Dyson Ornstein Uhlenbeck process

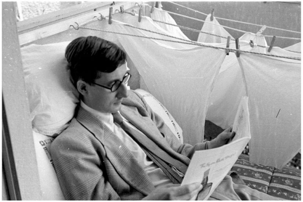
Cutoff phenomenon

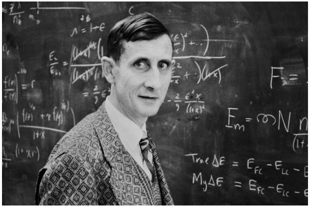
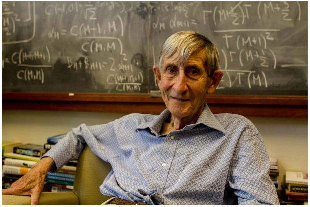
Jeanne Boursier, Djalil Chafaï*, Cyril Labbé

DMA, École normale supérieure – PSL
CEREMADE, Université Paris-Dauphine – PSL

UniMelb-Bielefeld RMT Seminar
Wednesday June 1 2022

Freeman J. Dyson (1923 – 2020)

A Brownian Motion Model for the Eigenvalues of a Random Matrix
Journal of Mathematical Physics 3 1191–1198 (1962)

Plan

Dyson Ornstein Uhlenbeck process DOU_β

- Interacting particle system $X_t^{n,1}, \ldots, X_t^{n,n}$ on $\mathbb{R}$

$$X_0^n = x_0^n, \quad \mathrm{d}X_t^n = \sqrt{\frac{2}{n}}\mathrm{d}B_t - \frac{1}{n}\nabla H(X_t^n)\mathrm{d}t$$

Dyson Ornstein Uhlenbeck process DOU_β

- Interacting particle system $X_t^{n,1}, \ldots, X_t^{n,n}$ on $\mathbb{R}$

$$X_0^n = x_0^n, \quad \mathrm{d}X_t^n = \sqrt{\frac{2}{n}}\mathrm{d}B_t - \frac{1}{n}\nabla H(X_t^n)\mathrm{d}t$$

- Configuration energy with Coulomb repulsion (singular)

$$H(x) = n\sum_{i=1}^{n} V(x_i) + \beta \sum_{i<j} \log\frac{1}{x_i - x_j}, \quad V(x) = \frac{x^2}{2}$$

Dyson Ornstein Uhlenbeck process DOU_β

- Interacting particle system $X_t^{n,1},\ldots,X_t^{n,n}$ on $\mathbb{R}$

$$X_0^n = x_0^n, \quad \mathrm{d}X_t^n = \sqrt{\frac{2}{n}}\mathrm{d}B_t - \frac{1}{n}\nabla H(X_t^n)\mathrm{d}t$$

- Configuration energy with Coulomb repulsion (singular)

$$H(x) = n\sum_{i=1}^n V(x_i) + \beta\sum_{i<j}\log\frac{1}{x_i - x_j}, \quad V(x) = \frac{x^2}{2}$$

- Convergence to equilibrium $X_t^n \xrightarrow[t\to\infty]{\mathrm{d}} P^n \propto \mathrm{e}^{-H(x)}\mathrm{d}x$

$$\mathrm{e}^{-H(x)} = \mathrm{e}^{-n\frac{|x|^2}{2}}\prod_{i<j}(x_i - x_j)^\beta$$

Dyson Ornstein Uhlenbeck process DOU_β

- Interacting particle system $X_t^{n,1}, \ldots, X_t^{n,n}$ on $\mathbb{R}$

$$X_0^n = x_0^n, \quad \mathrm{d}X_t^n = \sqrt{\frac{2}{n}}\mathrm{d}B_t - \frac{1}{n}\nabla H(X_t^n)\mathrm{d}t$$

- Configuration energy with Coulomb repulsion (singular)

$$H(x) = n\sum_{i=1}^{n} V(x_i) + \beta\sum_{i<j}\log\frac{1}{x_i - x_j}, \quad V(x) = \frac{x^2}{2}$$

- Convergence to equilibrium $X_t^n \xrightarrow[t\to\infty]{\mathrm{d}} P^n \propto \mathrm{e}^{-H(x)}\mathrm{d}x$

$$\mathrm{e}^{-H(x)} = \mathrm{e}^{-n\frac{|x|^2}{2}}\prod_{i<j}(x_i - x_j)^\beta$$

- We take $\beta = 0$ or $\beta \geq 1$ (preserves order $x_n < \cdots < x_1$)

High dimensional random matrices

- Random matrix cases $\beta \in \{1, 2, 4\}$: matrix OU

$$\mathrm{d}M_t = \sqrt{\frac{2}{n}}\mathrm{d}B_t - M_t\mathrm{d}t$$

High dimensional random matrices

- Random matrix cases $\beta \in \{1, 2, 4\}$: matrix OU

$$\mathrm{d}M_t = \sqrt{\frac{2}{n}}\mathrm{d}B_t - M_t\mathrm{d}t$$

  - Symmetric/Hermitian/Symplectic $n \times n$ matrices
    Real/Complex/Quaternion off-diagonal entries : $\mathbb{R}^\beta$

$$M_t \xrightarrow[t\to\infty]{\mathrm{d}} \mathrm{GOE/GUE/GSE} \propto \mathrm{e}^{-n\mathrm{Trace}(M^2)}\mathrm{d}M$$

High dimensional random matrices

- Random matrix cases $\beta \in \{1, 2, 4\}$: matrix OU

$$\mathrm{d}M_t = \sqrt{\frac{2}{n}}\mathrm{d}B_t - M_t\mathrm{d}t$$

  - Symmetric/Hermitian/Symplectic $n \times n$ matrices
    Real/Complex/Quaternion off-diagonal entries : $\mathbb{R}^{\beta}$

$$M_t \xrightarrow[t\to\infty]{\mathrm{d}} \mathrm{GOE/GUE/GSE} \propto \mathrm{e}^{-n\mathrm{Trace}(M^2)}\mathrm{d}M$$

  - Dyson : $(\mathrm{spectrum}(M_t))_{t\geq 0} \overset{\mathrm{d}}{=} \mathrm{DOU}_{\beta}$

DOU semigroup and generator

- Markov semigroup

$$\mathbb{E}(f(X_t^n) \mid X_0^n = x) = (\mathrm{e}^{t\,\mathrm{L}} f)(x)$$

DOU semigroup and generator

- Markov semigroup

$$\mathbb{E}(f(X_t^n) \mid X_0^n = x) = (\mathrm{e}^{t\,\mathrm{L}} f)(x)$$

- Infinitesimal generator

$$\mathrm{L} = \frac{\Delta - \nabla H \cdot \nabla}{n}$$

DOU semigroup and generator

- Markov semigroup

$$\mathbb{E}(f(X_t^n) \mid X_0^n = x) = (\mathrm{e}^{t\,\mathrm{L}} f)(x)$$

- Infinitesimal generator

$$\begin{aligned}\mathrm{L} &= \frac{\Delta - \nabla H \cdot \nabla}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n} \partial_{x_i}^2 - \sum_{i=1}^{n}\Big[V'(x_i) - \frac{\beta}{n}\sum_{j \neq i}\frac{1}{x_i - x_j}\Big]\partial_{x_i}\end{aligned}$$

DOU semigroup and generator

- Markov semigroup

$$\mathbb{E}(f(X_t^n) \mid X_0^n = x) = (\mathrm{e}^{t\,\mathrm{L}} f)(x)$$

- Infinitesimal generator

$$\begin{aligned}
\mathrm{L} &= \frac{\Delta - \nabla H \cdot \nabla}{n} \\
&= \frac{1}{n} \sum_{i=1}^{n} \partial_{x_i}^2 - \sum_{i=1}^{n} \Big[V'(x_i) - \frac{\beta}{n} \sum_{j \neq i} \frac{1}{x_i - x_j} \Big] \partial_{x_i} \\
&= \oplus_{i=1}^{n} \mathrm{L}_{x_i}^{\mathrm{OU}} + \frac{\beta}{2n} \sum_{j \neq i} \frac{\partial_{x_i} - \partial_{x_j}}{x_i - x_j}
\end{aligned}$$

DOU semigroup and generator

- Markov semigroup

$$\mathbb{E}(f(X_t^n) \mid X_0^n = x) = (\mathrm{e}^{t\,\mathrm{L}} f)(x)$$

- Infinitesimal generator

$$\begin{aligned}
\mathrm{L} &= \frac{\Delta - \nabla H \cdot \nabla}{n} \\
&= \frac{1}{n}\sum_{i=1}^n \partial_{x_i}^2 - \sum_{i=1}^n \Big[V'(x_i) - \frac{\beta}{n}\sum_{j\neq i}\frac{1}{x_i - x_j}\Big]\partial_{x_i} \\
&= \oplus_{i=1}^n \mathrm{L}_{x_i}^{\mathrm{OU}} + \frac{\beta}{2n}\sum_{j\neq i}\frac{\partial_{x_i} - \partial_{x_j}}{x_i - x_j}
\end{aligned}$$

