Concentration Inequalities for Gibbs random fields

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Gibbs measures are (non-product) measures on the configuration space $\mathscr{S}^{\mathbb{Z}^d}$, $d \geq 2$.

In this talk: $\mathscr{S} = \{-1, +1\}$ (spins) for simplicity but any finite set $\mathscr{S}$ is ok.

Abstract:

- At sufficiently high temperature, we have a Gaussian concentration bound.
  In fact, such a bound holds in Dobrushin's uniqueness regime.
- For some Gibbs measures at sufficiently low temperature, we have a 'stretched exponential' concentration bound.
- These bounds have many consequences.

Boltzmann-Gibbs kernel

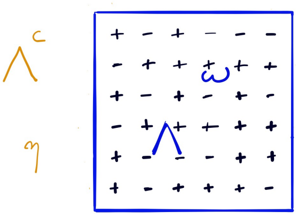
$$\gamma_{\Lambda}^{(\boldsymbol{\beta})}(\omega|\eta) = \frac{\exp\big(-\boldsymbol{\beta}\,\mathcal{H}_{\Lambda}(\omega|\eta)\big)}{Z_{\Lambda}^{(\boldsymbol{\beta})}(\eta)}, \quad \Lambda \Subset \mathbb{Z}^d, \omega, \eta \in \mathscr{S}^{\mathbb{Z}^d}.$$

$\rightsquigarrow$ (DLR equation) Gibbs measures on $\mathscr{S}^{\mathbb{Z}^d}$ depending on η in general.

Parameter $\beta \geq 0$: inverse temperature.

Special case: $\beta = 0$ (infinite temperature)

$\rightsquigarrow$ uniform **product measure** ($\rightsquigarrow$ Gaussian concentration bound).

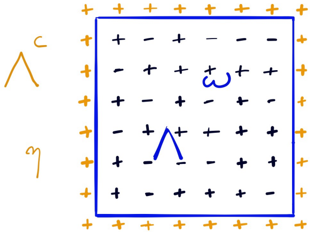


The ferromagnetic Ising model (Markov random field)

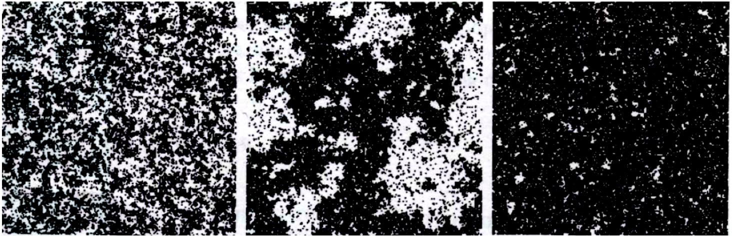
$$\mathcal{H}_\Lambda(\omega|\eta) = -\sum_{\substack{\boldsymbol{i},\boldsymbol{j}\in\Lambda \\ \|\boldsymbol{i}-\boldsymbol{j}\|_1=1}} \omega_{\boldsymbol{i}}\,\omega_{\boldsymbol{j}} - \sum_{\substack{\boldsymbol{i}\in\partial\Lambda,\, \boldsymbol{j}\notin\Lambda \\ \|\boldsymbol{i}-\boldsymbol{j}\|_1=1}} \omega_{\boldsymbol{i}}\,\eta_{\boldsymbol{j}}$$

$\eta_{\boldsymbol{j}} = +1, \forall \boldsymbol{j} \in \mathbb{Z}^d$ ("+-boundary condition"), gives rise to μ^+.

Fact ($d \geq 2$): there exists a unique Gibbs measure μ for all $\beta < \beta_c$, whereas there are several ones for all $\beta > \beta_c$, depending on η, in fact, two extremal ones: μ^+ and μ^- (*i.e.*, ergodic under the shift action).



