature (e.g. social networks). The spectral analysis of large preferential attachment graphs is open.

Another fascinating problem is the analogue of the Kesten-McKay theorem, for random oriented regular graphs, linked with the free sum of Haar unitary [8].

Concepts

Models:

- Hermitian versus non-Hermitian
- Reversible versus non-reversible
- Light tails versus heavy tails
- Transition matrices versus generators
- Complete graph versus structured graphs
- Sparsity via atom versus via heavy tails

[1]Hoi H. Nguyen has apparently proved this circular law very recently!

Questions:

- First order versus second order
- Singular values versus eigenvalues
- Bulk versus edge
- Most eigenvectors versus invariant vector (spatial and distributional (de)localization)
- Universality versus non-universality

Techniques:

- Perturbative methods for spectra
- Resolvent method
- Moments method
- Hermitization method
- Objective method
- Concentration of measure
- Asymptotic geometric analysis (least singular values)
- Non-linear aspects of linear algebra (Weyl, Schur, Sylvester, . . .)
- Poisson point processes
- Free probability

Caveat: the bibliography below lists our papers and some few related papers. Many other works on closely related models are cited in all these papers (and in Charles's HDR).

References

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