- Universality wrt β : spectrum, Poincaré, log-Sobolev

Wigner theorem and semi-circle law : scaling in n

- Empirical measure and exchangeability

$$\mu_t^n = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_t^{n,i}} \quad \text{and} \quad \mathbb{E}\mu_\infty^n \sim \frac{1}{n}\sum_{i=1}^{n} P^{n,i} = P^{n,1}$$

Wigner theorem and semi-circle law : scaling in n

- Empirical measure and exchangeability

$$\mu_t^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_t^{n,i}} \quad \text{and} \quad \mathbb{E}\mu_\infty^n \sim \frac{1}{n}\sum_{i=1}^{n} P^{n,i} = P^{n,1}$$

- Wigner theorem and semi-circle law

$$\mu_\infty^n \underset{n\to\infty}{\longrightarrow} \mu_\infty = \mathrm{SemiCircle}[-\sqrt{2\beta}, \sqrt{2\beta}]$$

Wigner theorem and semi-circle law : scaling in n

- Empirical measure and exchangeability

$$\mu_t^n = \frac{1}{n}\sum_{i=1}^n \delta_{X_t^{n,i}} \quad \text{and} \quad \mathbb{E}\mu_\infty^n \sim \frac{1}{n}\sum_{i=1}^n P^{n,i} = P^{n,1}$$

- Wigner theorem and semi-circle law

$$\mu_\infty^n \underset{n\to\infty}{\longrightarrow} \mu_\infty = \mathrm{SemiCircle}[-\sqrt{2\beta}, \sqrt{2\beta}]$$

- Long-time behavior & mean-field limit (when $\mu_0^n \underset{n\to\infty}{\longrightarrow} \mu_0$)

$$\begin{tikzcd} \mu_t^n \arrow[r, "t\to\infty"'] \arrow[d, "n\to\infty"'] & \mu_\infty^n \arrow[d, "n\to\infty"'] \\ \mu_t \arrow[r, "t\to\infty"'] & \mu_\infty \end{tikzcd}$$

Mean-field limit and free probability : scaling in t

- McKean-Vlasov evolution equation

$$\partial_t \int f \mathrm{d}\mu_t = -\int x f'(x) \mu_t(\mathrm{d}x) + \frac{\beta}{2} \iint \frac{f'(x) - f'(y)}{x - y} \mu_t(\mathrm{d}x) \mu_t(\mathrm{d}y)$$

Mean-field limit and free probability : scaling in t

- McKean-Vlasov evolution equation

$$\partial_t \int f \mathrm{d}\mu_t = -\int x f'(x) \mu_t(\mathrm{d}x) + \frac{\beta}{2} \iint \frac{f'(x) - f'(y)}{x - y} \mu_t(\mathrm{d}x) \mu_t(\mathrm{d}y)$$

- Cauchy–Stieltjes transform ($\mathrm{Im}\, z > 0$)

$$s_t(z) = \int \frac{\mu_t(\mathrm{d}x)}{x - z}$$

Mean-field limit and free probability : scaling in t

- McKean-Vlasov evolution equation

$$\partial_t \int f \mathrm{d}\mu_t = -\int x f'(x)\mu_t(\mathrm{d}x) + \frac{\beta}{2} \iint \frac{f'(x) - f'(y)}{x - y} \mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)$$

- Cauchy–Stieltjes transform ($\mathrm{Im}\, z > 0$)

$$s_t(z) = \int \frac{\mu_t(\mathrm{d}x)}{x - z}$$

- Complex Burgers equation (take $f(x) = \frac{1}{x-z}$)

$$\partial_t s_t = s_t + z\partial_z s_t + \beta s_t \partial_z s_t$$

Mean-field limit and free probability : scaling in t

- McKean-Vlasov evolution equation

$$\partial_t \int f \mathrm{d}\mu_t = -\int x f'(x)\mu_t(\mathrm{d}x) + \frac{\beta}{2} \iint \frac{f'(x)-f'(y)}{x-y}\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)$$

- Cauchy–Stieltjes transform ($\mathrm{Im}\, z > 0$)

$$s_t(z) = \int \frac{\mu_t(\mathrm{d}x)}{x-z}$$

- Complex Burgers equation (take $f(x) = \frac{1}{x-z}$)

$$\partial_t s_t = s_t + z\partial_z s_t + \beta s_t \partial_z s_t$$

- Free OU process and free Mehler formula

$$\mu_t = \mathrm{dil}_{\mathrm{e}^{-t}}\mu_0 \boxplus \mathrm{dil}_{\sqrt{1-\mathrm{e}^{-2t}}}\mu_\infty$$

Mean-field limit and free probability : scaling in t

- McKean-Vlasov evolution equation

$$\partial_t \int f \mathrm{d}\mu_t = -\int x f'(x)\mu_t(\mathrm{d}x) + \frac{\beta}{2}\iint \frac{f'(x)-f'(y)}{x-y}\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)$$

- Cauchy–Stieltjes transform ($\mathrm{Im}\, z > 0$)

$$s_t(z) = \int \frac{\mu_t(\mathrm{d}x)}{x-z}$$

- Complex Burgers equation (take $f(x) = \frac{1}{x-z}$)

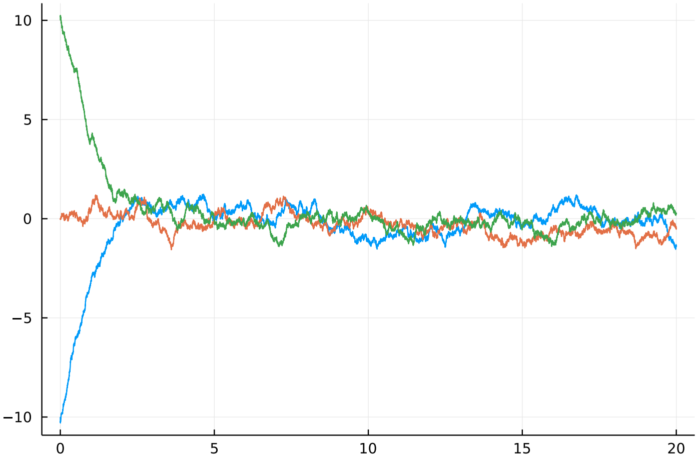
$$\partial_t s_t = s_t + z\partial_z s_t + \beta s_t \partial_z s_t$$

- Free OU process and free Mehler formula

$$\mu_t = \mathrm{dil}_{\mathrm{e}^{-t}}\mu_0 \boxplus \mathrm{dil}_{\sqrt{1-\mathrm{e}^{-2t}}}\mu_\infty$$

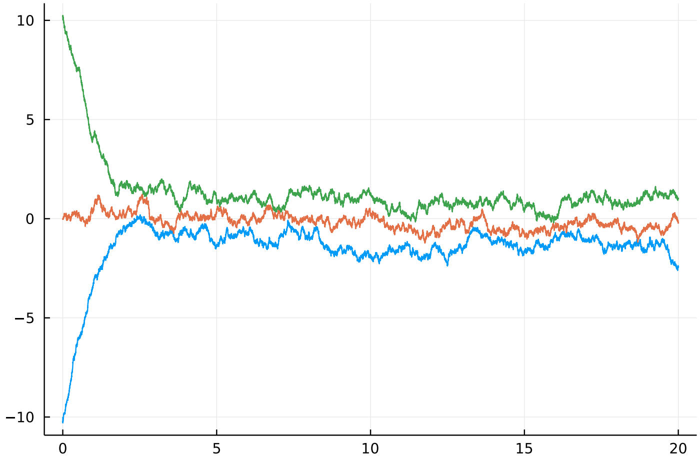
- $\mu_\infty = \mathrm{SemiCircle}[-\sqrt{2\beta}, \sqrt{2\beta}] = \mathrm{dil}_{\sqrt{\frac{\beta}{2}}}\mathrm{SemiCircle}[-2, 2]$

Numerical experiments



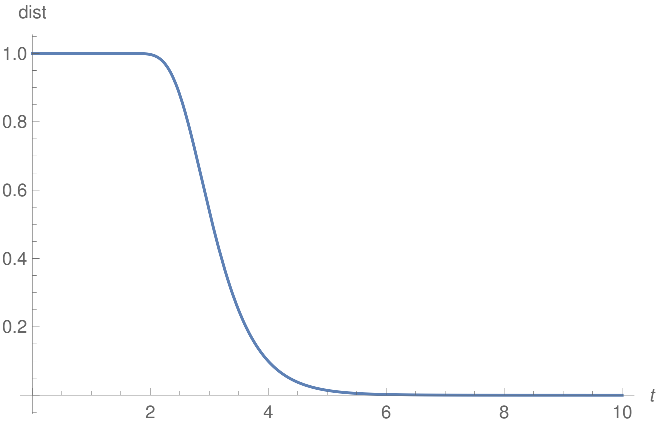
$n = 3$, $\beta = 0$: confinement and independence (OU)

Numerical experiments



$n = 3$, $\beta = 2$: confinement and repulsion (DOU)

Cutoff for OU : Hellinger distance $\mathrm{dist}(\mathrm{Law}(X_t^n) \mid P^n)$



$$n = 50,\ \beta = 0,\ \frac{|x_0^n|^2}{n} = 1,\ \log(50) \approx 3.91$$

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)\, c_n \\ 0 & \text{if } t_n = (1+\varepsilon)\, c_n \end{cases}$$

