About the use of weak transport costs for concentration and functionals inequalities in discrete spaces.

Based on :
- Transport-entropy inequalities on locally acting groups of permutations. Electron. J. Probab. 22 (2017), no. 62.
- Kantorovich duality for general transport costs and applications. J. Funct. Anal. 273 (2017), no. 11, 3327-3405.
Joint work with N. Gozlan, C. Roberto et P. Tetali.

Institut d'Études Scientifiques de Cargèse

Concentration of measure and its applications

May 2018

P-M. Samson
Université Paris-Est Marne-la-Vallée

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By Jensen's inequality,

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introduction
Marton's inequality
Talagrand's concentration
Kantorovich duality
for classical costs
for weak costs
Examples of weak cost
Marton's type of cost
Barycentric cost
Strassen's result
Martingale costs
Weak transport inequalities
Dual characterization
to concentration
Universal transport inequalities
Barycentric transport inequalities
examples
characterisation on $\mathbb{R}$
Transport inequality on the symmetric group
introduction
Ewens distribution
deviation inequalities
The Schrödinger minimization problem
definition
curvature in discrete spaces
functional inequalities
Examples in discrete

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Extension of Kantorovich duality to weak transport cost

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Definition : Weak optimal transport cost

P-M. Samson

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Extension of Kantorovich duality to weak transport cost

Definition : Weak optimal transport cost

Let us consider a measurable function

$$c: \begin{array}{ccc} \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) & \to & [0,+\infty] \\ (x,p) & \mapsto & c(x,p), \end{array}$$

The weak optimal cost, $\mathcal{T}_c(\nu|\mu)$, associated to c is defined by

$$\mathcal{T}_c(\nu|\mu) := \inf_{\substack{\pi \in \Pi(\mu,\nu) \\ \pi = \mu \oslash p}} \int c(x,p_x) d\mu(x), \qquad \mu,\nu \in \mathcal{P}_\gamma(\mathcal{X}),$$

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Particular case : a Talagrand's cost for $\gamma_0(u) = \mathbb{1}_{u\neq 0}$,

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and $c(x,p) = +\infty$ otherwise, with $\beta : \mathbb{R}^+ \to [0,+\infty]$, convex and $\beta(0) = 0$.

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used by Talagrand (1996) as a main ingredient to reach deviation inequalities for supremum of empirical processes with Bernstein's bounds, see also S. (2007).

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$$\overline{\mathcal{T}}_1(\nu|\mu) = \inf_{(X,Y)} \mathbb{E}[|X - \mathbb{E}[Y|X]|] = \sup_{\psi \text{ convex, 1-Lipschitz}} \left\{ \int \psi\, d\mu - \int \psi\, d\nu \right\}.$$

Application : A simple proof of a result by Strassen
Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^m)$; one says that μ is dominated by ν in the convex order sense, $\mu \preceq_C \nu$, if

$$\int \psi\, d\mu \leqslant \int \psi\, d\nu,$$

for all convex $\psi : \mathbb{R}^m \to \mathbb{R}$.

Theorem. [Strassen 1965]

Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^m)$.Then $\mu \preceq_C \nu$ if and only if there exists a martingale (X, Y) ($\mathbb{E}[Y|X] = X$), where X follows the law μ and Y the law ν.

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The dual Kantorovich Theorem for weak cost applies and we recover the duality result by Beighböck-Henry-Labordère-Penker (2013).

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$$f_0(x) := -R_c g_0(x) = \sup_p \inf_{\gamma \in \mathbb{R}} \left\{ -\int g_0 dp + \int w(x,y) dp(y) - \int \gamma \cdot (y-x) dp(y) \right\}$$

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Duality for martingale costs

Theorem : [B and al.,2013]

Let $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a upper semi-continuous function, bounded from above.

$$\sup_{\pi \in \Pi^{mart}(\mu,\nu)} \iint w \, d\pi = \inf_{f,g,h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},$$

where the infimum runs over all measurable bounded functions f, g, h such that for all $x, y \in \mathbb{R}$, $\quad w(x,y) \leqslant f(x) + g(y) + h(x)(y-x)$.

idea of the proof : The inequality $\leqslant$ is obvious since for all $\pi \in \Pi^{mart}(\mu,\nu)$, $\int h(x)(y-x)\, d\pi(x,y) = 0$. For the reverse inequality $\geqslant$: let $\varepsilon > 0$, $\omega = -w$.

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over all f_0, g_0, γ, $\quad f_0(x) + g_0(y) + \gamma(x) \cdot (y-x) + \varepsilon \geqslant w(x,y)$.

Applications of duality to transport-entropy inequalities and concentration

Definition : Weak transport-entropy inequality $\mathbf{T}_c(a_1, a_2)$

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ii) is a generalisation of the so-called (convex) τ-property introduced by Maurey (1990) to recover Talagrand's concentration results.

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By optimizing over all ν_1, ν_2 we get

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Choose $q = \lambda/n$, $\lambda > 0$, and use the weak convergence as $n \to +\infty$ of the binomial law $\mu_{\lambda/n,n}$ to the Poisson measure $p_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k \in \mathbb{N}$.

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$$\widehat{T}_2(\nu_2|\nu_1) := \inf_{\substack{\pi \in \Pi(\nu_1,\nu_2) \\ \pi = \nu_1 \oslash p}} \int \sum_{i=1}^{n} \left(\int \mathbb{1}_{\sigma(i)\neq\tau(i)} dp_\sigma(\tau) \right)^2 d\nu_1(\sigma).$$

By Cauchy-Schwarz inequality $\frac{1}{n} {W_1}^2(\nu_1,\nu_2) \leqslant \widehat{T}_2(\nu_2|\nu_1) \leqslant W_1(\nu_1,\nu_2),$

where W_1 is the Wasserstein distance on $\mathcal{P}(S_n)$ associated to d_H.

Theorem : [S. 2017]

For all $s \in (0,1)$,

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$$\left(\int_{S_n} e^{s\widehat{Q}\varphi} d\mu^\theta \right)^{1/s} \left(\int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leqslant 1,$$

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$$\mu^\theta(\sigma) = \mu^\theta(\sigma^{-1}) \quad \text{and} \quad \mu^\theta(\sigma) = \mu^\theta(t^{-1}\sigma t) \quad \forall t \in S_n.$$

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$$\begin{aligned}
\widetilde{Q}\varphi(\sigma) &= \inf_{p \in \mathcal{P}(S_n)} \left\{ \int \varphi(\tau) dp(\tau) + \frac{1}{20} \sum_{i=1}^n \left(\int 1_{\sigma(i) \neq \tau(i)} dp(\tau) \right)^2 \right\} \\
&\geqslant \varphi(\sigma) - \sup_p \sum_{i=1}^n \left[\alpha_i(\sigma) \int \mathbb{1}_{\sigma(i) \neq \tau(i)} dp(\tau) - \frac{1}{20} \left(\int 1_{\sigma(i) \neq \tau(i)} dp(\tau) \right)^2 \right] \\
&\geqslant \varphi(\sigma) - \sum_{i=1}^n \sup_{l \geqslant 0} \left\{ \alpha_i(\sigma) l - \frac{l^2}{20} \right\} \\
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Question : Is there a “good” notion of curvature in discrete setting from which we can recover

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It recovers the Lott-Sturm-Villani definition of Ricci curvature $\geqslant \kappa$.

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  Consequence : The best constant α in mLSI satisfies $\alpha \geqslant \kappa$.

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  According to Gao-Quastel (2003) Bobkov-Tetali (2006), $\alpha \geqslant Cn$.

Thank you.