From the mesoscopic to microscopic scale in random matrix theory

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Wigner's universality idea (1956). *Perhaps I am too courageous when I try to guess the distribution of the distances between successive levels (of energies of heavy nuclei). Theoretically, the situation is quite simple if one attacks the problem in a simpleminded fashion. The question is simply what are the distances of the characteristic values of a symmetric matrix with random coefficients.*

Gaussian Orthogonal Ensemble :
(a) Invariance by $H \mapsto U^* H U$, $U \in \mathrm{O}(N)$.
(b) Independence of the $H_{i,j}$'s, $i \leq j$.
The entries are Gaussian and the spectral density is

$$\frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\beta \frac{N}{4} \sum_i \lambda_i^2}$$

with $\beta = 1$. Semicircle law as $N \to \infty$. Eigenstates Haar-distributed.

Fundamental belief in universality : the macroscopic statistics (like the equilibrium measure) depend on the models, but the microscopic statistics are independent of the details of the systems except the symmetries.

- GOE : Hamiltonians of systems with time reversal invariance
- GUE : no time reversal symmetry (e.g. application of a magnetic field)
- GSE : time reversal but no rotational symmetry

Correlation functions. For a point process $\chi = \sum \delta_{\lambda_i}$:

$$\rho_k^{(N)}(x_1, \dots, x_k) = \lim_{\varepsilon \to 0} \varepsilon^{-k}\, \mathbb{P}\left(\chi(x_i, x_i + \varepsilon) = 1, 1 \le i \le k\right).$$

For deterministic systems, $\mathbb{P}$ is an averaging over the energy level in the semiclassical limit.

Gaudin, Dyson, Mehta : for any $E \in (-2, 2)$ then ($\beta = 2$ for example)

$$\rho_k^{(N)}\left(E + \frac{u_1}{N\varrho(x)}, \dots, E + \frac{u_k}{N\varrho(x)}\right) \underset{N\to\infty}{\longrightarrow} \det_{k \times k} \frac{\sin(\pi(u_i - u_j))}{\pi(u_i - u_j)}.$$

Wigner matrix : symmetric, Hermitian (or symplectic), entries have variance $1/N$, some large moment is finite.

The Wigner-Dyson-Mehta conjecture. Correlation functions of symmetric Wigner matrices (resp. Hermitian, symplectic) converge to the limiting GOE (resp. GUE, GSE).

Recently universality was proved under various forms.
Fixed (averaged) energy universality. For any $k \geq 1$, smooth $F : \mathbb{R}^k \to \mathbb{R}$, for arbitrarily small ε and $s = N^{-1+\varepsilon}$,

$$\lim_{N\to\infty} \frac{1}{\varrho(E)^k} \int_E^{E+s} \frac{\mathrm{d}x}{s} \int \mathrm{d}\mathbf{v} F(\mathbf{v}) \rho_k^{(N)} \left(x + \frac{\mathbf{v}}{N\varrho(E)} \right) \mathrm{d}\mathbf{v} = \int \mathrm{d}\mathbf{v} F(\mathbf{v}) \rho_k^{(\mathrm{GOE})} (\mathbf{v})$$

Johansson (2001)	Hermitian class, fixed E, Gaussian divisible entries
Erdős Schlein Péché Ramirez Yau (2009)	Hermitian class, fixed E Entries with density
Tao Vu (2009)	Hermitian class, fixed E Entries with 3rd moment=0
Erdős Schlein Yau (2010)	Any class, averaged E

Key input for all recent results : **rigidity of eigenvalues** (Erdős Schlein Yau) : $|\lambda_k - \gamma_k| \leq N^{-1+\varepsilon}$ in the bulk. Optimal rigidity ?

Jimbo, Miwa, Mori, Sato & condition number of ± 1 matrices ?

Related developments : gaps universality by Erdős Yau (2012).

