CONCENTRATION IN THE CHINESE RESTAURANT

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WHICH RESTAURANT?

- Customers sit at tables in a restaurant

  NUMBERS — PARTS — PARTITION

  $1, 2, \ldots, n$

- At time 1, customer 1 sits by herself:

$$\left[\mathcal{P}_1 = \{\{1\}\}\right]$$

AT TIME n

- At time n, customers labelled by $[n]$ are partitioned into tables. E.g. for $n = 10$.

$$\left[\mathcal{P}_{10} = \{ \{1,3\}, \{2\}, \{4,7,10\}, \{5,6,8,9\} \} \right]$$

- <u>Notation</u>:

$$\left[\mathcal{P}_n := \{ \underbrace{A_{i,n}}_{\text{parts}} : i \in [\underbrace{V_n}_{\text{\# of parts}}] \} \right]$$

MARKOVIAN JUMP TO $\mathcal{P}_{n+1}$

$\left[\theta, \alpha \in \mathbb{R} \text{ parameters} \right]$

(a) FOR $i = 1$ TO V_n

With probability $\dfrac{|A_{i,n}| - \alpha}{n+\theta}$, add $n+1$ to table $A_{i,n}$

$$\left[\mathcal{P}_{n+1} := \{ A_{j,n} : j \in [V_n] \setminus \{i\} \} \cup \{ A_{i,n} \cup \{n+1\} \} \right]$$

(b) With probability $\dfrac{\alpha V_n + \theta}{n+\theta}$, $n+1$ lands at new table.

$$\left[\mathcal{P}_{n+1} := \{ A_{i,n} : i \in [V_n] \} \cup \{ \{n+1\} \} \right]$$

WHY STUDY THIS ?

- Two-parameter generalization of Ewens' sampling formula in Genetics;
- Explicit examples in the theory of exchangeable random partitions (Kingman);
- Applications to Bayesian Nonparametrics, especially topic models (Blei et al).

REGIME # 1

$[\alpha = 0, \ \theta > -1]$ Ewens sampling

$$\mathbb{P}(V_{n+1} = V_n + 1 \mid \mathcal{F}_n) = \frac{1}{n+\theta}$$

$\Rightarrow \quad \mathbb{E}(V_n) \sim \log n$ (few parts)

- Part w/ k elements gets new with prob $\frac{k}{n}$.

rich get richer!

REGIME #2

$$\left[\ \alpha < 0,\ \theta = -m\alpha \text{ for } m \in \mathbb{N}\ \right]$$

$$\mathbb{P}(V_{n+1} = V_n + 1 \mid \mathcal{F}_n) = \frac{(-\alpha)(m - V_n)}{n + \theta}$$

$$\Rightarrow \quad V_n \to m \quad \text{a.s.}$$

- Urn process with m urns & funky start.

REGIME #3 (OUR FOCUS)

$$\left[\alpha > 0 \ , \ \theta > -\alpha\right]$$

$$\mathbb{P}(V_{n+1} = V_n + 1 \mid \mathcal{F}_n) = \frac{\alpha V_n + \theta}{n+\theta}$$

$$\Rightarrow \mathbb{E}(V_{n+1}) = \left(1 + \frac{\alpha}{n+\theta}\right)\mathbb{E}(V_n) + \frac{\theta}{n+\theta}$$

$$\Rightarrow \mathbb{E}(V_n) \asymp n^{\alpha}$$

(many parts!)

<u>In fact</u>: $\dfrac{V_n}{n^{\alpha}} \xrightarrow{a.s.} V^{\bullet}$ non-trivial r.v.

[Pitman]

WHAT WE STUDY

- **Recall** $\mathcal{F}_n = \{ A_{i,n} : i \in [V_n] \}$

- **Want** to understand

  (1) V_n # of tables

  (2) $N_n(k) := |\{ i \in [V_n] : |A_{i,n}| = k \}|$

  # of tables with k customers

- **Remarks:** $\sum_{k=1}^{n} N_n(k) = V_n$, $\sum_{k=1}^{n} k N_n(k) = n$.

A BRIEF ASIDE ON EXCHANGEABILITY

- $$\mathbb{P}(\mathcal{P}_n = \text{some partition } \mathcal{P})$$

$$= \frac{1}{Z_n}\left\{\prod_{i=1}^{|\mathcal{P}|}(\alpha i+\theta)\right\}\cdot\prod_{A\in\mathcal{P}}\prod_{j=1}^{|A|}(j-\alpha)$$

is <u>invariant under permutations</u> of $[n]$.

