Spectrum of large deformed classical Hermitian matrices and free probability theory

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December 5, 2014

Aim of this talk:

To show how free probability theory sheds light on spectral properties of deformed matricial models and provides A UNIFIED UNDERSTANDING of various phenomena

Notations

$B = B^* \in \mathcal{M}_N(\mathbb{C})$
Eigenvalues of B: $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_N(B)$,

The empirical distribution of these eigenvalues:

$$\mu_B := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(B)}$$

μ probability measure on $\mathbb{C}$, $z \in \mathbb{C} \setminus \mathrm{supp}(\mu)$, $\mathrm{g}_\mu(\mathrm{z}) = \int \frac{\mathrm{d}\mu(\mathrm{x})}{\mathrm{z} - \mathrm{x}}$

$\mathcal{M}$: the set of probability measures supported on the real line
$\mathcal{M}^+$: the set of probability measures supported on $[0;+\infty[$.
Free probability theory defines:

- a binary operation on $\mathcal{M}$:the free additive convolution $\mu \boxplus \nu$ for μ and ν in $\mathcal{M}$,
- binary operations on $\mathcal{M}^+$: the free multiplicative convolution $\mu \boxtimes \nu$ and the free rectangular convolution with ratio $c \in]0;1]$ $\mu \boxplus_c \nu$, for μ and ν in $\mathcal{M}^+$,

(cf Voiculescu, Maassen, Bercovici-Voiculescu, and Benaych-Georges)

For several matricial models where A_N and B_N are independent $N \times N$ Hermitian random matrices
(for instance when $\mathcal{L}(B_N) = \mathcal{L}(U_N B_N U_N^*)$ for any deterministic unitary matrix ("unitarily invariant")),
free probability provides a good understanding of the asymptotic global behaviour of the spectrum of $A_N + B_N$ and $A_N^{\frac{1}{2}} B_N A_N^{\frac{1}{2}}$ $(A_N \geq 0, B_N \geq 0)$

$$\mu_{A_N+B_N} \rightarrow_{N\rightarrow+\infty} \mu_a \boxplus \mu_b$$

$$\mu_{A_N^{\frac{1}{2}} B_N A_N^{\frac{1}{2}}} \rightarrow_{N\rightarrow+\infty} \mu_a \boxtimes \mu_b$$

where $\mu_{A_N} \rightarrow_{N\rightarrow+\infty} \mu_a$ and $\mu_{B_N} \rightarrow_{N\rightarrow+\infty} \mu_b$.

Pionnering work 90' of D. Voiculescu extended by several authors

For several matricial models where A_N and B_N are independent rectangular $n \times N$ random matrices such that $n/N \to c \in]0;1]$, rectangular free convolution provides a good understanding of the asymptotic global behaviour of the singular values of $A_N + B_N$:

$$\frac{1}{n} \sum_{s \text{ sing. val. of } A_N+B_N} \delta_s \to \nu_a \boxplus_c \nu_b.$$

(where $\displaystyle\frac{1}{n} \sum_{s \text{ sing. val. of } A_N} \delta_s \to \nu_a, \quad \frac{1}{n} \sum_{s \text{ sing. val. of } B_N} \delta_s \to \nu_b$)

(cf work of Benaych-Georges when B_N is invariant, in law, under multiplication, on the right and on the left, by any unitary matrix: "biunitarily invariant")

Additive free subordination property

For a probability measure τ on $\mathbb{R}$, $z \in \mathbb{C}\backslash\mathbb{R}$, $g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z-x}$.

Theorem (D.Voiculescu (93), P. Biane (98))

Let μ and ν be two probability measures on $\mathbb{R}$, there exists a unique analytic map $\omega_{\mu,\nu} : \mathbb{C}^+ \to \mathbb{C}^+$ such that

$$\forall z \in \mathbb{C}^+, g_{\mu\boxplus\nu}(z) = g_\nu(\omega_{\mu,\nu}(z)),$$

$\forall z \in \mathbb{C}^+, \Im\omega_{\mu,\nu}(z) \geq \Im z$ *and* $\lim_{y\uparrow+\infty} \frac{\omega_{\mu,\nu}(iy)}{iy} = 1.$
$\omega_{\mu,\nu}$ *is called the additive subordination map of $\mu \boxplus \nu$ with respect to ν.*

Multiplicative free subordination property

$$\Psi_\tau(z) = \int \frac{tz}{1-tz} d\tau(t) = \frac{1}{z} g_\tau\left(\frac{1}{z}\right) - 1,$$

for complex values of z such that $\frac{1}{z}$ is not in the support of τ.

Theorem (Biane (98))

Let $\tau \neq \delta_0$ and $\nu \neq \delta_0$ be two probability measures on $[0; +\infty[$. There exists a unique analytic map $F_{\tau,\nu}$ defined on $\mathbb{C} \setminus [0; +\infty[$ such that

$$\forall z \in \mathbb{C} \setminus [0; +\infty[, \ \Psi_{\nu \boxtimes \tau}(z) = \Psi_\nu(F_{\tau,\nu}(z))$$

and

$$\forall z \in \mathbb{C}^+, \ F_{\tau,\nu}(z) \in \mathbb{C}^+, \ F_{\tau,\nu}(\bar{z}) = \overline{F_{\tau,\nu}(z)}, \ \arg(F_{\tau,\nu}(z)) \geq \arg(z).$$

$F_{\tau,\nu}$ is called the multiplicative subordination map of $\tau \boxtimes \nu$ with respect to ν.

Rectangular free subordination property

τ probability measure on $\mathbb{R}^+$; $c \in]0;1]$.

$$M_\tau(z) = \int_{\mathbb{R}^+} \frac{t^2 z}{1 - t^2 z} d\tau(t), \quad H_\tau^{(c)}(z) := z\,(cM_\tau(z) + 1)\,(M_\tau(z) + 1).$$

Theorem (Belinschi&Benaych-Georges&Guionnet (2008))

Assume that τ is $\boxplus_c$ infinitely divisible. Then there exist two unique meromorphic functions ω_1, ω_2 on $\mathbb{C} \setminus \mathbb{R}^+$ so that

$$H_\tau^{(c)}(\omega_1(z)) = H_\nu^{(c)}(\omega_2(z)) = H_{\tau \boxplus_c \nu}^{(c)}(z),$$

$\omega_j(\overline{z}) = \overline{\omega_j(z)}$ *and* $\lim_{x \uparrow 0} \omega_j(x) = 0$, $j \in \{1;2\}$.