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)\, c_n \\ 0 & \text{if } t_n = (1+\varepsilon)\, c_n \end{cases}$$

- Choice of initial conditions S_0^n

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)\,c_n \\ 0 & \text{if } t_n = (1+\varepsilon)\,c_n \end{cases}$$

- Choice of initial conditions S_0^n
- Choice of distance or divergence dist

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)\,c_n \\ 0 & \text{if } t_n = (1+\varepsilon)\,c_n \end{cases}$$

- Choice of initial conditions S_0^n
- Choice of distance or divergence dist
- Related/Inspiring

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

- Choice of initial conditions S_0^n
- Choice of distance or divergence dist
- Related/Inspiring
  - Brownian motion on compact groups and on spheres

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

- Choice of initial conditions S_0^n
- Choice of distance or divergence dist
- Related/Inspiring
  - Brownian motion on compact groups and on spheres
  - Exclusion processes on finite sets

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)\, c_n \\ 0 & \text{if } t_n = (1+\varepsilon)\, c_n \end{cases}$$

- Choice of initial conditions S_0^n
- Choice of distance or divergence dist
- Related/Inspiring
  - Brownian motion on compact groups and on spheres
  - Exclusion processes on finite sets
  - High dimensional OU (discrete or continuous)

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

- Choice of initial conditions S_0^n
- Choice of distance or divergence dist
- Related/Inspiring
  - Brownian motion on compact groups and on spheres
  - Exclusion processes on finite sets
  - High dimensional OU (discrete or continuous)
- Critical time (guess) : $c_n \asymp \log(n)$

Expectation : cutoff phenomenon

- For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in S_0^n} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)\,c_n \\ 0 & \text{if } t_n = (1+\varepsilon)\,c_n \end{cases}$$

- Choice of initial conditions S_0^n
- Choice of distance or divergence dist
- Related/Inspiring
  - Brownian motion on compact groups and on spheres
  - Exclusion processes on finite sets
  - High dimensional OU (discrete or continuous)
- Critical time (guess) : $c_n \asymp \log(n)$
- Universality with respect to β

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\left(\frac{d\nu}{d\mu}\right) = \left\| \frac{d\nu}{d\mu} - 1 \right\|^2_{L^2(\mu)}$$

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\left(\frac{d\nu}{d\mu}\right) = \left\|\frac{d\nu}{d\mu} - 1\right\|^2_{L^2(\mu)}$$

$$\mathrm{Entropy}(\nu \mid \mu) = \int \log \frac{d\nu}{d\mu} d\nu = \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu$$

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\Big(\frac{d\nu}{d\mu}\Big) = \Big\|\frac{d\nu}{d\mu} - 1\Big\|^2_{L^2(\mu)}$$

$$\mathrm{Entropy}(\nu \mid \mu) = \int \log\frac{d\nu}{d\mu} d\nu = \int \frac{d\nu}{d\mu}\log\frac{d\nu}{d\mu} d\mu$$

$$\mathrm{Fisher}(\nu \mid \mu) = \int\Big|\nabla\log\frac{d\nu}{d\mu}\Big|^2 d\nu = 4\int\Big|\nabla\sqrt{\frac{d\nu}{d\mu}}\Big|^2 d\mu$$

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\Big(\frac{d\nu}{d\mu}\Big) = \Big\|\frac{d\nu}{d\mu} - 1\Big\|^2_{L^2(\mu)}$$

$$\mathrm{Entropy}(\nu \mid \mu) = \int \log\frac{d\nu}{d\mu}\,d\nu = \int \frac{d\nu}{d\mu}\log\frac{d\nu}{d\mu}\,d\mu$$

$$\mathrm{Fisher}(\nu \mid \mu) = \int\Big|\nabla\log\frac{d\nu}{d\mu}\Big|^2 d\nu = 4\int\Big|\nabla\sqrt{\frac{d\nu}{d\mu}}\Big|^2 d\mu$$

$$\mathrm{Wasserstein}^2(\mu, \nu) = \inf_{(X_\mu, X_\nu)} \mathbb{E}(\tfrac{1}{2}|X_\mu - X_\nu|^2) = \sup_{f\mathrm{BL}}\Big(\int Q_1(f)\,d\mu - \int f\,d\nu\Big)$$

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\left(\frac{d\nu}{d\mu}\right) = \left\|\frac{d\nu}{d\mu} - 1\right\|^2_{L^2(\mu)}$$

$$\mathrm{Entropy}(\nu \mid \mu) = \int \log \frac{d\nu}{d\mu} d\nu = \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu$$

$$\mathrm{Fisher}(\nu \mid \mu) = \int \left|\nabla \log \frac{d\nu}{d\mu}\right|^2 d\nu = 4 \int \left|\nabla \sqrt{\frac{d\nu}{d\mu}}\right|^2 d\mu$$

$$\mathrm{Wasserstein}^2(\mu, \nu) = \inf_{(X_\mu, X_\nu)} \mathbb{E}(\tfrac{1}{2}|X_\mu - X_\nu|^2) = \sup_{f \mathrm{BL}}\left(\int Q_1(f) d\mu - \int f d\nu\right)$$

$$\|\mu - \nu\|_{\mathrm{TV}} = \inf_{(X_\mu, X_\nu)} \mathbb{E}(1_{X_\mu \neq X_\nu}) = \sup_{\|f\|_\infty \leq \frac{1}{2}} \left(\int f d\mu - \int f d\nu\right)$$

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\left(\frac{d\nu}{d\mu}\right) = \left\|\frac{d\nu}{d\mu} - 1\right\|^2_{L^2(\mu)}$$

$$\mathrm{Entropy}(\nu \mid \mu) = \int \log \frac{d\nu}{d\mu} d\nu = \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu$$

$$\mathrm{Fisher}(\nu \mid \mu) = \int \left|\nabla \log \frac{d\nu}{d\mu}\right|^2 d\nu = 4 \int \left|\nabla \sqrt{\frac{d\nu}{d\mu}}\right|^2 d\mu$$

$$\mathrm{Wasserstein}^2(\mu, \nu) = \inf_{(X_\mu, X_\nu)} \mathbb{E}\left(\tfrac{1}{2}|X_\mu - X_\nu|^2\right) = \sup_{f \mathrm{BL}} \left(\int Q_1(f) d\mu - \int f d\nu\right)$$

$$\begin{aligned}\|\mu - \nu\|_{\mathrm{TV}} &= \inf_{(X_\mu, X_\nu)} \mathbb{E}(1_{X_\mu \neq X_\nu}) = \sup_{\|f\|_\infty \leq \frac{1}{2}} \left(\int f d\mu - \int f d\nu\right) \\ &= \sup_A |\nu(A) - \mu(A)| = \tfrac{1}{2}\|\varphi_\mu - \varphi_\nu\|_{L^1(\lambda)}\end{aligned}$$

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\Big(\frac{d\nu}{d\mu}\Big) = \Big\|\frac{d\nu}{d\mu} - 1\Big\|^2_{L^2(\mu)}$$

$$\mathrm{Entropy}(\nu \mid \mu) = \int \log\frac{d\nu}{d\mu}d\nu = \int \frac{d\nu}{d\mu}\log\frac{d\nu}{d\mu}d\mu$$

$$\mathrm{Fisher}(\nu \mid \mu) = \int\Big|\nabla\log\frac{d\nu}{d\mu}\Big|^2 d\nu = 4\int\Big|\nabla\sqrt{\frac{d\nu}{d\mu}}\Big|^2 d\mu$$

$$\mathrm{Wasserstein}^2(\mu,\nu) = \inf_{(X_\mu,X_\nu)}\mathbb{E}(\tfrac{1}{2}|X_\mu - X_\nu|^2) = \sup_{f\mathrm{BL}}\Big(\int Q_1(f)d\mu - \int f d\nu\Big)$$

$$\|\mu-\nu\|_{\mathrm{TV}} = \inf_{(X_\mu,X_\nu)}\mathbb{E}(1_{X_\mu\neq X_\nu}) = \sup_{\|f\|_\infty\leq\frac{1}{2}}\Big(\int f d\mu - \int f d\nu\Big)$$

$$= \sup_A|\nu(A)-\mu(A)| = \tfrac{1}{2}\|\varphi_\mu - \varphi_\nu\|_{L^1(\lambda)}$$

$$\mathrm{Hellinger}^2(\mu,\nu) = \tfrac{1}{2}\|\sqrt{\varphi_\mu} - \sqrt{\varphi_\nu}\|^2_{L^2(\lambda)}$$