- $\left[\mathcal{P}_\infty = \lim_{n\nearrow+\infty}\mathcal{P}_n\right]$ is <u>exchangeable</u>

(law invariant under finite permutations of $\mathbb{N}$)

De Finetti's Thm & offspring

KINGMAN'S THEORY

- $\mathcal{P}_\infty$ is "dust free" (no singleton parts)

$\Rightarrow$ $\exists$ <u>random probability measure</u> μ on $\mathbb{N}$

such that if $X_1, X_2, \ldots$ iid μ,

$$\left[\mathcal{P}_\infty \overset{d}{=} \{ \{ m \in \mathbb{N} : X_m = i \} : i \in \mathbb{N} \} \right]$$

(Chinese Restaurant) μ has <u>Poisson-Dirichlet</u> distribution w/ parameters θ, α. (Pitman)

CONSEQUENCES

- In the <u>GCRP</u> (θ, α):

  (0) Draw $\mu \sim PD(\theta, \alpha)$
  & $X_1, X_2, \ldots \sim \mu^{\otimes \mathbb{N}}$

  (1) V_n = # distinct color in $X_1, \ldots, X_n$

  (2) $N_n(k)$ = # of colors appearing exactly k times in $X_1, \ldots, X_n$

- <u>Exchangeability</u> $\Rightarrow$ These RV determine $\mathbb{P}_n$ up to random permutation.

ASYMPTOTIC RESULTS

Pitman
PTRF 1995
ST FLOUR

$$\frac{V_n}{n^\alpha} \xrightarrow[a.s.]{} V^\bullet \quad \text{with density on } (0,+\infty)$$

$$\frac{N_n(k)}{V_n} \xrightarrow[a.s.]{} c(k) \sim k^{-(1+\alpha)} \quad \text{for each } k$$

Favaro & Feng & Gao
EJP 2015; Sankhya 2018

Large & moderate deviations for

$$\mathbb{P}(N_n(k) \geq n^\alpha \beta_n), \quad \beta_n \text{ grows "fast"} \gg (\log n)^{1-\alpha}$$

A NONASYMPTOTIC RESULT

Thm. [O., Pereira, Ribeiro]

The following holds w/ prob $\geq 1-\delta$.

$$\forall n \in \mathbb{N}\setminus\{0\}$$

$$\left[\left|\frac{V_n}{\phi_n} - V_*\right| \leq C\,\frac{\log\log n + \log\left(\frac{1}{\delta}\right)}{n^{\alpha/2}}\right]$$

$$\phi_n = \frac{\Gamma(1+\theta)}{\Gamma(1+\theta+\alpha)} \cdot \frac{\Gamma(n+\alpha+\theta)}{\Gamma(n+\theta)} \asymp n^{\alpha}, \quad V_* = c\,V^{\bullet}$$

A 2nd NONASYMPTOTIC RESULT

Thm 2 [O., Pereira, Ribeiro] Fix $n \in \mathbb{N}$.

The following holds w/ probability $\geq 1-\delta$:

$$\forall k \leq \varepsilon n^{\frac{\alpha}{2\alpha+4}}$$

$$\left| N_n(k) - c(\alpha,\theta) \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} V_* n^{\alpha} \right|$$

$$\leq C \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} n^{\alpha} \varepsilon^{\alpha+2} \left(1 + \frac{\ln \frac{1}{\delta}}{\ln n} \right).$$

EASIER ASYMPTOTICS

Corollary Assume $\varepsilon = \varepsilon_n \to 0$. Then $\exists\, \delta_n \to 0$ such that

$$\mathbb{P}\left(\forall k \leq \varepsilon_n n^{\frac{\alpha}{2\alpha+4}} : \quad 1-\delta_n \leq \frac{N_n(k)}{c(k)\, V_+ n^{\alpha}} \leq 1+\delta_n\right)$$

converges to 1. Here,

$$c(k) = c(\theta, \alpha)\, \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} \sim \frac{c}{k^{1+\alpha}}.$$