Standard models

- **Wigner matrices**

$$X_N = \frac{1}{\sqrt{N}} W_N$$

$(W_N)_{ii}$, $\sqrt{2}\Re e((W_N)_{ij})_{i<j}$, $\sqrt{2}\Im m((W_N)_{ij})_{i<j}$ are i.i.d, with distribution μ with variance σ^2 and mean zero.

If $\mu = \mathcal{N}(0, \sigma^2)$, $W_N =: W_N^G$ is a $G.U.E$-matrix.

- **Wishart matrices**

$$X_N = \frac{1}{p} B_N B_N^*$$

B_N is a $N \times p(N)$ matrix, $(B_N)_{u,v} = Z_{u,v} + iY_{u,v}$ $Z_{u,v}$, $Y_{u,v}$, $u = 1, \dots, N$, $v = 1, \dots, p(N)$ are i.i.d, with distribution μ with variance $\frac{1}{2}$ and mean zero.

If $\mu = \mathcal{N}(0, \frac{1}{2})$, X_N is a $L.U.E$ matrix.

Convergence of the spectral measure

Theorem (Wigner (50'))

$$\mu_{\frac{W_N}{\sqrt{N}}} := \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(\frac{W_N}{\sqrt{N}})} \to \mu_{sc} \quad a.s \text{ when } \quad N \to +\infty$$

$$\frac{d\mu_{sc}}{dx}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\, 1_{[-2\sigma,2\sigma]}(x)$$

Theorem (Marchenko-Pastur (1967))

If $c_N := \frac{N}{p} \to c > 0$ *when* $N \to \infty$,

$$\mu_{\frac{B_N B_N^*}{p}} \to \mu_{MP} \quad a.s \text{ when } \quad N \to +\infty$$

$$\frac{d\mu_{MP}}{dx}(x) = \frac{1}{2\pi c x}\sqrt{(b-x)(x-a)}\, 1_{[a,b]}(x)$$

$a = (1-\sqrt{c})^2$, $b = (1+\sqrt{c})^2$. *and* $\mu_c(0) = 1 - \frac{1}{c}$ *if* $c > 1$.

No outlier

Theorem (Bai-Yin 1988)

If $\int x^4 d\mu(x) < +\infty$, *then*

$$\lambda_1\left(\frac{W_N}{\sqrt{N}}\right) \to 2\sigma \text{ and } \lambda_N\left(\frac{W_N}{\sqrt{N}}\right) \to -2\sigma \text{ a.s when } N \to +\infty.$$

Theorem (Geman 1980, Bai-Yin-Krishnaiah 1988, Bai-Silverstein-Yin 1988)

If $\int x^4 d\mu(x) < +\infty$,

$$\lambda_1\left(\frac{B_N B_N^*}{p}\right) \to (1+\sqrt{c})^2 \text{ a.s when } N \to +\infty.$$

$$\lambda_{\min(N,p)}\left(\frac{B_N B_N^*}{p}\right) \to (1-\sqrt{c})^2 \text{ a.s when } N \to +\infty.$$

Deformed models

A_N is a deterministic matrix such that $\sup_N \|A_N\| < \infty$.

- Deformed Wigner matrices W_N is a Wigner matrix and A_N is an Hermitian matrix such that $\mu_{A_N} \to_{N\to+\infty} \nu$ weakly.

$$M_N = \frac{W_N}{\sqrt{N}} + A_N$$

- Sample covariance matrices A_N is a non negative definite matrix such that $\mu_{A_N} \to_{N\to+\infty} \nu$ weakly.

$$M_N = {A_N}^{\frac{1}{2}} \frac{B_N B_N^*}{p} {A_N}^{\frac{1}{2}}.$$

- Information-Plus-Noise type matrices, $N \leq p(N)$, A_N is such that $\mu_{A_N A_N^*} \to_{N\to+\infty} \nu$ weakly.

$$M_N = \left(\frac{B_N}{\sqrt{p}} + A_N\right)\left(\frac{B_N}{\sqrt{p}} + A_N\right)^*.$$

Convergence of spectral measures

- Deformed Wigner matrices $\mu_{M_N} \rightarrow_{N\rightarrow+\infty} \mu_{dW}$ weakly.
  Pastur (72), Anderson&Guionnet&Zeitouni (2010)
- Sample covariance matrices $\mu_{M_N} \rightarrow_{N\rightarrow+\infty} \mu_{Scm}$ weakly.
  Marchenko&Pastur (67) Grenander&Silverstein(77), Wachter (78), Krishnaiah&Y.Q.Yin (83), Y.Q.Yin (86), Bai&Silverstein (95), Silverstein (95).
- Information-Plus-Noise type matrices $\mu_{M_N} \rightarrow_{N\rightarrow+\infty} \mu_{Ipn}$ weakly.
  Dozier&Silverstein (2007), Hachem&Loubaton&Najim (2007), Xie (2012)

Convergence of spectral measures

- Deformed Wigner matrices $\mu_{M_N} \to_{N\to+\infty} \mu_{dW}$ weakly.

$$\forall z \in \mathbb{C}^+, \quad g_{\mu_{dW}}(z) = \int \frac{1}{z - \sigma^2 g_{\mu_{dW}}(z) - t} d\nu(t).$$

- Sample covariance matrices $\mu_{M_N} \to_{N\to+\infty} \mu_{Scm}$ weakly.

$$\forall z \in \mathbb{C}^+, \quad g_{\mu_{Scm}}(z) = \int \frac{1}{z - t(1 - c + czg_{\mu_{Scm}}(z))} d\nu(t).$$

- Information-Plus-Noise type matrices $\mu_{M_N} \to_{N\to+\infty} \mu_{Ipn}$ weakly.

$$\forall z \in \mathbb{C}^+, \quad g_{\mu_{Ipn}}(z) = \int \frac{1}{(1 - cg_{\mu_{Ipn}}(z))z - \frac{t}{1-cg_{\mu_{Ipn}}(z)} - (1-c)} d\nu(t).$$

$\mu_{dW}, \mu_{Scm}, \mu_{Ipn}$ are **deterministic**, in general non explicit. They are **universal** (do not depend on the distribution of the entries of W_N or B_N) and only depend on A_N through the limiting spectral measure ν.