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\left(\frac{d\nu}{d\mu}\right) = \left\|\frac{d\nu}{d\mu} - 1\right\|^2_{L^2(\mu)}$$

$$\text{Kullback}(\nu \mid \mu) = \int \log \frac{d\nu}{d\mu} d\nu = \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu$$

$$\text{Fisher}(\nu \mid \mu) = \int \left|\nabla \log \frac{d\nu}{d\mu}\right|^2 d\nu = 4 \int \left|\nabla \sqrt{\frac{d\nu}{d\mu}}\right|^2 d\mu$$

$$\text{Wasserstein}^2(\mu, \nu) = \inf_{(X_\mu, X_\nu)} \mathbb{E}(\tfrac{1}{2}|X_\mu - X_\nu|^2) = \sup_{f\mathrm{BL}} \left(\int Q_1(f) d\mu - \int f d\nu\right)$$

$$\|\mu - \nu\|_{\mathrm{TV}} = \inf_{(X_\mu, X_\nu)} \mathbb{E}(1_{X_\mu \neq X_\nu}) = \sup_{\|f\|_\infty \leq \frac{1}{2}} \left(\int f d\mu - \int f d\nu\right)$$

$$= \sup_A |\nu(A) - \mu(A)| = \tfrac{1}{2}\|\varphi_\mu - \varphi_\nu\|_{L^1(\lambda)}$$

$$\text{Hellinger}^2(\mu, \nu) = \tfrac{1}{2}\|\sqrt{\varphi_\mu} - \sqrt{\varphi_\nu}\|^2_{L^2(\lambda)}$$

Monotonicity

- With $\nu_t = \mathrm{Law}(X_t^n)$ and $\mu = P^n$

$$\mathrm{dist}(\nu_t \mid \mu) \underset{t\to\infty}{\searrow} 0$$

Monotonicity

- With $\nu_t = \mathrm{Law}(X_t^n)$ and $\mu = P^n$

$$\mathrm{dist}(\nu_t \mid \mu) \underset{t\to\infty}{\searrow} 0$$

- Markovianity and convexity : if $\nu \ll \mu$ then

$$\mathrm{dist}(\nu \mid \mu) = \int \Phi\left(\frac{d\nu}{d\mu}\right) d\mu$$

$$\Phi(u) = \begin{cases} u^2 - 1 & \text{if dist} = \chi^2 \\ u\log(u) & \text{if dist} = \text{Entropy} \\ \frac{1}{2}|u-1| & \text{if dist} = \text{TV} \\ \frac{1}{2}(1-\sqrt{u}) & \text{if dist} = \text{Hellinger}^2 \end{cases}$$

Monotonicity

- With $\nu_t = \mathrm{Law}(X_t^n)$ and $\mu = P^n$

$$\mathrm{dist}(\nu_t \mid \mu) \underset{t\to\infty}{\searrow} 0$$

- Markovianity and convexity : if $\nu \ll \mu$ then

$$\mathrm{dist}(\nu \mid \mu) = \int \Phi\left(\tfrac{d\nu}{d\mu}\right) d\mu$$

$$\Phi(u) = \begin{cases} u^2 - 1 & \text{if dist} = \chi^2 \\ u\log(u) & \text{if dist} = \text{Entropy} \\ \frac{1}{2}|u-1| & \text{if dist} = \text{TV} \\ \frac{1}{2}(1-\sqrt{u}) & \text{if dist} = \text{Hellinger}^2 \end{cases}$$

- Fisher and Wasserstein : involve also convexity of V

Moments and cutoff

- Moments

$$m_k(t) = \mathbb{E}\left(\frac{\sum_{i=1}^{n}(X_t^{n,i})^k}{n}\right) = \mathbb{E}\int u^k \mu_t^n(\mathrm{d}u)$$

Moments and cutoff

- Moments

$$m_k(t) = \mathbb{E}\left(\frac{\sum_{i=1}^{n}(X_t^{n,i})^k}{n}\right) = \mathbb{E}\int u^k \mu_t^n(\mathrm{d}u)$$

- Long-time behavior

$$m_1(\infty) = 0 \quad \text{and} \quad m_2(\infty) = \frac{1+\frac{\beta}{2}(n-1)}{n} \sim \frac{\beta}{2}$$

Moments and cutoff

- Moments

$$m_k(t) = \mathbb{E}\left(\frac{\sum_{i=1}^n (X_t^{n,i})^k}{n}\right) = \mathbb{E}\int u^k \mu_t^n(\mathrm{d}u)$$

- Long-time behavior

$$m_1(\infty) = 0 \quad \text{and} \quad m_2(\infty) = \frac{1+\frac{\beta}{2}(n-1)}{n} \sim \frac{\beta}{2}$$

- Dynamics

$$m_1(t) = \mathrm{e}^{-t} m_1(0) \quad \text{and} \quad m_2(t) = \mathrm{e}^{-2t} m_2(0) + (1-\mathrm{e}^{-2t}) m_2(\infty)$$

Moments and cutoff

- Moments

$$m_k(t) = \mathbb{E}\left(\frac{\sum_{i=1}^n (X_t^{n,i})^k}{n}\right) = \mathbb{E}\int u^k \mu_t^n(\mathrm{d}u)$$

- Long-time behavior

$$m_1(\infty) = 0 \quad \text{and} \quad m_2(\infty) = \frac{1+\frac{\beta}{2}(n-1)}{n} \sim \frac{\beta}{2}$$

- Dynamics

$$m_1(t) = \mathrm{e}^{-t} m_1(0) \quad \text{and} \quad m_2(t) = \mathrm{e}^{-2t} m_2(0) + (1-\mathrm{e}^{-2t}) m_2(\infty)$$

- Eigenfunctions $\mathrm{e}^{t\mathrm{L}}\pi_k = \mathrm{e}^{-tk}\pi_k$

$$\pi_1(x) = \sum_{i=1}^n x_i \quad \text{and} \quad \pi_2(x) = \sum_{i=1}^n x_i^2 - \left(1 + \frac{\beta}{2}(n-1)\right)$$

Moments and cutoff

- Moments

$$m_k(t) = \mathbb{E}\left(\frac{\sum_{i=1}^{n}(X_t^{n,i})^k}{n}\right) = \mathbb{E}\int u^k \mu_t^n(\mathrm{d}u)$$

- Long-time behavior

$$m_1(\infty) = 0 \quad \text{and} \quad m_2(\infty) = \frac{1+\frac{\beta}{2}(n-1)}{n} \sim \frac{\beta}{2}$$

- Dynamics

$$m_1(t) = \mathrm{e}^{-t} m_1(0) \quad \text{and} \quad m_2(t) = \mathrm{e}^{-2t} m_2(0) + (1-\mathrm{e}^{-2t}) m_2(\infty)$$

- Eigenfunctions $\mathrm{e}^{t\mathrm{L}}\pi_k = \mathrm{e}^{-tk}\pi_k$

$$\pi_1(x) = \sum_{i=1}^{n} x_i \quad \text{and} \quad \pi_2(x) = \sum_{i=1}^{n} x_i^2 - \left(1 + \frac{\beta}{2}(n-1)\right)$$

- $\log(n)$ cutoff for $\mathbb{E}(\pi_k(X_t))$: dimension n versus e^{-kt} decay

Cutoff for DOU : processes

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$ and set

$$Z_t = \sum_{i=1}^{n} X_t^{n,i} \quad \text{and} \quad R_t = \sum_{i=1}^{n} (X_t^{n,i})^2 = |X_t^n|^2$$

Then $(Z_t)_{t\geq 0} \sim \mathrm{OU}$ and $(R_t)_{t\geq 0} \sim \mathrm{CIR}$

Cutoff for DOU : processes

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$ and set

$$Z_t = \sum_{i=1}^{n} X_t^{n,i} \quad \text{and} \quad R_t = \sum_{i=1}^{n} (X_t^{n,i})^2 = |X_t^n|^2$$

Then $(Z_t)_{t\geq 0} \sim$ OU and $(R_t)_{t\geq 0} \sim$ CIR

  - $Z_t, R_t \xrightarrow{\mathrm{d}} \mathcal{N}(0,1), \mathrm{Gamma}\left(\frac{n}{2} + \frac{\beta}{2}\frac{n(n-1)}{2}, \frac{n}{2}\right)$ as $t\to\infty$

Cutoff for DOU : processes

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$ and set

$$Z_t = \sum_{i=1}^{n} X_t^{n,i} \quad \text{and} \quad R_t = \sum_{i=1}^{n} (X_t^{n,i})^2 = |X_t^n|^2$$

Then $(Z_t)_{t\geq 0} \sim \mathrm{OU}$ and $(R_t)_{t\geq 0} \sim \mathrm{CIR}$

  - $Z_t, R_t \xrightarrow{\mathrm{d}} \mathcal{N}(0,1), \mathrm{Gamma}\left(\frac{n}{2} + \frac{\beta}{2}\frac{n(n-1)}{2}, \frac{n}{2}\right)$ as $t \to \infty$
  - $R \overset{\mathrm{d}}{=} |Z_*|^2$ with $Z_* \sim \mathrm{OU}_{\mathbb{R}^m}$ when $m = n + \beta\frac{n(n-1)}{2} \in \mathbb{N}$

Cutoff for DOU : processes

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$ and set

$$Z_t = \sum_{i=1}^{n} X_t^{n,i} \quad \text{and} \quad R_t = \sum_{i=1}^{n} (X_t^{n,i})^2 = |X_t^n|^2$$

