About random matrices

Joint works with
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Probability Seminar
Stanford University
Monday March 29, 2021

Outline

Random matrices

- **Wishart** $\approx$ 1930
  Empirical covariance matrices
  Mathematical statistics

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- **Voiculescu** ≈ 1990
  Freeness as a high dimensional phenomenon
  Operator algebra

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Girko matrices

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ & \vdots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$$

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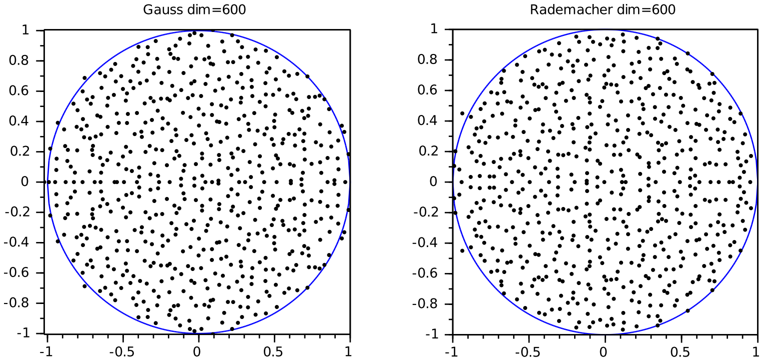
- Spectrum multiset and empirical spectral distribution

$$\{\lambda_1(A), \ldots, \lambda_n(A)\} = \{z \in \mathbb{C} : \det(A - zI) = 0\}$$

$$\mu_A = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(A)}$$

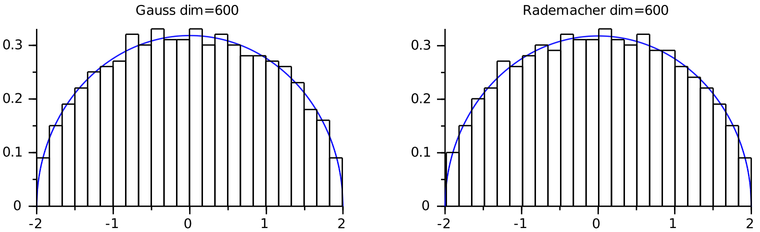
Universality $\frac{X}{\sqrt{n}}$ (Girko)



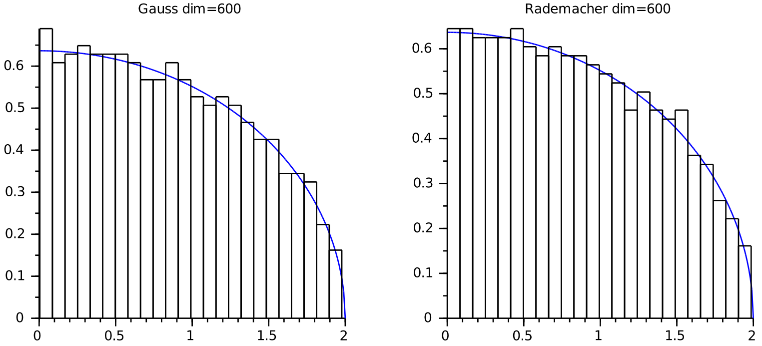

```
julia> eigvals(sign.(randn(dim,dim))/sqrt(dim))
```

Random but not independent - Collective phenomenon

Universality $\frac{X+X^*}{\sqrt{2n}}$ (Wigner)

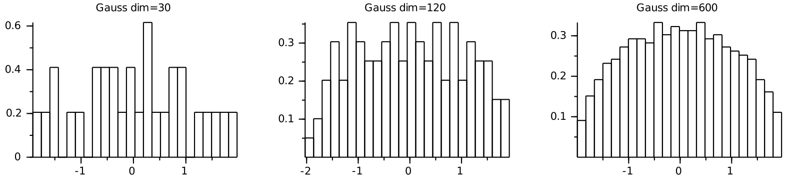


Universality $\frac{\sqrt{XX^*}}{\sqrt{n}}$ (Marchenko – Pastur)



High dimensional phenomenon

- High dimensional phenomenon (NLLLN, here in Wigner case)



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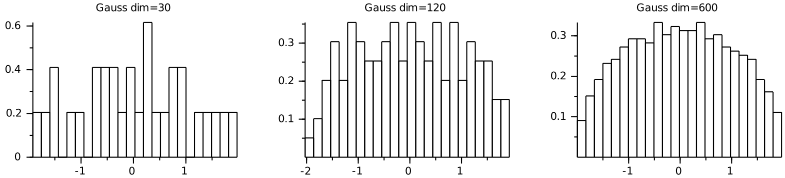