The gaps are much more stable statistics than the fixed energy ones :

$$\langle \lambda_i, \lambda_j \rangle \sim N^{-2} \log \frac{N}{1+|i-j|}, \text{ almost crystal. } \langle \lambda_{i+1} - \lambda_i \lambda_{j+1} - \lambda_j \rangle \sim \frac{N^{-2}}{1+|i-j|^2}$$

Theorem (B., Erdős, Yau, Yin, 2014).

(i) Fixed energy universality : for Wigner matrices from all symmetry classes.

(ii) Optimal fluctuations : Log-correlated Gaussian field.

The Dyson Brownian Motion (DBM, $\mathrm{d}H_t = \frac{\mathrm{d}B_t}{\sqrt{N}} - \frac{1}{2}H_t\mathrm{d}t$) is an essential interpolation tool, as in the Erdős Schlein Yau approach to universality, summarized as :

$$\begin{array}{ccc} & & H_0 \\ & & \updownarrow \\ \widetilde{H}_0 & \xrightarrow{(\mathrm{DBM})} & \widetilde{H}_t \end{array}$$

$\xrightarrow{(\mathrm{DBM})}$: for $t = N^{-1+\varepsilon}$, the eigenvaues of $\widetilde{H}_t$ satisfy averaged universality.

$\updownarrow$: Density argument. For any $t \ll 1$, there exists $\widetilde{H}_0$ s.t. the resolvents of H_0 and $\widetilde{H}_t$ have the same statistics on the microscopic scale.

What makes the Hermitian universality easier ? The $\xrightarrow{(\mathrm{DBM})}$ step is replaced by HCIZ formula : correlation functions of $\widetilde{H}_t$ are explicit only for $\beta = 2$.

A few facts about the general proof of fixed energy universality.

(i) A game coupling 3 Dyson Brownian Motions.

(ii) Homogenization allows to obtain microscopic statistics from mesoscopic ones.

(iii) Need of a second order type of Hilbert transform. Emergence of new explicit kernels for any Bernstein-Szegő measure. These include Wigner, Marchenko-Pastur, Kesten-McKay.

(iv) The relaxing time of DBM depends on the Fourier support of the test function : the step $\stackrel{\text{(DBM)}}{\longrightarrow}$ becomes the following.

$$\widetilde{F}(\boldsymbol{\lambda}, \Delta) = \sum_{i_1,\dots,i_k=1}^{N} F\left(\{N(\lambda_{i_j} - E) + \Delta, 1 \leq j \leq k\}\right)$$

Theorem. If $\operatorname{supp}\hat{F} \subset \mathrm{B}(0, 1/\sqrt{\tau})$, then for $t = N^{-\tau}$,

$$\mathbb{E}\,\widetilde{F}(\boldsymbol{\lambda}_t, 0) = \mathbb{E}\,\widetilde{F}(\boldsymbol{\lambda}^{(\mathrm{GOE})}, 0).$$

First step : coupling 3 DBM. Let $\mathbf{x}(0)$ be the eigenvalues of $\widetilde{H}_0$ and $\mathbf{y}(0), \mathbf{z}(0)$ those of two indepndent GOE.

$$\mathrm{d}x_i/\mathrm{d}y_i/\mathrm{d}z_i = \sqrt{\frac{2}{N}}\mathrm{d}B_i(t) + \frac{1}{N}\left(\sum_{j\neq i}\frac{1}{x_i/y_i/z_i - x_j/y_j/z_j} - \frac{1}{2}x_i/y_i/z_i\right)\mathrm{d}t$$

Let $\delta_\ell(t) = e^{t/2}(x_\ell(t) - y_\ell(t))$. Then we get the parabolic equation

$$\partial_t \delta_\ell(t) = \sum_{k\neq\ell}\mathcal{B}_{k\ell}(t)\left(\delta_k(t) - \delta_\ell(t)\right),$$

where $\mathcal{B}_{k\ell}(t) = \frac{1}{N(x_k(t)-x_\ell(t))(y_k(t)-y_\ell(t))} > 0$. By the de Giorgi-Nash-Moser method (+Caffarelli-Chan-Vasseur+Erdős-Yau), this PDE is Hölder-continuous for $t > N^{-1+\varepsilon}$, i.e. $\delta_\ell(t) = \delta_{\ell+1}(t) + \mathrm{O}(N^{-1-\varepsilon})$, i.e. gap universality.