Then $(Z_t)_{t\geq 0} \sim \mathrm{OU}$ and $(R_t)_{t\geq 0} \sim \mathrm{CIR}$

  - $Z_t, R_t \xrightarrow{\mathrm{d}} \mathcal{N}(0,1), \mathrm{Gamma}\left(\frac{n}{2} + \frac{\beta}{2}\frac{n(n-1)}{2}, \frac{n}{2}\right)$ as $t\to\infty$
  - $R \overset{\mathrm{d}}{=} |Z_*|^2$ with $Z_* \sim \mathrm{OU}_{\mathbb{R}^m}$ when $m = n + \beta\frac{n(n-1)}{2} \in \mathbb{N}$
  - $R = \mathrm{Trace}(M_t^2)$ and $m = \mathrm{d}^\circ\mathrm{freedom}(M_t)$ when $\beta \in \{1, 2, 4\}$

Cutoff for DOU : processes

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$ and set

$$Z_t = \sum_{i=1}^{n} X_t^{n,i} \quad \text{and} \quad R_t = \sum_{i=1}^{n} (X_t^{n,i})^2 = |X_t^n|^2$$

  Then $(Z_t)_{t\geq 0} \sim \mathrm{OU}$ and $(R_t)_{t\geq 0} \sim \mathrm{CIR}$

  - $Z_t, R_t \xrightarrow{\mathrm{d}} \mathcal{N}(0,1), \mathrm{Gamma}\left(\frac{n}{2} + \frac{\beta}{2}\frac{n(n-1)}{2}, \frac{n}{2}\right)$ as $t \to \infty$
  - $R \overset{\mathrm{d}}{=} |Z_*|^2$ with $Z_* \sim \mathrm{OU}_{\mathbb{R}^m}$ when $m = n + \beta\frac{n(n-1)}{2} \in \mathbb{N}$
  - $R = \mathrm{Trace}(M_t^2)$ and $m = \mathrm{d}^\circ\mathrm{freedom}(M_t)$ when $\beta \in \{1, 2, 4\}$

- **Proof :** Stroock–Varadhan local martingale

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathrm{L} f(X_s)\mathrm{d}s$$

$$\langle M \rangle_t = \int_0^t \Gamma(f)(X_s)\mathrm{d}s, \text{ take } \mathrm{L} f = -\lambda f, \text{ then } f \in \{\pi_1, \pi_2\}$$

Plan

Cutoff for OU : Mean-field case

- Theorem : if $\beta = 0$ and $\frac{|x_0^n|^2}{n} \asymp 1$ then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

Cutoff for OU : Mean-field case

- **Theorem** : if $\beta = 0$ and $\frac{|x_0^n|^2}{n} \asymp 1$ then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \frac{1}{2}\log(n) & \text{if dist} = \text{Wasserstein} \\ \log(n) & \text{if dist} \in \{\chi^2, \text{Entropy}, \text{TV}, \text{Hellinger}\} \\ \frac{3}{2}\log(n) & \text{if dist} = \text{Fisher} \end{cases}$$

Cutoff for OU : Mean-field case

- Theorem : if $\beta = 0$ and $\frac{|x_0^n|^2}{n} \asymp 1$ then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \frac{1}{2}\log(n) & \text{if dist} = \text{Wasserstein} \\ \log(n) & \text{if dist} \in \{\chi^2, \text{Entropy}, \text{TV}, \text{Hellinger}\} \\ \frac{3}{2}\log(n) & \text{if dist} = \text{Fisher} \end{cases}$$

- Other initial conditions ?

Cutoff for OU : General case

- Theorem : if $\beta = 0$ ~~and $\frac{|x_0^n|^2}{n} \asymp 1$~~ then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X^n_{t_n}) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

Cutoff for OU : General case

- **Theorem :** if $\beta = 0$ ~~and~~ $\cancel{\frac{|x_0^n|^2}{n} \asymp 1}$ then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}|x_0^n|) \vee \frac{1}{4}\log(n) & \text{if dist} \in \{\chi^2, \text{Entropy}, \text{TV}, \text{Hell.}\} \\ \log(n|x_0^n|) \vee \frac{1}{2}\log(n) & \text{if dist} = \text{Fisher} \end{cases}$$

Cutoff for OU : General case

- **Theorem** : if $\beta = 0$ ~~and~~ $\cancel{\frac{|x_0^n|^2}{n} \asymp 1}$ then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X^n_{t_n}) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}|x_0^n|) \vee \frac{1}{4}\log(n) & \text{if dist} \in \{\chi^2, \text{Entropy}, \text{TV}, \text{Hell.}\} \\ \log(n|x_0^n|) \vee \frac{1}{2}\log(n) & \text{if dist} = \text{Fisher} \end{cases}$$

- Bias versus variance

Cutoff for OU : General case

- Theorem : if $\beta = 0$ ~~and~~ $\cancel{\frac{|x_0^n|^2}{n}} \asymp 1$ then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X^n_{t_n}) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}|x_0^n|) \vee \frac{1}{4}\log(n) & \text{if dist} \in \{\chi^2, \text{Entropy}, \text{TV}, \text{Hell.}\} \\ \log(n|x_0^n|) \vee \frac{1}{2}\log(n) & \text{if dist} = \text{Fisher} \end{cases}$$

- Bias versus variance
- $\beta = 0$: $\mu^n_\infty = \mathcal{N}(0, \frac{1}{n}\mathrm{I}_n)$, $\mu_\infty = \delta_0$

Cutoff for OU : General case

- Theorem : if $\beta = 0$ ~~and~~ $\cancel{\frac{|x_0^n|^2}{n} \asymp 1}$ then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}|x_0^n|) \vee \frac{1}{4}\log(n) & \text{if dist} \in \{\chi^2, \text{Entropy}, \text{TV}, \text{Hell.}\} \\ \log(n|x_0^n|) \vee \frac{1}{2}\log(n) & \text{if dist} = \text{Fisher} \end{cases}$$

- Bias versus variance
- $\beta = 0$: $\mu_\infty^n = \mathcal{N}(0, \frac{1}{n}\mathrm{I}_n)$, $\mu_\infty = \delta_0$
- Wasserstein ?

Cutoff for OU : General case (Wasserstein)

- Theorem : if $\beta = 0$ then for all $\varepsilon \in (0,1)$

Cutoff for OU : General case (Wasserstein)

- **Theorem :** if $\beta = 0$ then for all $\varepsilon \in (0,1)$
  - if $\lim_{n\to\infty} |x_0^n| = \infty$ then cutoff with $c_n = \log(|x_0^n|)$:

$$\lim_{n\to\infty} \mathrm{Wasserstein}(\mathrm{Law}(X_{t_n}^n), P^n) = \begin{cases} \infty & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

Cutoff for OU : General case (Wasserstein)

- **Theorem :** if $\beta = 0$ then for all $\varepsilon \in (0,1)$
  - if $\lim_{n\to\infty} |x_0^n| = \infty$ then cutoff with $c_n = \log(|x_0^n|)$:

$$\lim_{n\to\infty} \mathrm{Wasserstein}(\mathrm{Law}(X_{t_n}^n), P^n) = \begin{cases} \infty & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

  - if $\lim_{n\to\infty} |x_0^n| = \alpha \in [0,\infty)$ then no cutoff : for all $t > 0$,

$$\lim_{n\to\infty} \mathrm{Wasserstein}^2(\mathrm{Law}(X_{t_n}^n), P^n) = 1 + \frac{1}{2}(\alpha^2 - 1)\mathrm{e}^{-2t} - \sqrt{1 - \mathrm{e}^{-2t}}$$

Cutoff for OU : General case (Wasserstein)

- **Theorem :** if $\beta = 0$ then for all $\varepsilon \in (0,1)$
  - if $\lim_{n\to\infty} |x_0^n| = \infty$ then cutoff with $c_n = \log(|x_0^n|)$:

$$\lim_{n\to\infty} \mathrm{Wasserstein}(\mathrm{Law}(X_{t_n}^n), P^n) = \begin{cases} \infty & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

  - if $\lim_{n\to\infty} |x_0^n| = \alpha \in [0,\infty)$ then no cutoff : for all $t > 0$,

$$\lim_{n\to\infty} \mathrm{Wasserstein}^2(\mathrm{Law}(X_{t_n}^n), P^n) = 1 + \tfrac{1}{2}(\alpha^2 - 1)\mathrm{e}^{-2t} - \sqrt{1 - \mathrm{e}^{-2t}}$$

- Reminds behavior of second moment m_2

Cutoff for OU : Proof 1/3

- OU : $\mathrm{d}Y_t = \sqrt{2\theta}\mathrm{d}B_t - Y_t\mathrm{d}t$, $\mathbb{R}^d$, $\eta_t = \mathrm{Law}(Y_t)$

$$\eta_t \underset{t\to\infty}{\longrightarrow} \eta_\infty = \mathcal{N}\big(0, \theta \mathrm{I}_d\big)$$

Cutoff for OU : Proof 1/3

- OU : $\mathrm{d}\,Y_t = \sqrt{2\theta}\mathrm{d}B_t - Y_t\mathrm{d}t$, $\mathbb{R}^d$, $\eta_t = \mathrm{Law}(Y_t)$

$$\eta_t \xrightarrow[t\to\infty]{} \eta_\infty = \mathcal{N}(0, \theta \mathrm{I}_d)$$

- Mehler formula : $Y_t = \mathrm{e}^{-t}\,Y_0 + \sqrt{2\theta}\int_0^t \mathrm{e}^{s-r}\mathrm{d}\,B_s$

$$\eta_t = \mathrm{dil}_{\mathrm{e}^{-t}}\eta_0 * \mathrm{dil}_{\sqrt{\theta(1-\mathrm{e}^{-2t})}}\eta_\infty$$