- Why this $\frac{1}{\sqrt{n}}$ normalization? Law of large numbers!

$$\frac{X}{\sqrt{n}} = \begin{pmatrix} \frac{X_{11}}{\sqrt{n}} & \cdots & \frac{X_{1n}}{\sqrt{n}} \\ & \vdots & \\ \frac{X_{n1}}{\sqrt{n}} & \cdots & \frac{X_{nn}}{\sqrt{n}} \end{pmatrix} \approx \text{Unitary}$$

Basic theorems

- Wigner

$$\mu_{\frac{X+X^*}{\sqrt{2n}}} \underset{n\to\infty}{\longrightarrow} \frac{\sqrt{4-x^2}}{2\pi}\mathbf{1}_{x\in[-2,2]}\mathrm{d}x$$

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- O1 : Bai – Silverstein (1980 – 2010), O2 : Erdős – Yau (2005 –)

Tools

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$$\int \lambda^r \mathrm{d}\mu_A(\lambda) = \frac{\mathrm{Trace}(A^r)}{n}$$

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  - Inversion

$$\mu_A = \frac{\Delta}{2\pi}(\log|\cdot| * \mu_A)$$

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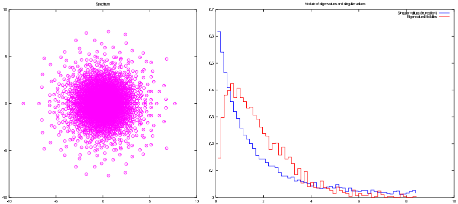
Outline

Girko – Lévy matrices : heavy tailed entries



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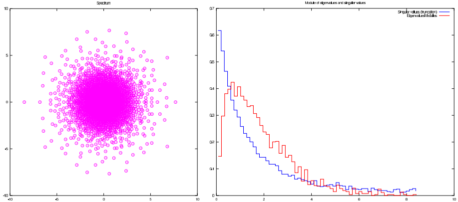

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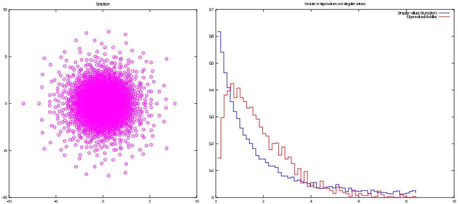

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- μ_α isotropic and not heavy tailed with density f_α

$$f_\alpha(z) \underset{|z|\to\infty}{\sim} c_\alpha\,|z|^{2(\alpha-1)}\,\mathrm{e}^{-\frac{\alpha}{2}|z|^\alpha}.$$

Operator convergence to Poisson Weighted Infinite Tree (Aldous)

Random regular graphs and free probability

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  - Voiculescu Free Central Limit Theorem

$$\frac{u_1+\cdots+u_d}{\sqrt{d}} \xrightarrow[d\to\infty]{\text{law}} \frac{\mathbf{1}_{|z|\leq 1}}{\pi}$$

Outline

Girko matrices: spectrum edge

- Light tails and heavy tails, outliers and point processes

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  - $\mathbb{E}[|X_{11}|^4] < \infty$: uniform law of large numbers $\max_i \sum_j |X_{ij}|^2$

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- Optimal theorem **without** any extra condition (2018, 2020)

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- Reciprocal characteristic polynomial : on $\{z \in \mathbb{C} : |z| < 1\}$

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  - $\alpha = \mathbb{E}[X_{11}^2]$, $|\alpha| \leq \mathbb{E}[|X_{11}|^2] = 1$ thus no zeros on $\{z \in \mathbb{C} : |z| < 1\}$

Proof outline

- Hurwitz phenomenon for zeros of random analytic functions

$$\inf_{|z|<1} \left|\det\left(1 - z\frac{X}{\sqrt{n}}\right)\right| > 0 \quad \text{as } n \to \infty$$

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- Tightness via orthogonal decomposition (→ Basak – Zeitouni)

$$\det\left(1 - z\frac{X}{\sqrt{n}}\right) = 1 + \sum_{k=1}^{n} (-z)^n \sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=k}} \frac{\det(X_{I,I})}{\sqrt{n}^k}$$

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- Joint CLT for fixed traces via moments (→ Janson – Nowicki)

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$$\inf_{|z|<1} \left| \det\left(1 - z\frac{X}{\sqrt{n}}\right) \right| > 0 \quad \text{as } n \to \infty$$