Phase transition for $d = 2$



β increases from left to right
'$+$' $\leftrightarrow$ black, '$-$' $\leftrightarrow$ white

$\beta_c = (1/2)\sinh^{-1}(1) \approx 0.4407$

The magnetization

$M_n(\omega) := \sum_{\boldsymbol{i} \in C_n} s_0(T_{\boldsymbol{i}}\, \omega)$, where $s_0(\omega) = \omega_0$, be the total magnetization in C_n, and where $(T_{\boldsymbol{i}}\, \omega)_{\boldsymbol{j}} = \omega_{\boldsymbol{j}-\boldsymbol{i}}$ (shift operator). Then

$$\frac{M_n(\omega)}{(2n+1)^d}$$

is the magnetization per spin in C_n. For any shift-invariant probability measure ν on $\mathscr{S}^{\mathbb{Z}^d}$,

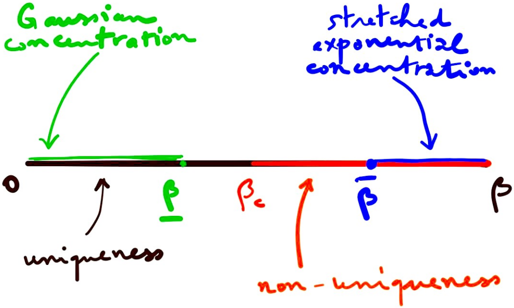
$$\mathbb{E}_\nu \left[\frac{M_n(\omega)}{(2n+1)^d} \right] = \mathbb{E}_\nu [s_0]$$

is the mean magnetization per site (magnetization, for short) wrt ν.

The following is well-known for the Ising model ($d \geq 2$):

- for $\beta < \beta_c$, $\mathbb{E}_\mu [s_0] = 0$;
- for $\beta > \beta_c$, $\mathbb{E}_{\mu^+} [s_0] \neq 0$.

Concentration for the Ising model



Let $F : \mathscr{S}^{\mathbb{Z}^d} \to \mathbb{R}$ and

$$\ell_{\boldsymbol{i}}(F) = \sup_{\omega \in \mathscr{S}^{\mathbb{Z}^d}} |F(\omega^{(\boldsymbol{i})}) - F(\omega)|, \;\; \boldsymbol{i} \in \mathbb{Z}^d,$$

where $\omega^{(\boldsymbol{i})}$ is obtained from ω by flipping the spin at $\boldsymbol{i}$.

Theorem: Gaussian concentration bound ($\beta < \underline{\beta}$)

Let μ be the (unique) Gibbs measure for the Ising model. There exists a constant $D > 0$ such that, for all functions F with $\sum_{\boldsymbol{i}\in\mathbb{Z}^d} \ell_{\boldsymbol{i}}(F)^2 < +\infty$, one has

$$\mathbb{E}_\mu\left[\exp(F - \mathbb{E}_\mu(F))\right] \leq \exp\left(D \sum_{\boldsymbol{i}\in\mathbb{Z}^d} \ell_{\boldsymbol{i}}(F)^2\right).$$

Remark. As shown by C. Külske, the Gaussian concentration bound holds in the Dobrushin uniqueness regime with $D = 2(1 - \mathfrak{c}(\gamma))^{-2}$, where $\mathfrak{c}(\gamma)$ is Dobrushin's contraction coefficient.

Recall that the Gaussian concentration implies that for all $u \geq 0$ one has

$$\mu\Big(\omega \in \mathscr{S}^{\mathbb{Z}^d} : |F(\omega) - \mathbb{E}_\mu[F]| \geq u\Big) \leq 2\exp\left(-\frac{u^2}{4D\sum_{\boldsymbol{i}\in\mathbb{Z}^d} \ell_{\boldsymbol{i}}(F)^2}\right).$$

Remark. All local functions satisfy $\sum_{\boldsymbol{i}\in\mathbb{Z}^d} \ell_{\boldsymbol{i}}(F)^2 < +\infty$.

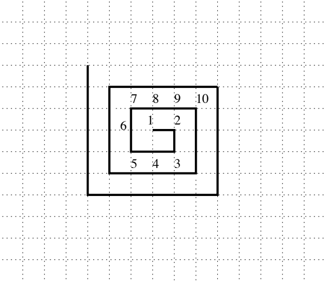
At sufficiently low temperature, we can gather all moment bounds to obtain the following. We denote by μ^+ the Gibbs measure for the $+$-phase of the Ising model.