COMMENTS & COMPARISONS

- Result on V_n seems new
- Result on $N_n(h)$ works simultaneously for $h = o\left(n^{\frac{\alpha}{2\alpha+4}}\right)$
- Finer scale but less precise than Favaro et al.
- Technical aspects: all based on martingales & recursions (preferential attachment)

THE ARGUMENT FOR V_n

$$\mathbb{E}\left(V_{n+1} - V_n \mid \mathcal{F}_n\right) = \frac{\alpha V_n + \theta}{n+\theta}$$

$$\Rightarrow \quad V_{n+1} = \left(\frac{n+\alpha+\theta}{n+\theta}\right) V_n + \frac{\theta}{n+\theta} + \underbrace{\xi_{n+1}}_{\mathbb{E}(\xi_{n+1} \mid \mathcal{F}_n) = 0}$$

$$\left[\frac{V_{n+1}}{\phi_{n+1}} = \frac{V_n}{\phi_n} + \frac{\theta}{(n+\theta)\phi_n} + \frac{\xi_{n+1}}{\phi_{n+1}}\right]$$

$\phi_n = \Theta(n^{\alpha})$

"noisy recursion"

WHAT CAN WE DO?

$$\frac{V_n}{\phi_n} = \underbrace{\sum_{i=1}^{n} \frac{\xi_i}{\phi_i}}_{\text{Martingale}} + \underbrace{c_n}_{\text{convergent}}$$

$\phi_i = \Theta(i^{\alpha})$

[How do we prove concentration for martingales ?]

(Our case) Expect $\frac{V_n}{\phi_n} \approx \frac{V_n}{n^{\alpha}} \approx$ const.

AZUMA HÖFFDING FAILS

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} \frac{\xi_i}{\phi_i}\right| \geq x\right) \leq 2\exp\left(-\frac{x^2}{2C}\right)$$

where $C := \sum_{i=1}^{n} \left\|\frac{\xi_i}{\phi_i}\right\|_\infty^2$

(Our case) $C = \Theta(n^{1-2\alpha})$

$$\therefore \sum_{i=1}^{n} \frac{\xi_i}{\phi_i} = O_{\mathbb{P}}\left(n^{\frac{1}{2}-\alpha}\right)$$

when $\alpha < \frac{1}{2}$.

FREEDMAN'S INEQUALITY ?

- Set $W_n := \sum_{i=1}^{n} \mathbb{E}\left(\left(\frac{\xi_i}{\phi_i}\right)^2 \middle| \mathcal{F}_{i-1}\right)$

$$m := \max_{1 \le i \le n} \left\| \frac{\xi_i}{\phi_i} \right\|_\infty .$$

Then

$$\boxed{\mathbb{P}\left(\left| \sum_{i=1}^{n} \frac{\xi_i}{\phi_i} \right| \geq x \,,\, W_n \leq \sigma^2 \right) \leq 2 \exp\left(- \frac{x^2}{2\sigma^2 + \frac{2mx}{3}} \right)}$$

SOMETHING IS FUNNY

- $$\xi_n = \left(V_{n+1} - \left(1 + \frac{\alpha}{n+\theta}\right) V_n - \frac{\theta}{n+\theta} \right)$$

$$\Rightarrow \mathbb{E}\left(\frac{\xi_n^2}{\phi_n^2} \middle| \mathcal{F}_{n-1} \right) \approx \frac{1}{(n+\theta)^{1+\alpha}} \left(\frac{V_{n-1}}{\phi_{n-1}} + 1 \right)$$

$$\Rightarrow W_n = \sum_{i=1}^{n} \mathbb{E}\left(\frac{\xi_i^2}{\phi_i^2} \middle| \mathcal{F}_{i-1} \right) \approx \sum_{i=1}^{n} \frac{1}{(i+\theta)^{1+\alpha}} \left(\frac{V_{i-1}}{\phi_{i-1}} + 1 \right)$$

[But we are using Freedman to control V_n!]

EXPONENTIAL TAIL

Lemma $$\mathbb{P}\left(\sup_{i\in\mathbb{N}} \frac{V_i}{\phi_i} \geq A\right) \leq C e^{-A/C}$$

where $C>0$ is fixed.