- Tightness via orthogonal decomposition ($\to$ Basak – Zeitouni)

$$\det\left(1 - z\frac{X}{\sqrt{n}}\right) = 1 + \sum_{k=1}^{n} (-z)^n \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I|=k}} \frac{\det(X_{I,I})}{\sqrt{n}^k}$$

- Reduction to bounded entries by truncation (!)
- Reduction to traces

$$\det\left(1 - z\frac{X}{\sqrt{n}}\right) = \exp\left(-\sum_{k=1}^{\infty} \frac{\mathrm{Trace}(X^k)}{\sqrt{n}^k} \frac{z^k}{k}\right)$$

- Joint CLT for fixed traces via moments ($\to$ Janson – Nowicki)
- Prefactor $\sqrt{1 - \alpha z^2}$ due to centering in CLT

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- CLT for linear statistics (Rider–Silverstein, Erdős et al)

$$n(U^{\mu_A}(z) - U^{\mu_\bullet}(z)) \xrightarrow[n\to\infty]{\mathrm{d}} G(z)$$

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- Convergence to Gaussian Analytic Function for $|z| > 1$

$$|\det(1-z^{-1}A)| = \mathrm{e}^{-n(U^{\mu_A}(z)-U^{\mu_\bullet}(z))} \xrightarrow[n\to\infty]{\mathrm{d}} \mathrm{e}^{-G(z)}.$$

Complex analytic link with CLT for linear statistics

- f analytic in a neighborhood of closed unit disc

$$\begin{aligned}\int f(\lambda)\mu_A(\mathrm{d}z) &= \frac{1}{2\pi \mathrm{i}}\int\Big(\oint \frac{f(z)}{z-\lambda}\mathrm{d}z\Big)\mu(\mathrm{d}\lambda)\\ &= \frac{1}{2\pi \mathrm{i}}\oint f(z)\Big(\int \frac{\mu(\mathrm{d}\lambda)}{z-\lambda}\Big)\mathrm{d}z\\ &= \frac{1}{2\pi \mathrm{i}}\oint f(z)(\log\det(z-A))'\mathrm{d}z.\end{aligned}$$

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- Compare with usual approach

$$\overline{\partial}\partial(\log|z| * \mu) = \Delta(\log|z| * \mu) = 2\pi\mu$$

$$\overline{\partial}\left(\tfrac{1}{z} * \mu\right) = \pi\mu$$

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- Work in progress : characteristic polynomial convergence

Outline

Ginibre$_{\mathbb{C}}$ matrices : exactly solvable model

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- Unitary invariance and Maxwell – Mehta characterization

$$\propto \mathrm{e}^{-\mathrm{Trace}(XX^*)}$$

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- Schur unitary decomposition $X = U(D+N)U^*$ gives

$$\mathrm{Trace}(XX^*) = \underbrace{\mathrm{Trace}(DD^*)}_{\sum_{i=1}^n |\lambda_i|^2} + \mathrm{Trace}(NN^*)$$

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- Rider : spectral radius $(\gamma_n=\log(n/(2\pi))-2\log(\log(n)))$

$$\sqrt{4n\gamma_n}\Big(\max_{1\leq i\leq n}|\lambda_i|-1-\sqrt{\frac{\gamma_n}{4n}}\Big)\xrightarrow[n\to\infty]{\mathrm{d}}\mathrm{Gumbel}=-\log\mathrm{Exp}$$

Ginibre$_{\mathbb{C}}$: Coulomb gas point of view

- Coulomb gas encoded with empirical measure $\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}$

$$\propto \mathrm{e}^{-n^2 \mathcal{E}_{\neq}(\mu_n)}$$

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- LDP – CLT linked by Hessian. Universality classes for LDP?

Spectral radius fluctuations : universality conjecture

- Rider : Ginibre$_{\mathbb{C}}$ with $\gamma_n = \log \frac{n}{2\pi} - 2\log\log n$

$$\sqrt{4n\gamma_n}\left(\frac{\rho(X)}{\sqrt{n}} - 1 - \sqrt{\frac{\gamma_n}{4n}}\right) \xrightarrow[n\to\infty]{\text{law}} \text{Gumbel} = -\log(\text{Expo})$$

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  - Coulomb gases on $\mathbb{C}$ with $V = Q(|\cdot|)$ and general β
    True for determinantal case $\beta = 2$ via Kostlan and Laplace

Spectral radius fluctuations : universality conjecture

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    True for determinantal case $\beta = 2$ via Kostlan and Laplace
  - Girko matrices with centered entries of unit variance
    Completely open!

Thank you!