Theorem: Stretched-exponential concentration bound ($\beta > \overline{\beta}$)

There exists $\varrho = \varrho(\beta) \in (0,1)$ and $c_\varrho > 0$ such that for all functions F with $\sum_{\boldsymbol{i}\in\mathbb{Z}^d} \ell_{\boldsymbol{i}}(F)^2 < +\infty$, for all $u \geq 0$, one has

$$\mu^+\Big(\omega \in \mathscr{S}^{\mathbb{Z}^d} : |F(\omega) - \mathbb{E}_{\mu^+}[F]| \geq u\Big) \leq 4 \exp\left(\frac{-\,c_\varrho\, u^\varrho}{\left(\sum_{\boldsymbol{i}\in\mathbb{Z}^d} \ell_{\boldsymbol{i}}(F)^2\right)^{\frac{\varrho}{2}}}\right).$$

The basic ingredients in proofs



Enumeration of $\mathbb{Z}^d$:

$$e : \mathbb{Z}^d \to \mathbb{N}$$

$$(\leq \boldsymbol{i}) := \{\boldsymbol{j} \in \mathbb{Z}^d : e(\boldsymbol{j}) \leq e(\boldsymbol{i})\}$$

$\mathcal{F}_{\leq \boldsymbol{i}}$: $\sigma-$field generated by $\omega_{\boldsymbol{j}}, \boldsymbol{j} \leq \boldsymbol{i}$

We have $\quad F - \mathbb{E}[F] = \sum_{\boldsymbol{i} \in \mathbb{Z}^2} \Delta_{\boldsymbol{i}}, \quad \Delta_{\boldsymbol{i}} := \mathbb{E}[F|\mathcal{F}_{\leq \boldsymbol{i}}] - \mathbb{E}[F|\mathcal{F}_{< \boldsymbol{i}}]$

and

$$\Delta_{\boldsymbol{i}} \leq \left(D^{\omega_{\leq \boldsymbol{i}}} \ell(F)\right)_{\boldsymbol{i}}$$

$$\text{where} \quad D^{\omega_{\leq \boldsymbol{i}}}_{\boldsymbol{i},\boldsymbol{j}} := \widehat{\mathbb{P}}_{\boldsymbol{i},+,-}\left(\omega^{(1)}_{\boldsymbol{j}} \neq \omega^{(2)}_{\boldsymbol{j}}\right)$$

where we maximally couple

$$\mathbb{P}(\cdot \,|\, \omega_{<\boldsymbol{i},+_{\boldsymbol{i}}}) \quad \text{and} \quad \mathbb{P}(\cdot \,|\, \omega_{<\boldsymbol{i},-_{\boldsymbol{i}}}).$$

Applications

Other models besides the standard Ising model: Potts, long-range Ising, etc.

- Ergodic sums in *arbitrarily shaped* volumes;
- Fluctuations in the Shannon-McMillan-Breiman theorem;
- First occurrence of a pattern of a configuration in another configuration;
- Bounding $\overline{d}$-distance by relative entropy;
- Fattening patterns;
- Speed of convergence of the empirical measure;
- Almost-sure central limit theorems.

Application 1: "speed" of convergence of the empirical measure

Take $\Lambda \Subset \mathbb{Z}^d$ and $\omega \in \mathscr{S}^{\mathbb{Z}^d}$ and let

$$\mathcal{E}_\Lambda(\omega) = \frac{1}{|\Lambda|} \sum_{\boldsymbol{i} \in \Lambda} \delta_{T_{\boldsymbol{i}}\,\omega}$$

where $(T_{\boldsymbol{i}}\,\omega)_{\boldsymbol{j}} = \omega_{\boldsymbol{j}-\boldsymbol{i}}$ (shift operator).

Let μ be an ergodic measure on $\mathscr{S}^{\mathbb{Z}^d}$. If $(\Lambda_n)_n$ is a sequence of cube $\uparrow \mathbb{Z}^d$ (more generally, a van Hove sequence), then

$$\mathcal{E}_{\Lambda_n}(\omega) \xrightarrow[\text{weakly}]{n\to\infty} \mu.$$

Question: If μ is a Gibbs measure, what is the "speed" of this convergence?