Free probabilistic interpretation

- Deformed Wigner matrices

$$\mu_{M_N} \rightarrow_{N\to+\infty} \mu_{dW} \text{ weakly}, \quad \mu_{dW} = \mu_{sc} \boxplus \nu$$

- Sample covariance matrices

$$\mu_{M_N} \rightarrow_{N\to+\infty} \mu_{Scm} \text{ weakly}, \quad \mu_{Scm} = \mu_{MP} \boxtimes \nu$$

- Information-Plus-Noise type matrices

$$\mu_{M_N} \rightarrow_{N\to+\infty} \mu_{Ipn} \text{ weakly}, \quad \mu_{Ipn} = (\sqrt{\mu_{MP}} \boxplus_c \sqrt{\nu})^2.$$

Equations satisfied by the limiting Stieltjes transforms $\Longleftrightarrow$ Free Subordination properties

- Deformed Wigner matrices

$$\forall z \in \mathbb{C}^+, \quad g_{\mu_{sc}\boxplus\nu}(z) = \int \frac{1}{z - \sigma^2 g_{\mu_{sc}\boxplus\nu}(z) - t} d\nu(t) = g_\nu(\omega_{\mu_{sc},\nu}(z)).$$

$$\omega_{\mu_{sc},\nu}(z) = z - \sigma^2 g_{\mu_{sc}\boxplus\nu}(z).$$

- Sample covariance matrices

$$\forall z \in \mathbb{C}^+, \quad g_{\mu_{MP}\boxtimes\nu}(z) = \int \frac{1}{z - t(1 - c + czg_{\mu_{MP}\boxtimes\nu}(z))} d\nu(t).$$

$$\rightarrow \quad \Psi_{\mu_{MP}\boxtimes\nu}\left(\frac{1}{z}\right) = \Psi_\nu\left(F_{\mu_{MP},\nu}\left(\frac{1}{z}\right)\right)$$

$$\Psi_\tau(z) = \int \frac{tz}{1 - tz} d\tau(t) = \frac{1}{z} g_\tau\left(\frac{1}{z}\right) - 1,$$

$$F_{\mu_{MP},\nu}(z) = z - cz + cg_{\mu_{MP}\boxtimes\nu}\left(\frac{1}{z}\right).$$

Equations satisfied by the limiting Stieltjes transforms $\Longleftrightarrow$ Free Subordination properties

- Information-Plus-Noise type matrices

$$\mu_{Ipn} = (\sqrt{\mu_{MP}} \boxtimes_c \sqrt{\nu})^2$$

$$\forall z \in \mathbb{C}^+, \quad g_{\mu_{Ipn}}(z) = \int \frac{1}{(1 - c g_{\mu_{Ipn}}(z))z - \frac{t}{1 - c g_{\mu_{Ipn}}(z)} - (1-c)} d\nu(t).$$

$$\rightarrow \quad H^{(c)}_{\sqrt{\mu_{Ipn}}}\left(\frac{1}{z}\right) = H^{(c)}_{\sqrt{\nu}}\left(\Omega_{\mu_{MP},\nu}\left(\frac{1}{z}\right)\right)$$

$$H^{(c)}_{\sqrt{\tau}}(z) = \frac{c}{z} g_\tau(\frac{1}{z})^2 + (1-c) g_\tau(\frac{1}{z}),$$

$$\Omega_{\mu_{MP},\nu}(z) = \frac{1}{\frac{1}{z}(1 - c g_{\mu_{Ipn}}(\frac{1}{z}))^2 - (1-c)(1 - c g_{\mu_{Ipn}}(\frac{1}{z}))}$$

Deep Studies of the limiting spectral measures

Support, density, behaviour of the density near its zeroes....

- Deformed Wigner matrices $\mu_{dW} = \mu_{sc} \boxplus \nu$
  P. Biane (1997)
- Sample covariance matrices $\mu_{Scm} = \mu_{MP} \boxtimes \nu$
  Choi&Silverstein (1995)
- Information-Plus-Noise type matrices $\mu_{Ipn} = (\sqrt{\mu_{MP}} \boxplus_c \sqrt{\nu})^2$
  Dozier&Silverstein (2007)

Characterization of the complement of the supports

(P.Biane 1997):

$$\mathcal{O} := \{u \in \mathbb{R} \setminus \text{support } \nu, \int \frac{1}{(u-x)^2} d\nu(x) < \frac{1}{\sigma^2}\}$$

$$\mathbb{R} \setminus \text{support } \mu_{sc} \boxplus \nu = h_{\mu_{sc},\nu}(\mathcal{O}).$$

$$h_{\mu_{sc},\nu} : z \mapsto z + \sigma^2 g_\nu(z).$$

$$\mathbb{R} \setminus \text{support } \mu_{sc} \boxplus \nu \underset{h_{\mu_{sc},\nu}}{\overset{\omega_{\mu_{sc}\nu}}{\rightleftarrows}} \mathcal{O},$$

The additive subordination map $\omega_{\mu_{sc},\nu}(z) = z - \sigma^2 g_{\mu_{sc} \boxplus \nu}(z)$

$h_{\mu_{sc},\nu}$ globally strictly increasing on $\mathcal{O}$.

- Deformed Wigner

$$\mathbb{R} \setminus \text{support } \mu_{sc} \boxplus \nu \;\underset{\phi_1}{\overset{\omega_{\mu_{sc}\nu}}{\rightleftarrows}}\; \mathcal{O}_1 \subset \mathbb{R} \setminus \text{support } \nu,$$

$$\theta \in \mathcal{O}_1, \quad \phi_1(\theta\ ) = \theta + \sigma^2 g_\nu(\theta).$$

- Sample covariance matrices

$$\mathbb{R} \setminus \{\text{support } \mu_{MP} \boxtimes \nu \cup \{0\}\} \;\underset{\phi_2}{\overset{x \mapsto \frac{1}{F_{\mu_{MP},\nu}(1/x)}}{\rightleftarrows}}\; \mathcal{O}_2 \subset \mathbb{R} \setminus \text{support } \nu,$$

$$\theta \in \mathcal{O}_2 \quad \phi_2(\theta) = \theta + c\theta \int \frac{t}{\theta - t} d\nu(t).$$

- Information-Plus-Noise type model

$$\mathbb{R} \setminus (\sqrt{\mu_{MP}} \boxtimes_c \sqrt{\nu})^2 \;\underset{\phi_3}{\overset{x \mapsto \frac{1}{\Omega_{\mu_{MP},\nu}(1/x)}}{\rightleftarrows}}\; \mathcal{O}_3 \subset \mathbb{R} \setminus \text{support } \nu,$$

$$\theta \in \mathcal{O}_3, \quad \phi_3(\theta) = \theta(1 + cg_\nu(\theta))^2 + (1 - c)(1 + cg_\nu(\theta))$$

Spiked models, localization of outliers

Seminal works on spiked models

Spiked finite rank deformation : $M_N = \frac{1}{\sqrt{N}} GUE(\sigma^2) + A_N$

$$A_N = \mathrm{diag}\,(\underbrace{0,\ldots,0}_{N-r \text{ times}}, \underbrace{\theta_1,\ldots,\theta_1}_{k_1 \text{ times}},\ldots,\underbrace{\theta_J,\ldots,\theta_J}_{k_J \text{ times}})$$

r: fixed, independent of N.
A_N : a deterministic Hermitian matrix of fixed finite rank r with r non-null eigenvalues (spikes) $\theta_1 > \cdots > \theta_r$ independent of N, the k_i independent of N.