Cutoff for OU : Proof 1/3

- OU : $\mathrm{d}\,Y_t = \sqrt{2\theta}\mathrm{d}B_t - Y_t\mathrm{d}t$, $\mathbb{R}^d$, $\eta_t = \mathrm{Law}(Y_t)$

$$\eta_t \underset{t\to\infty}{\longrightarrow} \eta_\infty = \mathcal{N}\big(0, \theta I_d\big)$$

- Mehler formula : $Y_t = \mathrm{e}^{-t}\,Y_0 + \sqrt{2\theta}\displaystyle\int_0^t \mathrm{e}^{s-r}\mathrm{d}B_s$

$$\eta_t = \mathrm{dil}_{\mathrm{e}^{-t}}\eta_0 * \mathrm{dil}_{\sqrt{\theta(1-\mathrm{e}^{-2t})}}\eta_\infty$$

- In particular if $\eta_0 = \delta_y$ then

$$\eta_t = \mathcal{N}\big(\mathrm{e}^{-t}y, \theta(1-\mathrm{e}^{-2t})I_d\big)$$

Cutoff for OU : Proof 2/3

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$\chi^2(\Gamma_1 \mid \Gamma_2) = \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2 \Sigma_2^{-1}|}} e^{\frac{1}{2}\Sigma_2^{-1}(I_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2})m \cdot m} - 1$$

Cutoff for OU : Proof 2/3

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$\chi^2(\Gamma_1 \mid \Gamma_2) = \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2\Sigma_2^{-1}|}} e^{\frac{1}{2}\Sigma_2^{-1}(\mathrm{I}_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2})m\cdot m} - 1$$

$$2\mathrm{Entropy}(\Gamma_1 \mid \Gamma_2) = \Sigma_2^{-1} m \cdot m + \mathrm{Tr}(\Sigma_2^{-1}\Sigma_1 - \mathrm{I}_n) + \log\det(\Sigma_2\Sigma_1^{-1})$$

Cutoff for OU : Proof 2/3

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$\chi^2(\Gamma_1 \mid \Gamma_2) = \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2\Sigma_2^{-1}|}} e^{\frac{1}{2}\Sigma_2^{-1}(\mathrm{I}_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2})m\cdot m} - 1$$

$$2\mathrm{Entropy}(\Gamma_1 \mid \Gamma_2) = \Sigma_2^{-1} m \cdot m + \mathrm{Tr}(\Sigma_2^{-1}\Sigma_1 - \mathrm{I}_n) + \log\det(\Sigma_2\Sigma_1^{-1})$$

$$\mathrm{Fisher}(\Gamma_1 \mid \Gamma_2) = |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}\Sigma_1 - 2\Sigma_2^{-1} + \Sigma_1^{-1})$$

Cutoff for OU : Proof 2/3

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$\begin{aligned}
\chi^2(\Gamma_1 \mid \Gamma_2) &= \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2\Sigma_2^{-1}|}}\mathrm{e}^{\frac{1}{2}\Sigma_2^{-1}(\mathrm{I}_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2})m\cdot m} - 1 \\
2\mathrm{Entropy}(\Gamma_1 \mid \Gamma_2) &= \Sigma_2^{-1} m \cdot m + \mathrm{Tr}(\Sigma_2^{-1}\Sigma_1 - \mathrm{I}_n) + \log\det(\Sigma_2\Sigma_1^{-1}) \\
\mathrm{Fisher}(\Gamma_1 \mid \Gamma_2) &= |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}\Sigma_1 - 2\Sigma_2^{-1} + \Sigma_1^{-1}) \\
&\overset{*}{=} |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}(\Sigma_2 - \Sigma_1)^2\Sigma_1^{-1})
\end{aligned}$$

Cutoff for OU : Proof 2/3

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$\chi^2(\Gamma_1 \mid \Gamma_2) = \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2\Sigma_2^{-1}|}} e^{\frac{1}{2}\Sigma_2^{-1}(I_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2})m\cdot m} - 1$$

$$2\mathrm{Entropy}(\Gamma_1 \mid \Gamma_2) = \Sigma_2^{-1} m \cdot m + \mathrm{Tr}(\Sigma_2^{-1}\Sigma_1 - I_n) + \log\det(\Sigma_2\Sigma_1^{-1})$$

$$\mathrm{Fisher}(\Gamma_1 \mid \Gamma_2) = |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}\Sigma_1 - 2\Sigma_2^{-1} + \Sigma_1^{-1})$$

$$\overset{*}{=} |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}(\Sigma_2 - \Sigma_1)^2\Sigma_1^{-1})$$

$$2\mathrm{Wasserstein}^2(\Gamma_1, \Gamma_2) = |m|^2 + \mathrm{Tr}\left(\Sigma_1 + \Sigma_2 - 2\sqrt{\sqrt{\Sigma_1}\Sigma_2\sqrt{\Sigma_1}}\right)$$

Cutoff for OU : Proof 2/3

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$\chi^2(\Gamma_1 \mid \Gamma_2) = \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2\Sigma_2^{-1}|}} e^{\frac{1}{2}\Sigma_2^{-1}(I_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2})m\cdot m} - 1$$

$$2\mathrm{Entropy}(\Gamma_1 \mid \Gamma_2) = \Sigma_2^{-1} m \cdot m + \mathrm{Tr}(\Sigma_2^{-1}\Sigma_1 - I_n) + \log\det(\Sigma_2\Sigma_1^{-1})$$

$$\begin{aligned}\mathrm{Fisher}(\Gamma_1 \mid \Gamma_2) &= |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}\Sigma_1 - 2\Sigma_2^{-1} + \Sigma_1^{-1}) \\ &\overset{*}{=} |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}(\Sigma_2 - \Sigma_1)^2\Sigma_1^{-1})\end{aligned}$$

$$\begin{aligned}2\mathrm{Wasserstein}^2(\Gamma_1, \Gamma_2) &= |m|^2 + \mathrm{Tr}\Big(\Sigma_1 + \Sigma_2 - 2\sqrt{\sqrt{\Sigma_1}\Sigma_2\sqrt{\Sigma_1}}\Big) \\ &\overset{*}{=} |m|^2 + \mathrm{Tr}((\sqrt{\Sigma_1} - \sqrt{\Sigma_2})^2)\end{aligned}$$

Cutoff for OU : Proof 2/3

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$\chi^2(\Gamma_1 \mid \Gamma_2) = \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2\Sigma_2^{-1}|}} e^{\frac{1}{2}\Sigma_2^{-1}(\mathrm{I}_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2})m\cdot m} - 1$$

$$2\mathrm{Entropy}(\Gamma_1 \mid \Gamma_2) = \Sigma_2^{-1} m \cdot m + \mathrm{Tr}(\Sigma_2^{-1}\Sigma_1 - \mathrm{I}_n) + \log\det(\Sigma_2\Sigma_1^{-1})$$

$$\mathrm{Fisher}(\Gamma_1 \mid \Gamma_2) = |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}\Sigma_1 - 2\Sigma_2^{-1} + \Sigma_1^{-1})$$

$$\stackrel{*}{=} |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}(\Sigma_2 - \Sigma_1)^2\Sigma_1^{-1})$$

$$2\mathrm{Wasserstein}^2(\Gamma_1, \Gamma_2) = |m|^2 + \mathrm{Tr}\Big(\Sigma_1 + \Sigma_2 - 2\sqrt{\sqrt{\Sigma_1}\Sigma_2\sqrt{\Sigma_1}}\Big)$$

$$\stackrel{*}{=} |m|^2 + \mathrm{Tr}((\sqrt{\Sigma_1} - \sqrt{\Sigma_2})^2)$$

$$\mathrm{Hellinger}^2(\Gamma_1, \Gamma_2) = 1 - \sqrt{\frac{\sqrt{\det(\Sigma_1\Sigma_2)}}{\det(\frac{\Sigma_1+\Sigma_2}{2})}} \exp\Big(-\frac{1}{4}(\Sigma_1 + \Sigma_2)^{-1} m \cdot m\Big)$$

Cutoff for OU : Proof 2/3

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$
\begin{aligned}
\chi^2(\Gamma_1 \mid \Gamma_2) &= \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2\Sigma_2^{-1}|}} e^{\frac{1}{2}\Sigma_2^{-1}(\mathrm{I}_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2})m\cdot m} - 1 \\
2\mathrm{Entropy}(\Gamma_1 \mid \Gamma_2) &= \Sigma_2^{-1} m \cdot m + \mathrm{Tr}(\Sigma_2^{-1}\Sigma_1 - \mathrm{I}_n) + \log\det(\Sigma_2\Sigma_1^{-1}) \\
\mathrm{Fisher}(\Gamma_1 \mid \Gamma_2) &= |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}\Sigma_1 - 2\Sigma_2^{-1} + \Sigma_1^{-1}) \\
&\overset{*}{=} |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}(\Sigma_2 - \Sigma_1)^2\Sigma_1^{-1}) \\
2\mathrm{Wasserstein}^2(\Gamma_1, \Gamma_2) &= |m|^2 + \mathrm{Tr}\Big(\Sigma_1 + \Sigma_2 - 2\sqrt{\sqrt{\Sigma_1}\Sigma_2\sqrt{\Sigma_1}}\Big) \\
&\overset{*}{=} |m|^2 + \mathrm{Tr}((\sqrt{\Sigma_1} - \sqrt{\Sigma_2})^2) \\
\mathrm{Hellinger}^2(\Gamma_1, \Gamma_2) &= 1 - \sqrt{\frac{\sqrt{\det(\Sigma_1\Sigma_2)}}{\det(\frac{\Sigma_1+\Sigma_2}{2})}} \exp\Big(-\tfrac{1}{4}(\Sigma_1 + \Sigma_2)^{-1} m \cdot m\Big) \\
\|\mu - \nu\|_{\mathrm{TV}} &\leq \sqrt{2\mathrm{Entropy}(\nu \mid \mu)}
\end{aligned}
$$