KANTOROVICH DISTANCE on the set of probability measures on $\mathscr{S}^{\mathbb{Z}^d}$:

$$d_{Kanto}(\mu_1, \mu_2) = \sup_{\substack{G:\mathscr{S}^{\mathbb{Z}^d} \to \mathbb{R} \\ G\ 1-\text{Lipshitz}}} \left(\mathbb{E}_{\mu_1}(G) - \mathbb{E}_{\mu_2}(G)\right)$$

where $|G(\omega) - G(\omega')| \leq d(\omega, \omega') = 2^{-k}$, where k is the sidelength of the largest cube in which ω and ω' coincide.

Lemma. Let μ be a probability measure and

$$F(\omega) = \sup_{\substack{G:\mathscr{S}^{\mathbb{Z}^d} \to \mathbb{R} \\ G\ 1-\text{Lipshitz}}} \left(\frac{1}{|\Lambda|} \sum_{\boldsymbol{i} \in \Lambda} G(T_{\boldsymbol{i}}\omega) - \mathbb{E}_{\mu}(G) \right).$$

Then

$$\sum_{\boldsymbol{i} \in \mathbb{Z}^d} \ell_{\boldsymbol{i}}(F)^2 \leq \frac{c_d}{|\Lambda|}$$

where $c_d > 0$ depends only on d.

Proof

Ising model at high & low temperature

Gaussian concentration for the empirical measure ($\beta < \underline{\beta}$)

Let μ be the (unique) Gibbs measure of the Ising model. There exists a constant $C > 0$ such that, for all $\Lambda \Subset \mathbb{Z}^d$ and for all $u \geq 0$, one has

$$\mu\Big\{\omega \in \mathscr{S}^{\mathbb{Z}^d} : \Big| d_{Kanto}(\mathcal{E}_\Lambda(\omega), \mu) - \mathbb{E}_\mu\big[d_{Kanto}(\mathcal{E}_\Lambda(\cdot), \mu)\big]\Big| \geq u\Big\}$$

$$\leq 2\exp\big(-C\,|\Lambda|u^2\big).$$

We denote by μ^+ the Gibbs measure for the $+$-phase of the Ising model.

Stretched-exponential concentration for the empirical measure ($\beta > \overline{\beta}$)

There exist $\varrho = \varrho(\beta) \in (0,1)$ and a constant $c_\varrho > 0$ such that, for all $\Lambda \Subset \mathbb{Z}^d$ and for all $u \geq 0$, one has

$$\mu^+\Big\{\omega \in \mathscr{S}^{\mathbb{Z}^d} : \Big| d_{\textit{Kanto}}(\mathcal{E}_\Lambda(\omega), \mu^+) - \mathbb{E}_{\mu^+}\big[d_{\textit{Kanto}}(\mathcal{E}_\Lambda(\cdot), \mu^+)\big]\Big| \geq u\Big\}$$

$$\leq 4 \, \exp\left(-c_\varrho |\Lambda|^{\frac{\varrho}{2}} u^\varrho\right).$$

Can we estimate $\mathbb{E}_\mu\left[d_{Kanto}(\mathcal{E}_\Lambda(\cdot),\mu)\right]$?

Let

$$\mathscr{L} = \left\{G : \mathscr{S}^{\mathbb{Z}^d} \to \mathbb{R} : G \text{ 1-Lipschitz}\right\}$$

and

$$\mathcal{Z}_G^\Lambda := \frac{1}{|\Lambda|}\sum_{\boldsymbol{i}\in\Lambda}(G\circ T_{\boldsymbol{i}} - \mathbb{E}_\mu(G)),\ \Lambda \Subset \mathbb{Z}^d.$$

Then

$$\mathbb{E}_\mu\left[d_{Kanto}(\mathcal{E}_\Lambda(\cdot),\mu)\right] = \mathbb{E}_\mu\left(\sup_{G\in\mathscr{L}}\mathcal{Z}_G^\Lambda\right).$$

Notice that we have functions defined on a Cantor space, which is really different from the case of, say, $[0,1]^k \subset \mathbb{R}^k$.