This is not enough for fixed energy universality.

Second step : homogenization. The continuum-space analogue of our parabolic equation is

$$\partial_t f_t(x) = (\mathcal{K} f_t)(x) := \int_{-2}^{2} \frac{f_t(y) - f_t(x)}{(x-y)^2} \varrho(y) \mathrm{d}y.$$

$\mathcal{K}$ is some type of second order Hilbert transform.

Theorem. Let f_0 be a smooth continuous-space extension of $\delta(0)$: $f_0(\gamma_\ell) = \delta_\ell(0)$. Then for any small $\tau > 0$ ($t = N^{-\tau}$) thre exists $\varepsilon > 0$ such that

$$\delta_\ell(t) = \left(e^{t\mathcal{K}} f_0\right)_\ell + \mathrm{O}(N^{-1-\varepsilon}).$$

Proof. Key inputs are the rigidity of the eigenvalues and optimal Wegner estimates (for level-repulsion).

Third step : the continuous-space kernel.

1. For the translation invariant equation

$$\partial_t g_t(x) = \int_{\mathbb{R}} \frac{g_t(y) - g_t(x)}{(x-y)^2} \mathrm{d}y,$$

   the fundamental solution is the Poisson kernel $p_t(x,y) = \frac{c_t}{t+(x-y)^2}$.

2. For us, t will be close to 1, so the edge curvture cannot be neglected. Fortunately, $\mathcal{K}$ can be fully diagonalized and ($x = 2\cos\theta$, $y = 2\cos\phi$)

$$k_t(x,y) = \frac{c_t}{|e^{\mathrm{i}(\theta+\phi)} - e^{-t/2}|^2 |e^{\mathrm{i}(\theta-\phi)} - e^{-t/2}|^2}.$$

   Called the Mehler kernel by Biane in free probability context. Here it appears as a second-order Hilbert transform fundamental solution.

3. Explicit kernels can be obtained for all Bernstein-Szegő measures,

$$\varrho(x) = \frac{c_{\alpha,\beta}(1-x^2)^{1/2}}{(\alpha^2 + (1-\beta^2)) + 2\alpha(1+\beta)x + 4\beta x^2}.$$

Fourth step : microscopic from mesoscopic. Homogenization yields

$$\delta_\ell(t) = \int k_t(x,y) f_0(y) \varrho(y) \mathrm{d}y + \mathrm{O}(N^{-1-\varepsilon})$$

The LHS is microscopic-type of statistics, the RHS is mesoscopic. This yields, up to negligible error,

$$N x_\ell(t) = N y_\ell(t) - \Psi_t(\mathbf{y}_0) + \Psi_t(\mathbf{x}_0),$$

where $\Psi_t(\mathbf{x}_0) = \sum h(N^\tau(x_i(0) - E))$ for some smooth h. We wanted to prove

$$\mathbb{E}\,\widetilde{F}(\mathbf{x}_t, 0) = \mathbb{E}\,\widetilde{F}(\mathbf{z}_t, 0) + \mathrm{o}(1).$$

We reduced it to

$$\mathbb{E}\,\widetilde{F}(\mathbf{y}_t, -\Psi_t(\mathbf{y}_0) + \Psi_t(\mathbf{x}_0)) = \mathbb{E}\,\widetilde{F}(\mathbf{y}_t, \Psi_t(\mathbf{y}_0) + \Psi_t(\mathbf{z}_0)) + \mathrm{o}(1).$$

where $\Psi_t(\mathbf{y}_0)$, $\Psi_t(\mathbf{x}_0)$ and $\Psi_t(\mathbf{z}_0)$ are mesoscopic observables and independent.

Fifth step and conclusion : CLT for GOE beyond the natural scale. Do $\Psi_t(\mathbf{x}_0)$ and $\Psi_t(\mathbf{y}_0)$ have the same distribution ? No, their variance depend on their fourth moment.

