About confined particles with singular pair repulsion

Djalil Chafaï

Université Paris-Dauphine

Talking Across Fields
Toulouse, France
Friday March 28, 2014

Outline

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Kac about Boltzmann

Ludwig Boltzmann

1844 – 1906

Boltzmann summarized most (but not all) of his work in a two volume treatise Vorlesungen uber Gastheorie. This is one of the greatest books in the history of exact sciences and the reader is strongly advised to consult it. **Mark Kac (1959)**.

Mechanical view of nature – Order and randomness

If you ask me about my innermost conviction whether our century will be called the century of iron or the century of steam or electricity, I answer without hesitation: It will be called the century of the mechanical view of Nature, the century of Darwin. **Ludwig Boltzmann (1886)**.

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- Continuous entropy:

$$\mathcal{E}(f) = -\int f(t) \log f(t)\, dt = -H(f)$$

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- $\nabla\mathcal{E}(f) = -(1 + \log(f))$

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- History: . . . , Kac, Lanford, Cercignani, Villani, St-Raymond, . . .

Classical Central Limit Theorem

Theorem (Classical CLT)

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  - Proof (2004): Artstein-Ball-Barthe-Naor

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- Entropy concavity along Markov semigroup ($\rightarrow$ curvature)
Stam, . . . , Bakry-Émery, . . . , Villani, Sturm, . . .

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- Ex.: $\mathcal{A} = \mathcal{L}(\ell^2_{\mathbb{C}}(F_n))$, $\tau(a) = \langle a\delta_e, \delta_e\rangle$, $L_{g_i^{\pm 1}}$ are free

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- $P^{2m}_{F_n}(e,e) = \frac{1}{(2n)^{2m}}\langle(\sum_{k=1}^n(L_{g_k^{+1}} + L_{g_k^{-1}}))^{2m}\delta_e, \delta_e\rangle \sim_{n\to\infty} \frac{1}{(2n)^m}\frac{1}{1+m}\binom{2m}{m} \quad (n>1)$
  Kesten distribution (PhD thesis, 1958)!

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- Shlyakhtenko: $\mathcal{E}$ is monotonic along Voiculescu Free CLT!

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- Density of eigenvalues (change of variables & Jacobian)

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- Penalized Voiculescu entropy!

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- Universality (Coulomb gases): Deift, Saff & Totik, ...

Beyond random matrices, how about particles in $\mathbb{R}^d$?

Interacting Particles System

- Particles at positions $x_1, \ldots, x_N$ in $\mathbb{R}^d$ with charge $1/N$

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$$
\begin{aligned}
E_N(x_1, \ldots, x_N) &= \sum_{i=1}^{N} \frac{1}{N} V(x_i) + \sum_{1 \leqslant i < j \leqslant N} \frac{1}{N^2} W(x_i, x_j) \\
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- For $d \leqslant 2$, Random Normal Matrix Ensemble

$$M = U \mathrm{Diag}(x_1, \ldots, x_N) U^* \text{ with } U \text{ Haar independent of } x.$$

Randomness

- Boltzmann measure at inverse temperature $\beta_N > 0$

$$\frac{dP_N(x_1,\ldots,x_N)}{dx_1\cdots dx_N} = \frac{e^{-\beta_N E_N(x_1,\ldots,x_N)}}{Z_N} = \prod_{i=1}^{N} f_1(x_i) \prod_{1\leqslant i<j\leqslant N} f_2(x_i,x_j)$$

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- Infinitesimal generator ($\mathbb{R}^{dN}$): $L = \beta_N^{-1}\Delta - \nabla E_N \cdot \nabla$

Examples

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- Riesz interaction, $0 < \alpha < d$ (Coulomb if $d \geqslant 3$ and $\alpha = 2$)

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Motivation: physical control problem

- Empirical measure

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- How to tune external field V and cooling scheme β_N s.t.

$$\lim_{N\to\infty} \mu_N = \mu_* \quad ?$$

General idea

- Limiting energy functional (quadratic form)

$$\mu \in \mathcal{M}_1 \mapsto I(\mu) = \int V(x)\, d\mu(x) + \frac{1}{2} \iint W(x,y)\, d\mu(x) d\mu(y).$$

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Large Deviations Principle on $(\mathcal{M}_1, d)$

Theorem (Large Deviations Principle, C.-Gozlan-Zitt 2013)

- *I is lower semi-continuous with compact level sets*

$\to$ BenArous-Guionnet, BenArous-Zeitouni, Hiai-Petz, Hardy.

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- **Cooling scheme.** $\beta_N \gg N \log(N)$ (RMT: $\beta_N = N^2$)

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About $\beta_N \gg N\log(N)$

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Rate function analysis

- I is convex iff W is weakly positive

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$$DU^\mu = -\mu \quad \text{and} \quad \mu_* \approx DV$$

Rate function in the Riesz case

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Thank you for your attention!