Theorem

Let μ be a probability measure on $\mathscr{S}^{\mathbb{Z}^d}$ satisfying the Gaussian concentration bound. Then

$$\mathbb{E}_\mu \left[d_{Kanto} \left(\mathcal{E}_\Lambda(\cdot), \mu \right) \right] \preceq \begin{cases} |\Lambda|^{-\frac{1}{2}(1+\log|\mathscr{S}|)^{-1}} & \text{if} \quad d = 1 \\ \exp\left(-\frac{1}{2} \left(\frac{\log|\Lambda|}{\log|\mathscr{S}|} \right)^{1/d} \right) & \text{if} \quad d \geq 2. \end{cases}$$

For (a_Λ) and (b_Λ) indexed by finite subsets of $\mathbb{Z}^d$ we denote $a_\Lambda \preceq b_\Lambda$ if, for every sequence (Λ_n) such that $|\Lambda_n| \to +\infty$ as $n \to +\infty$, we have $\limsup_n \frac{\log a_{\Lambda_n}}{\log b_{\Lambda_n}} \leq 1$.

It is possible to get *bounds* but they are really messy.

Application 2: Almost-sure central limit theorems (only part of the story)

This application shows that one can also get *limit theorems* out of concentration inequalities.

Informal statement:
If you know that the central limit theorem holds for some function $f : \mathscr{S}^{\mathbb{Z}^d} \to \mathbb{R}$ wrt to a shift-invariant probability measure, and if you can prove that this measure satisfies a *moment concentration bound of order* 2, then the almost-sure central limit theorem holds in the sense of Kantorovich distance.

Given $f : \mathscr{S}^{\mathbb{Z}^d} \to \mathbb{R}$ and ν a shift-invariant probability measure on $\mathscr{S}^{\mathbb{Z}^d}$, the usual form of the CLT is: for all $u \in \mathbb{R}$

$$\lim_{n\to\infty} \nu \left\{ \omega \in \mathscr{S}^{\mathbb{Z}^d} : \frac{\sum_{\boldsymbol{i}\in C_n} f(T_{\boldsymbol{i}}\omega)}{(2n+1)^{\frac{d}{2}}} \leq u \right\} = G_{0,\sigma_f}\big((-\infty, u]\big)$$

where

$$\sigma_f^2 = \sum_{\boldsymbol{i}\in\mathbb{Z}^d} \int f \cdot f \circ T_{\boldsymbol{i}} \,\mathrm{d}\nu \in (0, +\infty)$$

and where G_{0,σ_f} is the Gaussian measure with mean 0 and variance σ_f.

The CLT can be re-written as

$$\lim_{n\to\infty} \mathbb{E}_\nu \left[\mathbb{1}_{\left\{ \sum_{\boldsymbol{i}\in C_n} f(T_{\boldsymbol{i}}\cdot)/(2n+1)^{\frac{d}{2}} \leq u \right\}} \right] = G_{0,\sigma_f}\big((-\infty, u]\big).$$

The ASCLT consists in replacing $\mathbb{E}_\nu$ by a point-wise logarithmic average and get an almost-sure version of the CLT: for all $u \in \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbb{1}_{\left\{ \sum_{\boldsymbol{i}\in C_n} f(T_{\boldsymbol{i}}\,\omega)/(2n+1)^{\frac{d}{2}} \leq u \right\}} = G_{0,\sigma_f}\big((-\infty, u]\big)$$

for ν-a.e. ω.

We will only formulate two results for $f = s_0$ (magnetization).

To state the theorems, define

$$d_{Kanto}(\nu_1, \nu_2) = \sup\left(\mathbb{E}_{\nu_1}(g) - \mathbb{E}_{\nu_2}(g)\right)$$

where the sup is taken over all functions $g : \mathbb{R} \to \mathbb{R}$ that are 1-Lipschitz.

Metrizes the weak topology on the set of probability measures on $\mathbb{R}$ with a first moment.