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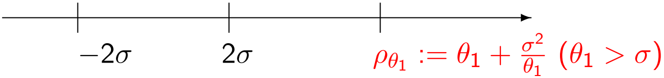
A_N : a deterministic Hermitian matrix of fixed finite rank r with r non-null eigenvalues (spikes) $\theta_1 > \cdots > \theta_r$ independent of N, the k_i independent of N.

$\Longrightarrow$ Convergence of the spectral measure $\mu_{M_N} := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i(M_N)}$ towards the semi-circular distribution μ_{sc}.

Theorem (Péché 2006)

- *If* $\theta_1 \leq \sigma$, $\lambda_1(M_N) \to 2\sigma$
- *If* $\theta_1 > \sigma$, $\lambda_1(M_N) \to \rho_{\theta_1}$ *with* $\rho_{\theta_1} := \theta_1 + \frac{\sigma^2}{\theta_1}$.



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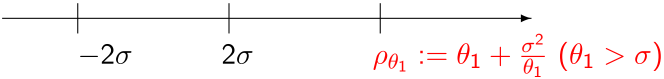

Actually if for some i, $|\theta_i| > \sigma$ then exactly k_i eigenvalues of M_N converge towards $\rho_{\theta_i} := \theta_i + \frac{\sigma^2}{\theta_i} \in]-\infty; -2\sigma[\cup]2\sigma; +\infty[$.

When A_N has finite rank, analog B.B.P phase transition phenomena proved for

$A_N + B_N$	$(I_N + A_N)^{1/2} B_N (I_N + A_N)^{1/2}$ $I_N + A_N > 0$	$(A_N + B_N)(A_N + B_N)^*$
$B_N = GUE$ Péché (2006) $B_N =$ Wigner Féral&Péché (2007) C.&Donati-Martin&Féral (2009) Pizzo&Renfrew&Soshnikov (2013), Knowles&Yin (2014)	$B_N =$ L.U.E Baik&Ben Arous&Péché (2005) $B_N =$ Wishart Baik&Silverstein (2006)	B_N Ginibre matrix Loubaton&Vallet (2011)

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B_N unitarily invariant $\mu_{B_N} \to_{N\to+\infty} \mu$ without outlier Benaych-Georges&Rao(2010)	B_N unitarily invariant $\mu_{B_N} \to_{N\to+\infty} \mu$ $B_N \geq 0$ without outlier Benaych-Georges&Rao(2010)	B_N biunitarily invaria $\mu_{B_N B_N^*} \to_{N\to+\infty} \mu$ without outlier Benaych-Georges&Ra (2010)

Free subordination properties shed light on these phenomena and provide a UNIFIED UNDERSTANDING, allowing to extend them to non-finite rank deformations.

A_N Hermitian deterministic.$\mu_{A_N} \rightarrow_{N \to +\infty} \nu$ compactly supported.
The eigenvalues of A_N:

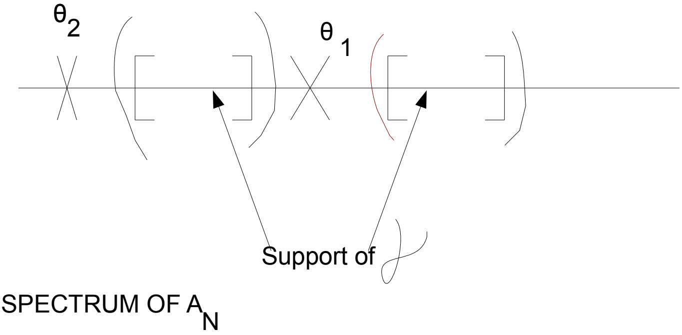
- $N - r$ (r fixed) eigenvalues $\beta_i(N)$ such that

$$\max_{i=1}^{N-r} \mathrm{dist}(\beta_i(N), \mathrm{supp}(\nu)) \rightarrow_{N \to \infty} 0$$

- a **finite** number J of **fixed** (independent of N) eigenvalues **(SPIKES)** $\theta_1 > \ldots > \theta_J$, $\forall i = 1, \ldots, J$, $\theta_i \notin \mathrm{supp}(\nu)$, each θ_j having a fixed multiplicity k_j, $\sum_j k_j = r$.

For large N, the $\beta_i(N)$ are inside

an ε neighborhood of the support of



Naive intuition for general additive deformed models:

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$$g_{\mu_{M_N}}(z) \approx g_{\mu_{A_N}}(\omega_{\mu,\nu}(z))$$

If $\rho \notin \mathrm{support}\ \mu \boxplus \nu$ is a solution of $\omega_{\mu,\nu}(\rho) = \theta_i$ for some $i \in \{1, \ldots, J\}$,

$\rho \notin \mathrm{support}\ \mu \boxplus \nu$ BUT $g_{\mu_{M_N}}(\rho) \approx g_{\mu_{A_N}}(\omega_{\mu,\nu}(\rho))$ explodes!

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$\Longrightarrow$ The spikes θ_i's of the perturbation A_N that may generate outliers in the spectrum of M_N belong to $\omega_{\mu,\nu}(\mathbb{R} \setminus \text{support } \mu \boxplus \nu)$

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$\Longrightarrow$ for large N, the θ_i's such that the equation

$$\omega_{\mu,\nu}(\rho) = \theta_i$$

has solutions ρ outside $\mathrm{support}\ \mu \boxplus \nu$ generate eigenvalues of M_N in a neighborhood of each of these ρ...