Proof Define $\tau_A := \inf\{i\in\mathbb{N} : V_i/\phi_i \geq A\}$.

$$\mathbb{P}(\tau_A < +\infty) = \lim_n \mathbb{P}(\tau_A \leq n)$$

$$= \lim_n \mathbb{P}\left(\frac{V_{\tau_A\wedge n}}{\phi_{\tau_A\wedge n}} \geq A\right)$$

SO WHAT?

$$\left[\left\{ \frac{V_{\tau_A \wedge n}}{\phi_{\tau_A \wedge n}} \geq A \right\} \right]$$

- **Recall:**

$$\frac{V_{\tau_A \wedge n}}{\phi_{\tau_A \wedge n}} = \underbrace{c_{\tau_A \wedge n}}_{\text{constant}} + \underbrace{\sum_{i=1}^{n \wedge \tau_A} \frac{\xi_i}{\phi_i}}_{\text{Martingale}}$$

- **Quadratic variation:**

$$\sum_{i=1}^{n \wedge \tau_A} \mathbb{E}\left(\left(\frac{\xi_i}{\phi_i} \right)^2 \middle| \mathcal{F}_{i-1} \right) \approx \sum_{i=1}^{n \wedge \tau_A} \frac{1}{(i+1)^{\alpha+1}} \underbrace{\frac{V_{i-1}}{\phi_{i-1}}}_{\text{Bounded by } A!}$$

$$\therefore \left[W_{n \wedge \tau_A} \lesssim A \right]$$

$\Rightarrow$

FREEDMAN

Want bound on $\lim_{n\to+\infty}(\dots)$

$$\mathbb{P}\left(\frac{V_{\tau_A\wedge n}}{\phi_{\tau_A\wedge n}} \geq A\right)$$

$$\leq \mathbb{P}\left(\sum_{i=1}^{n\wedge\tau_A} \frac{\xi_i}{\phi_i} \geq A-c\right)$$

$$\leq \exp\left(-\frac{(A-c)^2}{2A + 2A/3}\right) \lesssim e^{-A}$$

stopped quadratic variation

uniform in n

SUMMING UP THE ARGUMENT

(1) Freedman gives Bennett-type bounds on martingale $\approx V_m/\phi_m$

(2) However, the variance proxy is

$$W_m = \sum_{i=1}^{m} \mathbb{E}\left(\frac{\xi_i^2}{\phi_i^2} \,\middle|\, \mathcal{F}_{i-1}\right) \lesssim \max_{i \le m-1} \frac{V_i}{\phi_i}$$

(3) Leads to subexponential tail bounds for $\sup_{i \ge m} \frac{V_i}{\phi_i}$ once we stop.

RECALL 1st THEOREM

Thm [O., Pereira, Ribeiro]

The following holds w/ prob $\geq 1-\delta$.

$$\forall n \in \mathbb{N}\setminus\{0\}$$

$$\left| \frac{V_n}{\phi_n} - V_* \right| \leq C \, \frac{\log\log n + \log\left(\frac{1}{\delta}\right)}{n^{\alpha/2}}$$

$$\phi_n = \frac{\Gamma(1+\theta)}{\Gamma(1+\theta+\alpha)} \cdot \frac{\Gamma(n+\alpha+\theta)}{\Gamma(n+\theta)} \simeq n^{\alpha} \, . \quad V_* = c\, V^{\bullet}$$

PROOF ELEMENTS

- <u>Copy</u> the stopped martingale argument & apply to $\frac{V_m}{\phi_m} - \frac{V_n}{\phi_n}$, $m \geq n$.

- <u>Obtain</u> $\mathbb{P}\left(\exists m \geq n \ \left|\frac{V_m}{\phi_m} - \frac{V_n}{\phi_n}\right| \geq \frac{c(A+1)}{n^{\alpha/2}}\right) \leq e^{-A}$

- <u>Union bound</u> over $n = 2^k$ w/ $A \simeq \log\left(k^2/\delta\right)$.

ON THE PROOF OF 2nd THEOREM

Thm 2 [O., Pereira, Ribeiro] Fix $n \in \mathbb{N}$.