Cutoff for OU : Proof 2/3

If $\Gamma_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\Gamma_2 = \mathcal{N}(\mu_2, \Sigma_2)$ in $\mathbb{R}^n$ then with $m = m_1 - m_2$:

$$\chi^2(\Gamma_1 \mid \Gamma_2) = \sqrt{\frac{|\Sigma_2|}{|2\Sigma_1 - \Sigma_1^2 \Sigma_2^{-1}|}} e^{\frac{1}{2}\Sigma_2^{-1}(I_n + 2\Sigma_1^{-1}\Sigma_2^{-1} - \Sigma_2^{-2}) m \cdot m} - 1$$

$$2\mathrm{Entropy}(\Gamma_1 \mid \Gamma_2) = \Sigma_2^{-1} m \cdot m + \mathrm{Tr}(\Sigma_2^{-1}\Sigma_1 - I_n) + \log\det(\Sigma_2 \Sigma_1^{-1})$$

$$\mathrm{Fisher}(\Gamma_1 \mid \Gamma_2) = |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}\Sigma_1 - 2\Sigma_2^{-1} + \Sigma_1^{-1})$$

$$\stackrel{*}{=} |\Sigma_2^{-1} m|^2 + \mathrm{Tr}(\Sigma_2^{-2}(\Sigma_2 - \Sigma_1)^2 \Sigma_1^{-1})$$

$$2\mathrm{Wasserstein}^2(\Gamma_1, \Gamma_2) = |m|^2 + \mathrm{Tr}\Big(\Sigma_1 + \Sigma_2 - 2\sqrt{\sqrt{\Sigma_1}\Sigma_2\sqrt{\Sigma_1}}\Big)$$

$$\stackrel{*}{=} |m|^2 + \mathrm{Tr}((\sqrt{\Sigma_1} - \sqrt{\Sigma_2})^2)$$

$$\mathrm{Hellinger}^2(\Gamma_1, \Gamma_2) = 1 - \sqrt{\frac{\sqrt{\det(\Sigma_1\Sigma_2)}}{\det(\frac{\Sigma_1 + \Sigma_2}{2})}} \exp\Big(-\tfrac{1}{4}(\Sigma_1 + \Sigma_2)^{-1} m \cdot m\Big)$$

$$\|\mu - \nu\|_{\mathrm{TV}} \le \sqrt{2\mathrm{Entropy}(\nu \mid \mu)}$$

$$\mathrm{Hellinger}^2(\mu, \nu) \le \|\mu - \nu\|_{\mathrm{TV}} \le \mathrm{Hellinger}(\mu, \nu)\sqrt{2 - \mathrm{Hellinger}(\mu, \nu)^2}$$

Cutoff for OU : Proof 3/3

If $\beta = 0$ and $X_0^n = x_0^n$ then

Cutoff for OU : Proof 3/3

If $\beta = 0$ and $X_0^n = x_0^n$ then

$$\chi^2(\mathrm{Law}(X_t^n) \mid P^n) = \frac{1}{(1-\mathrm{e}^{-4t})^{n/2}} \exp\Big(n|x_0^n|^2 \frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}}\Big) - 1$$

$$\mathrm{Entropy}(\mathrm{Law}(X_t^n) \mid P^n) = \tfrac{1}{2}\Big(n|x_0^n|^2\mathrm{e}^{-2t} - n\mathrm{e}^{-2t} - n\log(1-\mathrm{e}^{-2t})\Big)$$

$$\mathrm{Fisher}(\mathrm{Law}(X_t^n) \mid P^n) = n^2|x_0^n|^2\mathrm{e}^{-2t} + n^2\frac{\mathrm{e}^{-4t}}{1-\mathrm{e}^{-2t}}$$

$$\mathrm{Hellinger}^2(\mathrm{Law}(X_t^n), P^n) = 1 - \exp\Big(-\frac{n}{4}\frac{|x_0^n|^2\mathrm{e}^{-2t}}{2-\mathrm{e}^{-2t}} + \frac{n}{4}\log\Big(4\frac{1-\mathrm{e}^{-2t}}{(2-\mathrm{e}^{-2t})^2}\Big)\Big)$$

$$\mathrm{Wasserstein}^2(\mathrm{Law}(X_t^n), P^n) = 1 + \tfrac{1}{2}(|x_0^n|^2 - 1)\mathrm{e}^{-2t} - \sqrt{1-\mathrm{e}^{-2t}}$$

Cutoff for OU : Proof 3/3

If $\beta = 0$ and $X_0^n = x_0^n$ then

$$\chi^2(\mathrm{Law}(X_t^n) \mid P^n) = \frac{1}{(1-\mathrm{e}^{-4t})^{n/2}} \exp\Big(n|x_0^n|^2 \frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}}\Big) - 1$$

$$\mathrm{Entropy}(\mathrm{Law}(X_t^n) \mid P^n) = \tfrac{1}{2}\Big(n|x_0^n|^2\mathrm{e}^{-2t} - n\mathrm{e}^{-2t} - n\log(1-\mathrm{e}^{-2t})\Big)$$

$$\mathrm{Fisher}(\mathrm{Law}(X_t^n) \mid P^n) = n^2|x_0^n|^2\mathrm{e}^{-2t} + n^2\frac{\mathrm{e}^{-4t}}{1-\mathrm{e}^{-2t}}$$

$$\mathrm{Hellinger}^2(\mathrm{Law}(X_t^n), P^n) = 1 - \exp\Big(-\frac{n}{4}\frac{|x_0^n|^2\mathrm{e}^{-2t}}{2-\mathrm{e}^{-2t}} + \frac{n}{4}\log\Big(4\frac{1-\mathrm{e}^{-2t}}{(2-\mathrm{e}^{-2t})^2}\Big)\Big)$$

$$\mathrm{Wasserstein}^2(\mathrm{Law}(X_t^n), P^n) = 1 + \tfrac{1}{2}(|x_0^n|^2 - 1)\mathrm{e}^{-2t} - \sqrt{1-\mathrm{e}^{-2t}}$$

Melted noise and dimension : $\mathrm{d}X_t^n = \sqrt{\frac{2}{n}}\mathrm{d}B_t - X_t^n\mathrm{d}t$ in $\mathbb{R}^n$

Cutoff for OU : Proof 3/3

If $\beta = 0$ and $X_0^n = x_0^n$ then

$$\chi^2(\mathrm{Law}(X_t^n) \mid P^n) = \frac{1}{(1-\mathrm{e}^{-4t})^{n/2}} \exp\Big(n|x_0^n|^2 \frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}}\Big) - 1$$

$$\mathrm{Entropy}(\mathrm{Law}(X_t^n) \mid P^n) = \tfrac{1}{2}\Big(n|x_0^n|^2\mathrm{e}^{-2t} - n\mathrm{e}^{-2t} - n\log(1-\mathrm{e}^{-2t})\Big)$$

$$\mathrm{Fisher}(\mathrm{Law}(X_t^n) \mid P^n) = n^2|x_0^n|^2\mathrm{e}^{-2t} + n^2\frac{\mathrm{e}^{-4t}}{1-\mathrm{e}^{-2t}}$$

$$\mathrm{Hellinger}^2(\mathrm{Law}(X_t^n), P^n) = 1 - \exp\Big(-\frac{n}{4}\frac{|x_0^n|^2\mathrm{e}^{-2t}}{2-\mathrm{e}^{-2t}} + \frac{n}{4}\log\Big(4\frac{1-\mathrm{e}^{-2t}}{(2-\mathrm{e}^{-2t})^2}\Big)\Big)$$

$$\mathrm{Wasserstein}^2(\mathrm{Law}(X_t^n), P^n) = 1 + \tfrac{1}{2}(|x_0^n|^2 - 1)\mathrm{e}^{-2t} - \sqrt{1-\mathrm{e}^{-2t}}$$

Melted noise and dimension : $\mathrm{d}X_t^n = \sqrt{\frac{2}{n}}\mathrm{d}B_t - X_t^n \mathrm{d}t$ in $\mathbb{R}^n$

$\log(n)$ cutoff for $\mathrm{dist}(\mathrm{Law}(X_t^n) \mid P^n)$: n versus e^{-t}

Plan

Cutoff for DOU: Random matrix case

- Theorem : Assume that $\beta \in \{1, 2, 4\}$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0, 1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon) c_n \\ 0 & \text{if } t_n = (1+\varepsilon) c_n \end{cases}$$

Cutoff for DOU: Random matrix case

- **Theorem :** Assume that $\beta \in \{1, 2, 4\}$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0, 1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}\, a_n) & \text{if dist} = \text{Wasserstein} \\ \log(n a_n) & \text{if dist} \in \{\text{Entropy}, \text{TV}, \text{Hellinger}\} \end{cases}$$

Cutoff for DOU: Random matrix case

- **Theorem :** Assume that $\beta \in \{1, 2, 4\}$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0, 1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon) c_n \\ 0 & \text{if } t_n = (1+\varepsilon) c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}\, a_n) & \text{if dist} = \text{Wasserstein} \\ \log(n a_n) & \text{if dist} \in \{\text{Entropy}, \text{TV}, \text{Hellinger}\} \end{cases}$$

- **Proof :** OU sandwich (trace Z, matrix M) + dist contract.