High-temperature Ising model

Theorem

Let $\beta < \underline{\beta}$. Then, for μ-a.e. $\omega \in \mathscr{S}^{\mathbb{Z}^d}$, we have

$$\lim_{N\to\infty} d_{Kanto}\left(\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\,\delta_{M_n(\omega)/(2n+1)^{\frac{d}{2}}}, G_{0,\sigma^2}\right) = 0$$

where

$$\sigma^2 = \sum_{\boldsymbol{i}\in\mathbb{Z}^d}\int s_0 \cdot s_0 \circ T_{\boldsymbol{i}}\,\mathrm{d}\mu \in (0,\infty).$$

Low-temperature Ising model

Theorem

Let $\beta > \overline{\beta}$. Then, for μ^+-a.e. $\omega \in \mathscr{S}^{\mathbb{Z}^d}$, we have

$$\lim_{N\to\infty} d_{Kanto}\left(\frac{1}{\ln N}\sum_{n=1}^{N}\frac{1}{n}\,\delta_{(M_n(\omega)-\mathbb{E}_{\mu^+}[s_0])/(2n+1)^{\frac{d}{2}}}, G_{0,\sigma^2}\right) = 0$$

where

$$\sigma^2 = \sum_{\boldsymbol{i}\in\mathbb{Z}^d}\int s_0 \cdot s_0 \circ T_{\boldsymbol{i}}\,\mathrm{d}\mu^+ \in (0,\infty).$$

Some open questions

1. 'Close the gap' between $\underline{\beta}$ and $\overline{\beta}$.
2. Write the proof in the low temperature regime in the setting of Pirogov-Sinai theory.
3. Get the optimal ϱ in

$$\exp\left(\frac{-\,c_{\varrho}\,u^{\varrho}}{\left(\sum_{\boldsymbol{i}\in\mathbb{Z}^d}\ell_{\boldsymbol{i}}(F)^2\right)^{\frac{\varrho}{2}}}\right).$$

References

- J.-R. Chazottes, P. Collet, F. Redig. *On concentration inequalities and their applications for Gibbs measures in lattice systems.* J. Stat. Phys. (2017).

  https://arxiv.org/abs/1610.06502
- J.-R. Chazottes, P. Collet, C. Külske, F. Redig. *Concentration inequalities for random fields via coupling*. Probab. Theory Relat. Fields (2007). https://arxiv.org/abs/math/0503483
- C. Külske. *Concentration inequalities for functions of Gibbs fields with application to diffraction and random Gibbs measures.* Commun. Math. Phys. (2003).

  http://www.ruhr-uni-bochum.de/imperia/md/content/mathematik/

  lehrstuhl-ii/kuelske-pub/autocorrdobrushin-fulltext.pdf

DLR equation

μ is a Gibbs measure for a given potential Φ if, for all $\Lambda \Subset \mathbb{Z}^d$ and for all $A \in \mathfrak{B}(\mathscr{S}^{\mathbb{Z}^d})$

$$\mu(A) = \int d\mu(\eta) \sum_{\omega' \in \Lambda} \gamma_\Lambda(\omega'|\eta)\, \mathbb{1}_A(\omega'_\Lambda \eta_{\Lambda^c})$$

where Φ is a real-valued function having two arguments: a finite subset of $\mathbb{Z}^d$ and a configuration $\omega \in \mathscr{S}^{\mathbb{Z}^d}$, and where

$$\mathcal{H}_\Lambda(\omega|\eta) = \sum_{\Lambda' \cap \Lambda \neq \emptyset} \Phi(\Lambda', \omega_\Lambda \eta_{\mathbb{Z}^d \setminus \Lambda})$$

where Λ' runs through the set of finite subsets of $\mathbb{Z}^d$.