The following holds w/ probability $\geq 1-\delta$:

$$\forall k \leq \varepsilon n^{\frac{\alpha}{2\alpha+4}}$$

$$\left| N_n(k) - c(\alpha,\theta)\frac{\Gamma(k-\alpha)}{\Gamma(k+1)} V_* n^{\alpha} \right|$$

$$\leq C \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} n^{\alpha} \varepsilon^{\alpha+2}\left(1+\frac{\ln\frac{1}{\delta}}{\ln n}\right).$$

CASE $k=1$

$$\left[\begin{array}{c} N_n(1) \\ = \text{\# of tables w/1} \\ \text{customer at} \\ \text{time } n \end{array}\right]$$

$$\mathbb{E}\left(N_{n+1}(1) - N_n(1) \,\middle|\, \mathcal{F}_n\right)$$

$$= \frac{1}{n+\theta}\Big\{ \underbrace{\alpha V_n + \theta}_{\text{new table}} - \underbrace{(1-\alpha) N_n(1)}_{\text{put new customer in table w/one}} \Big\}$$

$$\Rightarrow N_{n+1}(1) = \left(\frac{n+\theta-(1-\alpha)}{n+\theta}\right) N_n(1) + \frac{\alpha V_n + \theta}{n+\theta} + \xi_{n+1}(1)$$

with $\mathbb{E}\left(\xi_{n+1}(1) \,\middle|\, \mathcal{F}_n\right) = 0.$

CASE $k>1$

$N_n(k)$ = # of tables w/ k customers at time n

$$\mathbb{E}\left(N_{n+1}(k) - N_n(k) \mid \mathcal{F}_n\right) =$$

$$= \frac{1}{n+\theta}\Big\{ \underbrace{(k-1-\alpha)N_n(k-1)}_{\text{add to table w/ } k-1} - \underbrace{(k-\alpha)N_n(k)}_{\text{add to table with } k} \Big\}$$

$$\Rightarrow \quad N_{n+1}(k) = \left(1 - \frac{k-\alpha}{n+\theta}\right) N_n(k) + \frac{(k-1-\alpha)}{n+\theta} N_n(k-1)$$

$$+ \quad \xi_{n+1}(k)$$

with $\mathbb{E}\left(\xi_{n+1}(k) \mid \mathcal{F}_n\right) = 0$.

NOISY RECURSIONS

- Set $\begin{cases} X_n(0) := \dfrac{V_n}{\phi_n} \\ X_n(k) := \dfrac{N_n(k)}{\Psi_n(k)} \end{cases}$.

  ($\Psi_n(k)$ chosen wisely)

- Obtain:

$$\left[X_n(k) = X_{n-1}(k) + c(k) X_{n-1}(k-1) + \text{"noise"} \right]$$

$$\mathbb{E}(\ldots \mid \mathcal{F}_{n-1}) = 0$$

NOISY RECURSIONS (II)

- We get:

$$\left[X_n(k) = c(k) \sum_{i=k-1}^{n-1} X_i(k-1) + \left(\text{martingale}\right) \right]$$

- <u>Our approach</u>: work this out recursively in k, starting from $k=0$.

$$\left(\text{Recall } X_n(0) = \frac{V_n}{\phi_n} \checkmark \right)$$

Messy & not obviously optimal ...

POTENTIAL ALTERNATIVE APPROACH

- $$\vec{N}_n = \begin{pmatrix} N_n(1) \\ \vdots \\ N_n(k) \end{pmatrix}$$

[Brightwell & Luczak EJP '12
PREFERENTIAL ATTACHMENT]

- $$\vec{N}_{n+1} = \left(I + \frac{A}{n+\theta}\right)\vec{N}_n + \left(\frac{\alpha V_n + \theta}{n+\theta}\right) e_1 + (\text{noise})$$

where

$$A = \begin{pmatrix} -(1-\alpha) & & & & 0 \\ (1-\alpha) & -(2-\alpha) & & & \\ & (2-\alpha) & -(3-\alpha) & & \\ & & (3-\alpha) & -(4-\alpha) & \\ 0 & & & & \ddots \end{pmatrix}$$

BACK TO WHAT WE DO

$$\left[X_n(k) = c(k) \sum_{i=k-1}^{n-1} X_i(k-1) + (\text{martingale}) \right]$$

- Show <u>martingale ≈ 0</u> w.h.p..
- Implies <u>concentration</u> of $X_n(k)$ around the <u>sum</u> $c(k) \sum_{i=k-1}^{n-1} X_i(k-1)$.