A stronger result holds : $\mathbb{E}\,\widetilde{F}(\mathbf{y}_t, -\Psi_t(\mathbf{y}_0)+c)$ does not depend on the constant c.

We know that $\mathbb{E}\,\widetilde{F}(\mathbf{y}_t, -\Psi_t(\mathbf{y}_0)+\Psi_t(\mathbf{z}_0)+c) = \mathbb{E}\,\widetilde{F}(\mathbf{y}_t, -\Psi_t(\mathbf{y}_0)+\Psi_t(\mathbf{z}_0))$ for all c (why ?).

Exercise : let X be a random variable. If $\mathbb{E}\,g(X+c)=0$ for all c, is it true that $g\equiv 0$?

Not always. But true if X is Gaussian (by Fourier).

Lemma. $\mathbb{E}\left(e^{\mathrm{i}\lambda\Psi_t(\mathbf{z}(0))}\right) = e^{-\frac{\lambda^2}{2}\tau\log N} + \mathrm{O}(N^{-1/100})$.

The proof uses algebraic ideas of Johansson and rigidity of β-ensembles.

By Parseval, proof when the support of $\hat{F}$ has size $1/\sqrt{\tau}$. This is why DBM needs to be run till time almost 1.

What is the optimal rigidity of eigenvalues ?

Theorem (Gustavsson, O'Rourke). Let $\boldsymbol{\lambda}$ be the ordered eigenvalues of a Gaussian ensemble, k_0 a bulk index and $k_{i+1} \sim k_i + N^{\theta_i}$, $0 < \theta_i < 1$. Then the nornalized eigenvalues fluctuations

$$X_i = \frac{\lambda_{k_i} - \gamma_{k_i}}{\frac{\sqrt{\log N}}{N}} \sqrt{\beta(4 - \gamma_{k_i}^2)}$$

converge to a Gaussian vector with vovariance

$$\Lambda_{ij} = 1 - \max\{\theta_k, i \leq k < j\}.$$

In particlar, $\lambda_i - \gamma_i$ has fluctuations $\frac{\sqrt{\log N}}{N}$.

Proof : determinantal point processes a la Costin-Lebowitz (GUE) + decimation relations (GOE, GSE).

One application of the homogenization/coupling : the same log-correlated Gaussian limit for any Wigner matrix.

Sketch. By homogenization we have

$$\frac{N(x_\ell(t)-\gamma_\ell)}{\sqrt{\log N}} = \frac{N(y_\ell(t)-\gamma_\ell)}{\sqrt{\log N}} + \frac{\Psi_t(\mathbf{y}(0))}{\sqrt{\log N}} - \frac{\Psi_t(\mathbf{x}(0))}{\sqrt{\log N}}.$$

The fluctuations of $\Psi_t(\mathbf{y}(0))$ are of order $\sqrt{\tau \log N}$. The fluctuations of $\Psi_t(\mathbf{x}(0))$ are of the same order $\sqrt{\tau \log N}$. Take arbitrarily small τ and the result follows.

Matrix models : eigenstates statistics

Delocalization. For any u_k, eigenvector of a generalized Wigner matrix,

$$\sup_{\alpha} |u_k(\alpha)| \leq (\log N)^C N^{-\frac{1}{2}}$$

for large enough N (Erdős, Schlein, Yau, Yin). Relies on the analysis of $G(z) = (H - z)^{-1}$. Delocalization for non-Hermitian random matrices by Rudelson-Vershynin, with another technique.

Microscopic scale : normality.

(i) The entries $(\sqrt{N} u_k(\alpha))_\alpha$ converge to i.i.d. Gaussian provided that the first 4 moments of H_{ij}'s match the Gaussian ones (Knowles-Yin, Tao-Vu, 2011).

(ii) For any $\mathbf{q} \in \mathbb{R}^N$, $\sqrt{N}\langle \mathbf{q}, u_k \rangle$ converges to a Gaussian if the first 5 moments match the Gaussian ones (Tao-Vu, 2011).