Cutoff for DOU: Random matrix case

- **Theorem :** Assume that $\beta \in \{1,2,4\}$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}\, a_n) & \text{if dist} = \text{Wasserstein} \\ \log(n a_n) & \text{if dist} \in \{\text{Entropy}, \text{TV}, \text{Hellinger}\} \end{cases}$$

- Proof : OU sandwich (trace Z, matrix M) + dist contract.
- Cutoff should be controlled by $|x_0^n - \rho^n|$ instead of $|x_0^n|$

Plan

Cutoff for DOU: general case

- Theorem : Assume that $\beta = 0$ or $\beta \geq 1$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

Cutoff for DOU: general case

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon) c_n \\ 0 & \text{if } t_n = (1+\varepsilon) c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}\, a_n) & \text{if dist} = \text{Wasserstein} \\ \log(n a_n) & \text{if dist} \in \{\cancel{\text{Entropy}}, \text{TV}, \text{Hellinger}\} \end{cases}$$

Cutoff for DOU: general case

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}\, a_n) & \text{if dist} = \text{Wasserstein} \\ \log(n a_n) & \text{if dist} \in \{\cancel{\text{Entropy}}, \text{TV}, \text{Hellinger}\} \end{cases}$$

- Lower bound : Contraction to OU Z

Cutoff for DOU: general case

- **Theorem :** Assume that $\beta = 0$ or $\beta \geq 1$. Let (a_n) be such that $\inf(a_n) > 0$. Then for all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0^n \in [-a_n, a_n]} \mathrm{dist}(\mathrm{Law}(X_{t_n}^n) \mid P^n) = \begin{cases} \max & \text{if } t_n = (1-\varepsilon)c_n \\ 0 & \text{if } t_n = (1+\varepsilon)c_n \end{cases}$$

where

$$c_n = \begin{cases} \log(\sqrt{n}\,a_n) & \text{if dist} = \text{Wasserstein} \\ \log(n a_n) & \text{if dist} \in \{\cancel{\text{Entropy}}, \text{TV}, \text{Hellinger}\} \end{cases}$$

- Lower bound : Contraction to OU Z
- Upper bound : LSI, regularization, coupling ($\sim$ exclusion)

Cutoff for DOU : Proof for general case (1/2)

- Optimal log-Sobolev inequality

$$\mathrm{Entropy}(\nu \mid P^n) \leq \frac{1}{2n}\mathrm{Fisher}(\nu \mid P^n)$$

Cutoff for DOU : Proof for general case (1/2)

- Optimal log-Sobolev inequality

$$\mathrm{Entropy}(\nu \mid P^n) \leq \frac{1}{2n}\mathrm{Fisher}(\nu \mid P^n) = -\frac{1}{2}\int \log(f)\mathrm{L}f\,\mathrm{d}P^n$$

- Exponential entropy decay

$$\mathrm{Entropy}(\mathrm{Law}(X_t^n) \mid P^n) \leq \mathrm{e}^{-2t}\mathrm{Entropy}(\mathrm{Law}(X_0^n) \mid P^n)$$

Cutoff for DOU : Proof for general case (1/2)

- Optimal log-Sobolev inequality

$$\mathrm{Entropy}(\nu \mid P^n) \leq \frac{1}{2n}\mathrm{Fisher}(\nu \mid P^n) = -\frac{1}{2}\int \log(f)\mathrm{L}f\,\mathrm{d}P^n$$

- Exponential entropy decay

$$\mathrm{Entropy}(\mathrm{Law}(X_t^n) \mid P^n) \leq \mathrm{e}^{-2t}\mathrm{Entropy}(\mathrm{Law}(X_0^n) \mid P^n)$$

- Regularization Y^n of X^n (smoothed $Y_0^{n,i} \geq X_0^{n,i} = x_0^{n,i}$)

$$\mathrm{Entropy}(\mathrm{Law}(Y_t^n) \mid P^n) \leq C(n|x_0^n|^2 + n^2\log(n))\mathrm{e}^{-2t}$$

Cutoff for DOU : Proof for general case (2/2)

- Coalescent coupling preserving order $Y_t^{n,i} \geq X_t^{n,i}$

$$\begin{aligned}\|\mathrm{Law}(Y_t^n) - \mathrm{Law}(X_t^n)\|_{\mathrm{TV}} &\leq \mathbb{P}(Y_t^n \neq X_t^n) \\ &\leq \mathbb{P}(\inf\{s \geq 0 : A_s^n = 0\} > t)\end{aligned}$$

Cutoff for DOU : Proof for general case (2/2)

- Coalescent coupling preserving order $Y_t^{n,i} \geq X_t^{n,i}$

$$
\begin{aligned}
\|\mathrm{Law}(Y_t^n) - \mathrm{Law}(X_t^n)\|_{\mathrm{TV}} &\leq \mathbb{P}(Y_t^n \neq X_t^n) \\
&\leq \mathbb{P}(\inf\{s \geq 0 : A_s^n = 0\} > t)
\end{aligned}
$$

- "Area" $A_t = \sum_{i=1}^n (Y_t^{n,i} - X_t^{n,i})$ is an AR sort of OU

$$
\mathrm{d}A_t = -A_t \mathrm{d}t + \mathrm{d}M_t
$$

Cutoff for DOU : Proof for general case (2/2)

- Coalescent coupling preserving order $Y_t^{n,i} \geq X_t^{n,i}$

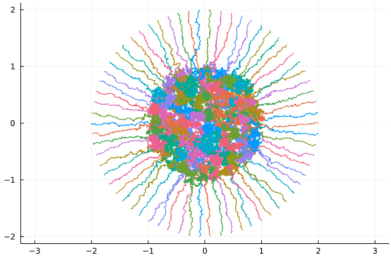
$$\begin{aligned}\|\mathrm{Law}(Y_t^n) - \mathrm{Law}(X_t^n)\|_{\mathrm{TV}} &\leq \mathbb{P}(Y_t^n \neq X_t^n) \\ &\leq \mathbb{P}(\inf\{s \geq 0 : A_s^n = 0\} > t)\end{aligned}$$

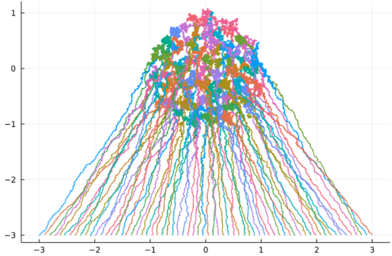
- "Area" $A_t = \sum_{i=1}^n (Y_t^{n,i} - X_t^{n,i})$ is an AR sort of OU

$$\mathrm{d}A_t = -A_t \mathrm{d}t + \mathrm{d}M_t$$

- Submartingale in $[0,1]$ $\mathrm{e}^{-\lambda A - \frac{\lambda^2}{2}\langle A \rangle}$

Thank you for your attention!





Selected bibliography and open problems

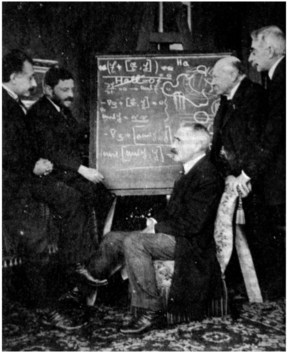
- Bibliography
  - Dyson, Anderson–Guionnet–Zeitouni, Erdős–Yau
  - Voiculescu, Rogers–Shi, Biane
  - Lassalle, Baker–Forrester
  - Lachaud, Barrera–Jara
  - Ané et al, Bakry–Gentil–Ledoux, Villani
  - Saloff-Coste, Méliot, Lacoin
  - C.–Lehec, Bolley–C.–Fontbona, Lu–Mattingly
- Problems
  - V : Exactly solvable cases (Hermite/Laguerre/Jacobi)
  - V : General strong convex case (Bakry–Émery or KLS)
  - Better initial conditions (ρ^n), other distances (Fisher, ...)
  - Non-convex interactions (such as planar DOU dynamics)

Leonard Salomon Ornstein (1880 – 1941)
George Eugene Uhlenbeck (1900 – 1988)

On the theory of Brownian Motion
Physical Reviews 36 (5) 823–841 (1930)

Paul Langevin (1872 – 1946)



Sur la théorie du mouvement brownien

Comptes-rendus de l'Académie des sciences (9 mars 1908)