The particular case of spiked Deformed Wigner model

P.Biane 1997:

$$\omega_{\mu_{sc},\nu}(\mathbb{R}\backslash \text{support } \mu_{sc}\boxplus\nu) = \{u \in \mathbb{R}\backslash \text{support } \nu, \int \frac{1}{(u-x)^2} d\nu(x) < \frac{1}{\sigma^2}\}$$

$$\mathbb{R} \setminus \text{support } \mu_{sc} \boxplus \nu \underset{h_{\omega_{\mu_{sc},\nu}}}{\overset{\omega_{\mu_{sc},\nu}}{\rightleftarrows}} \mathcal{O}, \qquad h_{\mu_{sc},\nu} : z \mapsto z + \sigma^2 g_\nu(z).$$

The particular case of spiked Deformed Wigner model

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$$\omega_{\mu_{sc},\nu}(\mathbb{R}\backslash\text{support } \mu_{sc}\boxplus\nu) = \{u \in \mathbb{R}\backslash\text{support } \nu, \int \frac{1}{(u-x)^2} d\nu(x) < \frac{1}{\sigma^2}\}$$

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Previous conjecture becomes:

$\Longrightarrow$ If $\int \frac{1}{(\theta_i - x)^2} d\nu(x) < \frac{1}{\sigma^2}$, θ_i generates outliers in a neigborhood of $\rho = \theta_i + \sigma^2 g_\nu(\theta_i) \in \mathbb{R} \setminus \text{support } \mu_{sc} \boxplus \nu$.

The particular case of spiked Deformed Wigner model

P.Biane 1997:

$$\omega_{\mu_{sc},\nu}(\mathbb{R}\backslash\text{support } \mu_{sc}\boxplus\nu) = \{u \in \mathbb{R}\backslash\text{support } \nu, \int \frac{1}{(u-x)^2} d\nu(x) < \frac{1}{\sigma^2}\}$$

$$\mathbb{R} \setminus \text{support } \mu_{sc} \boxplus \nu \;\underset{h_{\omega_{\mu_{sc},\nu}}}{\overset{\omega_{\mu_{sc},\nu}}{\rightleftarrows}}\; \mathcal{O}, \qquad h_{\mu_{sc},\nu} : z \mapsto z + \sigma^2 g_\nu(z).$$

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Remark

When A_N has finite rank, $\nu = \delta_0$, this condition corresponds to $|\theta_i| > \sigma$ and then $\rho = \theta_i + \frac{\sigma^2}{\theta_i}$.

General spiked deformed models

Solving the problem of outliers consists in solving an equation involving the free subordination function and the spikes of the perturbation

General spiked deformed models

$M_N = A_N + B_N$ $\mu_{A_N} \to_{N\to+\infty} \nu$ $\mu_{B_N} \to_{N\to+\infty} \mu$ $\theta \in$ Spect(A_N) θ multiplicity k_i $\theta \notin \mathrm{supp}(\nu)$	$M_N = A_N^{1/2} B_N A_N^{1/2}$ $\mu_{A_N A_N^*} \to_{N\to+\infty} \nu$ $\mu_{B_N B_N^*} \to_{N\to+\infty} \mu$ $\theta \in$ Spect(A_N) θ multiplicity k_i $\theta > 0, \theta \notin \mathrm{supp}(\nu)$	$M_N = (A_N + B_N)(A_N + B_N)^*$ $\mu_{A_N A_N^*} \to_{N\to+\infty} \nu$ $\mu_{B_N B_N^*} \to_{N\to+\infty} \mu$ $\theta \in$ Spect$(A_N A_N^*)$ θ multiplicity k_i $\theta > 0, \theta \notin \mathrm{supp}(\nu)$
$\mu_{M_N} \to_{N\to+\infty} \mu \boxplus \nu$	$\mu_{M_N} \to_{N\to+\infty} \mu \boxtimes \nu$	$\mu_{M_N} \to_{N\to+\infty} (\sqrt{\mu} \boxplus_c \sqrt{\nu})^2$
$g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z-x}$	$\Psi_\tau(z) = \frac{1}{z} g_\tau(\frac{1}{z}) - 1$	$H^{(c)}_{\sqrt{\tau}} = \frac{c}{z} g_\tau(\frac{1}{z})^2 + (1-c) g_\tau(\frac{1}{z})$
$g_{\mu\boxplus\nu}(z) = g_\nu(\omega_{\mu,\nu}(z))$	$\Psi_{\mu\boxtimes\nu}(z) = \Psi_\nu(F_{\mu,\nu}(z))$	$H^{(c)}_{\sqrt{\mu}\boxplus_c\sqrt{\nu}}(z) = H^{(c)}_{\sqrt{\nu}}(\Omega_{\mu,\nu}(z))$
k_i outliers of M_N in the neighborhood of each ρ s.t $\omega_{\mu,\nu}(\rho) = \theta$	k_i outliers of M_N in the neighborhood of each ρ s.t $\frac{1}{F_{\mu,\nu}(1/\rho)} = \theta$	k_i outliers of M_N in the neighborhood of each ρ s.t $\frac{1}{\Omega_{\mu,\nu}(1/\rho)} = \theta$

When A_N has full rank, such results are proved for

$A_N + B_N$	$(A_N)^{1/2} B_N (A_N)^{1/2}$ $A_N > 0$	$(A_N + B_N)(A_N + B_N)$
B_N = Wigner C.&D-M.&F.&F. (2011)	B_N = Wishart Rao&Silverstein (2010) *Bai&Yao*(2012)	B_N i.i.d matrix A_N diagonal C. (2013)
$B_N = U_N D_N U_N^*$ U_N Haar, D_N deterministic $\mu_{D_N} \rightarrow_{N \to +\infty} \mu$ B.&B.&C.&F. (2012)	$B_N = U_N D_N U_N^*$ U Haar, $D_N \geq 0$ deterministic $\mu_{D_N} \rightarrow_{N \to +\infty} \mu$ B.&B.&C.&F. (2014)	

(C.&D-M.&F.&F.=C.&Donati-Martin&Féral&Février)

(B.&B.&C.&F.= Belinschi&Bercovici&C.&Février)

FOR ALL DEFORMED MODELS IN THE PREVIOUS ARRAY, if θ_i has multiplicity k_i in the spectrum of the deformation, then for each ρ which is a solution of the corresponding subordination equation (for instance in the additive case $\omega_{\mu,\nu}(\rho) = \theta_i$), almost surely, for all large N, there are exactly k_i eigenvalues of M_N in a neighborhood of ρ.