The proof relies on resolvent expansion, moment matching, comparison with GOE/GUE.

Theorem (B., Yau (2013)).

(1) (i) and (ii) hold for generalized Wigner matrices.

(2) Probabilistic version of QUE, at any scale. For any (N-dependent) $I \subset [\![1, N]\!]$, k and (fixed) δ,

$$\mathbb{P}\left(\frac{N}{|I|}\left|\sum_{\alpha\in I}|u_k(\alpha)|^2 - \frac{1}{N}\right| > \delta\right) \leq C\left(N^{-\varepsilon} + |I|^{-1}\right).$$

Remark. Rudnick&Sarnak's QUE conjecture : for any negatively curved compact Riemannian manifold $\mathcal{M}$, the eigenstates become equidistributed :

$$\int_A |\psi_k(x)|^2\mu(\mathrm{d}x) \underset{k\to\infty}{\longrightarrow} \int_A \mu(\mathrm{d}x).$$

The Dyson vector flow

Coupled eigenvalues/eigenvectors dynamics when the entrie of H are Brownian motions :

$$\mathrm{d}\lambda_k = \frac{\mathrm{d}B_{kk}}{\sqrt{N}} + \left(\frac{1}{N}\sum_{\ell\neq k}\frac{1}{\lambda_k-\lambda_\ell}\right)\mathrm{d}t$$

$$\mathrm{d}u_k = \frac{1}{\sqrt{N}}\sum_{\ell\neq k}\frac{\mathrm{d}B_{k\ell}}{\lambda_k-\lambda_\ell}u_\ell - \frac{1}{2N}\sum_{\ell\neq k}\frac{\mathrm{d}t}{(\lambda_k-\lambda_\ell)^2}u_k$$

Let $c_{k\ell} = \frac{1}{N}\frac{1}{(\lambda_k-\lambda_\ell)^2}$. If all $c_{k\ell}$'s were equal, $U = (u_1,\dots,u_N)$ would be the Brownian motion on the unitary group.

Such eigenvector flows were discovered by Norris, Rogers, Williams (Brownian motion on GL_N), Bru (real Wishart), Anderson, Guionnet, Zeitouni (symmetric and Hermitian).

Relaxation of the Dyson vector flow : first try

Conditionally on the trajectory $(\lambda_i(t), 1 \leq i \leq N)_{t\geq 0}$, the Dyson vector flow has generator

$$L = \sum_{k<\ell} c_{k\ell}(t)(u_k \cdot \partial_{u_\ell} - u_\ell \cdot \partial_{u_k})^2 \text{``} = \Delta\text{''}$$

where Δ is the Laplace-Beltrami for the metric g defined by

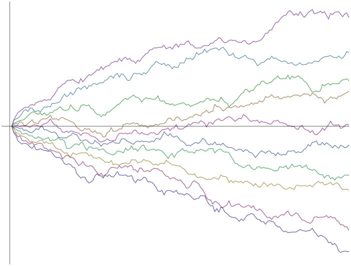
$$\langle E_\alpha, E_\beta\rangle^{(g)} = \frac{2}{c_{ij}} \mathbb{1}_{\alpha=\beta},\ \alpha = (i,j).$$

If $\mathrm{Ricci}^{(g)} \geq c > 0$, the relaxation time is at most $1/c$ (Bakry, Émery). Here,

$$\frac{\mathrm{Ricci}^{(g)}_{\mathrm{Id}}(E_\alpha, E_\alpha)}{\langle E_\alpha, E_\alpha\rangle^g} = \frac{1}{N} \sum_{k \notin \{i,j\}} \frac{1}{(\lambda_i - \lambda_k)(\lambda_k - \lambda_j)}.$$

Not even positive, and time-dependent metric. No general relaxation theory taking initial conditions into account.