Boltzmann-Gibbs kernel

Dobrushin contraction coefficient

Let

$$C_{\boldsymbol{i},\boldsymbol{j}}(\gamma) = \sup_{\substack{\omega,\omega' \in \mathscr{S}^{\mathbb{Z}^d} \\ \omega_{\mathbb{Z}^d \setminus \boldsymbol{j}} = \omega'_{\mathbb{Z}^d \setminus \boldsymbol{j}}}} \|\gamma_{\{\boldsymbol{i}\}}(\cdot|\omega) - \gamma_{\{\boldsymbol{i}\}}(\cdot|\omega')\|_\infty .$$

Then in our context $C_{\boldsymbol{i},\boldsymbol{j}}$ only depends on $\boldsymbol{i} - \boldsymbol{j}$ and we define

$$\mathfrak{c}(\gamma) = \sum_{\boldsymbol{i} \in \mathbb{Z}^d} C_{0,\boldsymbol{i}}(\gamma).$$

Dobrushin's uniqueness regime: $\mathfrak{c}(\gamma) < 1$.

◂ Gaussian concentration bound

van Hove sequence

A sequence $(\Lambda_n)_n$ of nonempty finite subsets of $\mathbb{Z}^d$ is said to tend to infinity in the sense of van Hove if, for each $\boldsymbol{i} \in \mathbb{Z}^d$, one has

$$\lim_{n\to+\infty} |\Lambda_n| = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{|(\Lambda_n + \boldsymbol{i})\backslash\Lambda_n|}{|\Lambda_n|} = 0.$$

◂ Empirical measure

Proof of the Lemma

Let $\omega, \omega' \in \mathscr{S}^{\mathbb{Z}^d}$ and $G : \mathscr{S}^{\mathbb{Z}^d} \to \mathbb{R}$ be a 1-Lipschitz function. Without loss of generality, we can assume that $\mathbb{E}_\mu(G) = 0$. We have

$$\sum_{\boldsymbol{i}\in\Lambda} G(T_{\boldsymbol{i}}\,\omega) \leq \sum_{\boldsymbol{i}\in\Lambda} G(T_{\boldsymbol{i}}\,\omega') + \sum_{\boldsymbol{i}\in\Lambda} d(T_{\boldsymbol{i}}\,\omega, T_{\boldsymbol{i}}\,\omega').$$

Taking the supremum over 1-Lipschitz functions thus gives

$$F(\omega) - F(\omega') \leq \sum_{\boldsymbol{i}\in\Lambda} d(T_{\boldsymbol{i}}\,\omega, T_{\boldsymbol{i}}\omega').$$

We can interchange ω and ω' in this inequality, whence

$$|F(\omega) - F(\omega')| \leq \sum_{\boldsymbol{i}\in\Lambda} d(T_{\boldsymbol{i}}\,\omega, T_{\boldsymbol{i}}\,\omega').$$

Now we assume that there exists $\boldsymbol{k} \in \mathbb{Z}^d$ such that $\omega_{\boldsymbol{j}} = \omega'_{\boldsymbol{j}}$ for all $\boldsymbol{j} \neq \boldsymbol{k}$. This means that $d(T_{\boldsymbol{i}}\,\omega, T_{\boldsymbol{i}}\,\omega') \leq 2^{-\|\boldsymbol{k}-\boldsymbol{i}\|_\infty}$ for all $\boldsymbol{i} \in \mathbb{Z}^d$, whence

$$\ell_{\boldsymbol{k}}(F) \leq \sum_{\boldsymbol{i}\in\Lambda} 2^{-\|\boldsymbol{k}-\boldsymbol{i}\|_\infty}.$$

Therefore, using Young's inequality,

$$\begin{aligned}\sum_{\boldsymbol{i}\in\mathbb{Z}^d} \ell_{\boldsymbol{i}}(F)^2 &\leq \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left(\sum_{\boldsymbol{i}\in\mathbb{Z}^d} \mathbb{1}_\Lambda(\boldsymbol{i})\, 2^{-\|\boldsymbol{k}-\boldsymbol{i}\|_\infty} \right)^2 \\ &\leq \sum_{\boldsymbol{i}\in\mathbb{Z}^d} \mathbb{1}_\Lambda(\boldsymbol{i}) \times \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} 2^{-\|\boldsymbol{k}\|_\infty} \right)^2.\end{aligned}$$

We thus obtain the desired estimate with $c_d = \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} 2^{-\|\boldsymbol{k}\|_\infty} \right)^2$. ■

◂ Kantorovich distance