For matricial models in the first row of the previous array, given one spike θ there is at most one solution ρ for the corresponding equation and everything is explicit:

- Deformed Wigner

$$\mathbb{R} \setminus \text{support } \mu_{sc} \boxplus \nu \quad \underset{\phi_1}{\overset{\omega_{\mu_{sc}\nu}}{\underset{\longleftarrow}{\longrightarrow}}} \quad \mathcal{O}_1 \subset \mathbb{R} \setminus \text{support } \nu,$$

$$\theta \in \mathcal{O}_1, \quad \rho = \phi_1(\theta\ ) = \theta + \sigma^2 g_\nu(\theta).$$

- Sample covariance matrices

$$\mathbb{R} \setminus \{\text{support } \mu_{MP} \boxtimes \nu \cup \{0\}\} \quad \underset{\phi_2}{\overset{x \mapsto \frac{1}{F_{\mu_{MP},\nu}(1/x)}}{\underset{\longleftarrow}{\longrightarrow}}} \quad \mathcal{O}_2 \subset \mathbb{R} \setminus \text{support } \nu,$$

$$\theta \in \mathcal{O}_2 \quad \rho = \phi_2(\theta) = \theta + c\theta \int \frac{t}{\theta - t} d\nu(t).$$

- Information-Plus-Noise type model

$$\mathbb{R} \setminus (\sqrt{\mu_{MP}} \boxtimes_c \sqrt{\nu})^2 \quad \underset{\phi_3}{\overset{x \mapsto \frac{1}{\Omega_{\mu_{MP},\nu}(1/x)}}{\underset{\longleftarrow}{\longrightarrow}}} \quad \mathcal{O}_3 \subset \mathbb{R} \setminus \text{support } \nu,$$

$$\theta \in \mathcal{O}_3, \quad \rho = \phi_3(\theta) = \theta(1 + c g_\nu(\theta))^2 + (1 - c)(1 + c g_\nu(\theta))$$

BUT

CONCERNING SOME MODELS OF THE LAST ROW OF THE PREVIOUS ARRAY (deformations of unitarily invariant matrices), the restriction to the real line of some subordination maps may be many-to-one so that for one θ_i, there may exist several distinct ρ solving the corresponding subordination equation.
$\Longrightarrow$ For such models, a single spiked eigenvalue of A_N may generate several outliers of M_N.

Example: Deformed GUE

$$M_N = GUE(N, \frac{1}{4N}) + \mathrm{diag}(\underbrace{-1, \ldots, -1}_{\frac{N-1}{2}}, \underbrace{1, \ldots, 1}_{\frac{N}{2}}, 10)$$

$\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, $\sigma^2 = \frac{1}{4}$ and $\theta = 10$.

$\int \frac{1}{(10-x)^2} d\nu(x) < 4$,

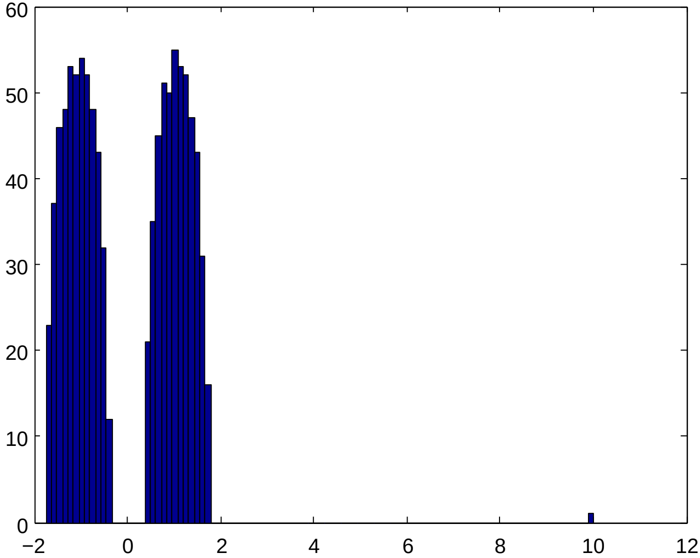
$$g_{\mu_{sc} \boxplus \nu}(z) = g_\nu(\omega^{(1)}_{\mu_{sc},\nu}(z))$$

$\omega^{(1)}_{\mu_{sc},\nu}$ is injective on $\mathbb{R} \setminus \mathrm{support}\ \mu_{sc} \boxplus \nu$

$\omega^{(1)}_{\mu_{sc},\nu}(\rho) = 10$ has 1 solution

$$\rho = 10 + \frac{1}{4}\left(\frac{1}{2}\frac{1}{10-1} + \frac{1}{2}\frac{1}{10-1}\right) \approx 10,05.$$

N=1000



Example

$$W^G := GUE(N-1, \frac{1}{4(N-1)}), \quad U_N \text{ Haar matrix independent from } W^G$$

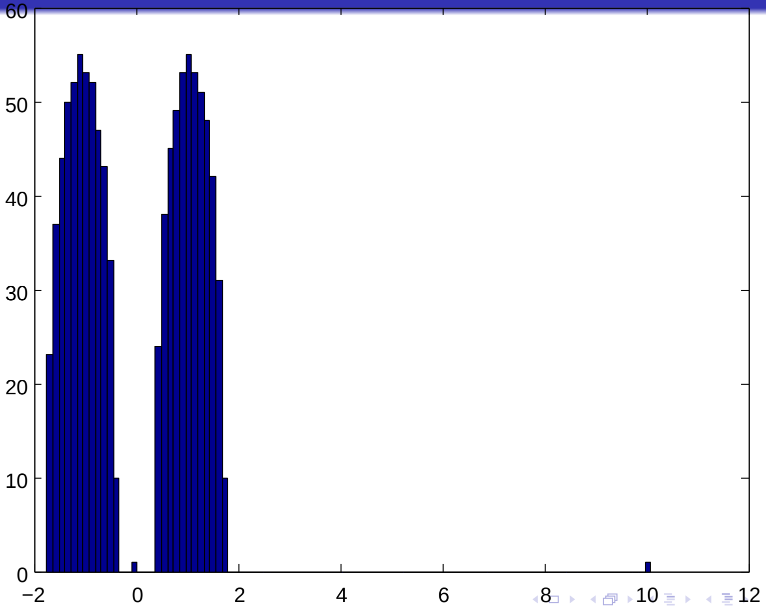
$$M_N = \begin{pmatrix} W^G & (0) \\ (0) & 10 \end{pmatrix} + U_N \mathrm{diag}(\underbrace{-1, \ldots, -1}_{\frac{N}{2}}, \underbrace{1, \ldots, 1}_{\frac{N}{2}}) U_N^*$$

This is not a spiked deformed GUE model and now, the spike $\theta = 10$ is associated to the matrix approximating the semicircular distribution.!!!!!!!!

$$g_{\mu_{sc} \boxplus \nu}(z) = g_\nu(\omega^{(1)}_{\mu_{sc},\nu}(z)) = g_{\mu_{sc}}(\omega^{(2)}_{\nu,\mu_{sc}}(z))$$

$\omega^{(1)}_{\mu_{sc},\nu}$ is injective on $\mathbb{R} \setminus \mathrm{support}\ \mu_{sc} \boxplus \nu$ but $\omega^{(2)}_{\nu,\mu_{sc}}$ may be many to one. $\omega^{(2)}_{\nu,\mu_{sc}}(\rho) = 10$ has 2 solutions ρ_1 and ρ_2.