A random walk in a dynamic random environment

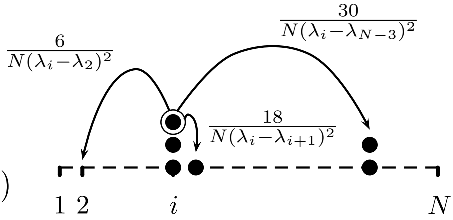


Definition of the (real) eigenvector moment flow.

The eigenvalues trajectory is a parameter ($c_{i,j}(t) = \frac{1}{N}\frac{1}{(\lambda_i(t)-\lambda_j(t))^2}$).

Configuration $\boldsymbol{\eta}$ of n points on $[\![1, N]\!]$. Number of particles at x : η_x. Configuration obtained by moving a particle from i to j : $\boldsymbol{\eta}^{ij}$.
Dynamics given by $\partial_t f = \mathscr{B}(t) f$ where

$$\mathscr{B}(t)f(\boldsymbol{\eta}) = \sum_{i\neq j} c_{ij}(t) 2\eta_i(1+2\eta_j)\left(f(\boldsymbol{\eta}^{i,j}) - f(\boldsymbol{\eta})\right)$$



Properties of the eigenvector moment flow

Let $z_k = \sqrt{N}\langle \mathbf{q}, u_k \rangle$, random and time dependent. For a configuration $\boldsymbol{\eta}$ with j_k points at i_k, let

$$f_{t,\boldsymbol{\lambda}}(\boldsymbol{\eta}) = \mathbb{E}\left(\prod_k z_{i_k}^{2j_k} \mid \boldsymbol{\lambda}\right) / \, \mathbb{E}\left(\prod_k \mathscr{N}_{i_k}^{2j_k}\right).$$

Fact 1 : $\partial_t f_{t,\boldsymbol{\lambda}}(\boldsymbol{\eta}) = \mathscr{B}(t) f_{t,\boldsymbol{\lambda}}(\boldsymbol{\eta})$.

QUE+Normality of the eigenvectors hold, it is equivalent to fast relaxation to equilibrium of the eigenvector moment flow.
This PDE analysis is made possible thanks to an explicit reversible measure for $\mathscr{B}$

Fact 2 :

- GOE : $\pi(\boldsymbol{\eta}) = \prod_{x=1}^N \phi(\eta_x)$ where $\phi(k) = \prod_{i=1}^k \left(1 - \frac{1}{2k}\right)$
- GUE : π is uniform

Relaxation to equilibrium

Goal : for $t \gg N^{-1}$, $\sup_{\boldsymbol{\eta}\subset \text{bulk}} |f_{\boldsymbol{\lambda},t}(\boldsymbol{\eta})-1| \leq N^{-\varepsilon}$.

Key tool : a maximum principle. If $S_t=\sup_{\boldsymbol{\eta}}(f_t(\boldsymbol{\eta})-1)$ is always obtained for a configuration supported in the bulk, then

$$S_t' \leq -N^{1-\varepsilon}S_t+N^{1-\varepsilon}.$$

The bulk condition does not hold. Development of a *local* maximum principle.

Proof of the maximum inequality.
For $n=1$, if $S_t=\sup_k(f_t(k)-1)=f_t(k_0)-1$, then for any $\eta>0$

$$\begin{aligned}
S_t' &= \frac{1}{N}\sum_{k\neq k_0}\frac{f_t(k)-f_t(k_0)}{(\lambda_k-\lambda_{k_0})^2}\\
&\leq \sum_{k\neq k_0}\frac{\mathbb{E}\left(u_k(t)^2\mid \boldsymbol{\lambda}\right)-f_t(k_0)}{(\lambda_k-\lambda_{k_0})^2+\eta^2}\\
&\leq \frac{1}{\eta}\mathbb{E}(\mathfrak{Im}\langle q,G(\lambda_{k_0}+\mathrm{i}\eta)q\rangle\mid\boldsymbol{\lambda})-f_t(k_0)\frac{1}{N\eta}\mathfrak{Im}\mathrm{Tr}G(\lambda_{k_0}+\mathrm{i}\eta)
\end{aligned}$$

One concludes by the local semicirle law for $\eta=N^{-1+\varepsilon}$.