TODAY WE STICK TO $k=1$

$$\left[N_{n+1}(1) = \left(\frac{n+\theta-(1-\alpha)}{n+\theta} \right) N_n(1) + \frac{\alpha V_n + \theta}{n+\theta} + \xi_{n+1}(1) \right]$$

- $$\xi_{n+1}(1) = \eta_{n+1} - \mathbb{E}(\eta_{n+1} \mid \mathcal{F}_n),$$

where $$\eta_{n+1} := \begin{cases} 1 & \text{if } V_{n+1} = V_n + 1 \\ -1 & \text{if } N_{n+1}(1) = N_n(1) - 1 \\ 0 & \text{otherwise} \end{cases}$$

GETTING A SIMPLER RECURSION

- $$N_{n+1}(1) = \left(\frac{n+\theta-(1-\alpha)}{n+\theta}\right) N_n(1) + \text{other terms}$$

- Define $$\left[\Psi_n(1) := \prod_{j=1}^{n-1}\left(\frac{j+\theta+\alpha-1}{j+\theta}\right)\right]$$

& $$\left[X_n(1) := \Psi_n(1)^{-1} N_n(1)\right]$$ Then:

$$\left[X_{n+1}(1) = X_n(1) + \frac{\alpha V_n + \theta}{(n+\theta)\,\Psi_{n+1}(1)} + \frac{\xi_{n+1}(1)}{\Psi_{n+1}(1)}\right]$$

OBTAIN TWO PARTS

- $$X_n(1) = \underbrace{\sum_{i=0}^{n-1} \frac{\alpha V_i + \theta}{(i+\theta)\,\Psi_{i+1}(1)}}_{\text{sum}} + \left(\text{martingale}\right)$$

- Recall $$\left| \frac{V_i}{\phi_i} - V_* \right| \lesssim C\left(\frac{\log\log i + \ln\left(\frac{1}{\delta}\right)}{i^{\alpha/2}} \right)$$

& note $\Psi_{i+1}(1) \sim i^{\alpha-1}$

$$\Rightarrow \left[(\text{Sum}) \sim A(1) \cdot V_* \, n \right] \quad \text{w.h.p.}$$

THE MARTINGALE

$$M_n(1) := \sum_{i=1}^{n} \frac{\xi_i(1)}{\Psi_i(1)}$$

where $\Psi_i(1) \sim i^{\alpha-1}$, $\xi_i(1) = (\Delta N_n(1) - \text{mean})$.

$\Rightarrow$ Quadratic variation:

$$W_n(1) \lesssim \sum_{i=1}^{n} \frac{(V_i + 1)}{i^{2\alpha-2}(i+\theta)} \approx V_* \cdot n^{1-\alpha}$$

COMBINING THE PARTS

- Up to small error, with prob. $\geq 1-\delta$:

$$\frac{X_n(1)}{\underbrace{\Psi_n(1)}_{n^{\alpha-1}}} = A(1)\, V_* \, n \pm C\left(n^{1-\frac{\alpha}{2}}\sqrt{\ln\tfrac{1}{\delta}} + \ln\tfrac{1}{\delta}\right)$$

$$\Rightarrow \quad N_n(1) = B(1)\, V_* \, n^{\alpha} \pm C\left(n^{\frac{\alpha}{2}}\sqrt{\ln\tfrac{1}{\delta}} + n^{\alpha-1}\ln\left(\tfrac{1}{\delta}\right)\right)$$

recall $0<\alpha<1$

HIGHER k

$$\left[\left| N_n(k) - c(\alpha,\theta)\frac{\Gamma(k-\alpha)}{\Gamma(k+1)} V_* n^{\alpha} \right| \leq C \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} n^{\alpha} \varepsilon^{\alpha+2}\left(1+\frac{\ln 1/\delta}{\ln n}\right). \right]$$

Carry induction in k to show:

$$X_n(k) = \frac{k-1-\alpha}{k-1+\theta}\sum_{i=k}^{n} X_i(k-1) \pm C(k)\left(\ln 1/\delta\right)^{\frac{1}{2}} n^{k-\frac{\alpha}{2}} \pm D(k)\ln 1/\delta \; n^{k-\alpha}$$

HIGHER k

- **Main difficulties**

(1) Combinatorics of coefficients

(2) "Self-reference" in quadratic variation.

OPEN PROBLEMS

- $GCRP(\theta, \alpha)$ <u>mixture of iid</u>

  $\Rightarrow$ any better argument from this (info about $PD(\theta, \alpha)$?)

- Make the <u>matrix approach</u> work

- Want <u>concentration for $k = o(n^{1-\varepsilon})$</u>.

THANKS !