N=1000



More funny...(Belinschi&Bercovici&C.&Février (2014))

$M_N = U_N B_N U_N^* + A_N$, U_N Haar unitary, A_N, B_N deterministic diagonal

$$\mu_{B_N} \to \mu, \quad \mu_{A_N} \to \nu$$

$\theta \notin \mathrm{supp}(\nu)$, with multiplicity k in the spectrum of A_N

$\alpha \notin \mathrm{supp}(\mu)$, with multiplicity l in the spectrum of B_N

whereas the other eigenvalues are uniformly close to the limiting supports.

$$g_{\mu \boxplus \nu}(z) = g_\nu(\omega^{(1)}_{\mu,\nu}(z)) = g_\mu(\omega^{(2)}_{\nu,\mu}(z)).$$

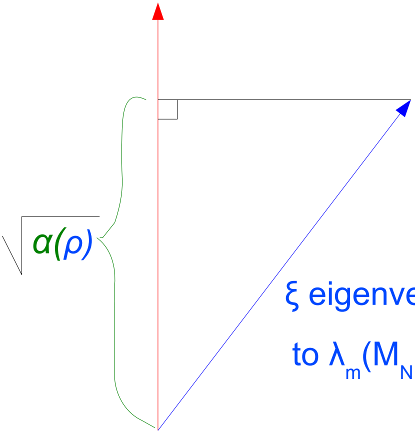
If there exists $\rho \in \mathbb{R} \setminus \mathrm{supp}(\mu \boxplus \nu)$ such that

$$\begin{cases} \omega^{(2)}_{\nu,\mu}(\rho) = \alpha \\ \omega^{(1)}_{\mu,\nu}(\rho) = \theta \end{cases}$$

then for all large N, there are $k + l$ outliers of M_N in a neighborhood of ρ.

$k_j=1$ When N goes to infinity,

V eigenvector associated to $\theta_j=\lambda_p(A_N)$



ξ eigenvector associated to $\lambda_m(M_N)$ in $]\rho-\delta\,;\,\rho+\delta[$

$$\text{where} \quad \alpha(\rho) = \begin{cases} \frac{1}{\omega'_{\mu,\nu}(\rho)} & \text{if } M_N = X_N + A_N \\ \frac{\rho F_{\mu,\nu}(1/\rho)}{F'_{\mu,\nu}(1/\rho)} & \text{if } M_N = A_N^{1/2} X_N A_N^{1/2} \end{cases}$$

This result is proved

- for $X_N + A_N$ when X_N is a Wigner matrix [C. 2011] and when the distribution of X_N is unitarily invariant [Benaych-Georges&Rao (2010) Belinschi&Bercovici& C.& Février (2014)]
- for $A_N^{1/2} X_N A_N^{1/2}$ when X_N is a Wishart matrix [C. 2011] and when the distribution of X_N is unitarily invariant [Benaych-Georges&Rao (2010) Belinschi&Bercovici& C.& Février (2014)]

(for information-plus-noise type models, results of Benaych-Georges-Rao (2012) dealing with finite rank perturbations)

“Deterministic fundamental measure”

Deformed Wigner matrices

W_N: a Wigner matrix , A_N: Hermitian deterministic.

$$M_N = \frac{W_N}{\sqrt{N}} + A_N$$

The deterministic measure

$$\mu_{A_N} \boxplus \mu_{sc}$$

plays a central role in the study of the spectrum.

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- "No eigenvalue outside the support of this measure"
- "Exact separation phenomenon" involving this measure

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plays a central role in the study of the spectrum.

- "No eigenvalue outside the support of this measure"
- "Exact separation phenomenon" involving this measure
- Universality of the fluctuations at some edges of the support of this measure

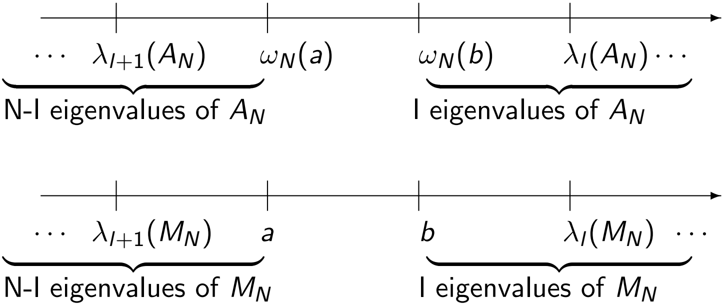
Exact separation phenomenon for deformed Wigner model

$\omega_N(z) = z - \sigma^2 g_{\mu_{sc} \boxplus \mu_{A_N}}(z)$ (the subordination map of $\mu_{sc} \boxplus \mu_{A_N}$ w.r.t μ_{A_N})

Then, almost surely, for large N,

$$[a, b] \subset \mathbb{R} \setminus \mathrm{support}\ (\mu_{sc} \boxplus \mu_{A_N}) \longleftrightarrow [\omega_N(a), \omega_N(b)]$$

$$\text{gap in } \mathrm{Spect}(M_N) \longleftrightarrow \text{gap in } \mathrm{Spect}(A_N)$$



Exact separation phenomena

involving the additive, multiplicative, rectangular subordination maps

- Deformed Wigner matrices
  C.&Donati-Martin&Féral&Février (2011)
- Sample Covariance matrices
  Bai&Silverstein (1999)
- Information-Plus-Noise type models
  Loubaton&Vallet (2011) C. (2014)

FLUCTUATIONS AT EDGES

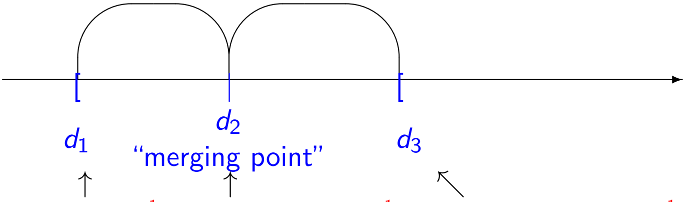
$$M_N = GUE(N, \frac{\sigma^2}{N}) + A_N, \quad A_N = \mathrm{diag}(\beta_1, \ldots, \beta_{N-r}, \theta_1, \ldots, \theta_J)$$

$\mu_{A_N} \to \nu$, ν compactly supported.

- $\max_{i=1}^{N-r} \mathrm{dist}(\beta_i(N), \mathrm{supp}(\nu)) \to_{N\to\infty} 0$
- a **finite** number J of **fixed** (independent of N) eigenvalues **(SPIKES)** $\theta_1 > \ldots > \theta_J$, $\forall i = 1, \ldots, J$, $\theta_i \notin \mathrm{supp}(\nu)$, each θ_j having a fixed multiplicity k_j.

Assumption: $\forall u \in \mathrm{support}(\nu), \quad \int \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2}.$

Example, p density of $\mu_{sc} \boxplus \nu$



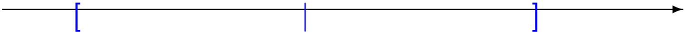
$p(x) \sim C|d_1 - x|^{\frac{1}{2}}$ $p(x) \sim C|d_2 - x|^{\frac{1}{3}}$ $p(x) \sim C|d_3 - x|^{\frac{1}{2}}$ (Biane 97)

For ϵ small enough, for all large N,

$\exists! d_1(N)$ left edge of $\mu_{A_N} \boxplus \mu_{sc}$ in $]d_1 - \epsilon; d_1 + \epsilon[$

$\exists! d_2(N)$ "merging point" of $\mu_{A_N} \boxplus \mu_{sc}$ in $]d_2 - \epsilon; d_2 + \epsilon[$

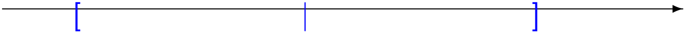
$\exists! d_3(N)$ right edge of $\mu_{A_N} \boxplus \mu_{sc}$ in $]d_3 - \epsilon; d_3 + \epsilon[$



$d_1(N)$ — $d_2(N)$ "merging point" — $d_3(N)$

$\uparrow$ $\uparrow$ $\uparrow$

Deformed GUE C.&Péché (2014)	Tracy-Widom at scale $N^{\frac{2}{3}}$	Pearcey at scale $N^{\frac{3}{4}}$	Tracy-Widom at scale $N^{\frac{2}{3}}$



	$d_1(N)$	$d_2(N)$ "merging point"	$d_3(N)$
	↑	↑	↑
Deformed GUE C.&Péché (2014)	Tracy-Widom at scale $N^{\frac{2}{3}}$	Pearcey at scale $N^{\frac{3}{4}}$	Tracy-Widom at scale $N^{\frac{2}{3}}$
W+D Lee&Schnelli (2014) W+A Knowles&Yin (2014)	Tracy-Widom at scale $N^{\frac{2}{3}}$		Tracy-Widom at scale $N^{\frac{2}{3}}$

	$d_1(N)$	$d_2(N)$ "merging point"	$d_3(N)$
	$\uparrow$	$\uparrow$	$\uparrow$
Deformed GUE C.&Péché (2014)	Tracy-Widom at scale $N^{\frac{2}{3}}$	Pearcey at scale $N^{\frac{3}{4}}$	Tracy-Widom at scale $N^{\frac{2}{3}}$
W+D Lee&Schnelli (2014) W+A Knowles&Yin (2014)	Tracy-Widom at scale $N^{\frac{2}{3}}$		Tracy-Widom at scale $N^{\frac{2}{3}}$

$\Longrightarrow$ Universality of the fluctuations around the edges $d_i(N)$ of $\mu_{A_N} \boxplus \mu_{sc}$

Considering fluctuations around d_i (instead of $d_i(N)$) may imply making assumption on the rate of convergence of $g_{\mu_{A_N}}$ towards g_ν.
Scherbina (2011)

Remark

Previous works of Brezin&Hikami (1998), Aptekarev&Bleher&Kuijilars (2004), (2005), Adler&Cafasso&Van Moerbeke (2007), (2011) when $\mu_{A_N} = \nu$ *is a finite combination of Dirac Delta masses.*

Fluctuation of outliers

$$M_N = GUE(N, \frac{\sigma^2}{N}) + A_N, \quad A_N = \mathrm{diag}(\beta_1, \ldots, \beta_{N-r}, \theta_1, \ldots, \theta_J)$$

$\mu_{A_N} \to \nu$, ν compactly supported.

- $\max_{i=1}^{N-r} \mathrm{dist}(\beta_i(N), \mathrm{supp}(\nu)) \to_{N\to\infty} 0$
- a **finite** number J of **fixed** (independent of N) eigenvalues **(SPIKES)** $\theta_1 > \ldots > \theta_J$, $\forall i = 1, \ldots, J$, $\theta_i \notin \mathrm{supp}(\nu)$, each θ_j having a fixed multiplicity k_j.

Let θ_i be such that $\int \frac{d\nu(x)}{(\theta_i - x)^2} < 1$ and $\rho_{\theta_i} = h_{\mu_{sc},\nu}(\theta_i)$. Then, for $\epsilon > 0$ small enough, for all large N, $\mathrm{supp}(\mu_{sc} \boxplus \mu_{A_N})$ has a unique connected component $[L_i(N); D_i(N)]$ inside $]\rho_{\theta_i} - \epsilon; \rho_{\theta_i} + \epsilon[$. Moreover, the k_i outliers of M_N close to ρ_{θ_i} fluctuate at rate $\sqrt{N}$ around $\frac{L_i(N)+D_i(N)}{2}$ as the eigenvalues of a $k_i \times k_i$ GUE.

Some remarks

Remark

Analog results at soft edges for Sample covariance matrices by Hachem&Hardy&Najim (2014), Lee&Schnelli (2014), Bao&Pan&Zhou (2014) and for outliers of Sample covariance matrices by Bai&Yao (2012)

Some remarks

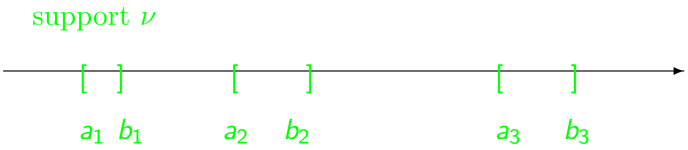
Remark

Analog results at soft edges for Sample covariance matrices by Hachem&Hardy&Najim (2014), Lee&Schnelli (2014), Bao&Pan&Zhou (2014) and for outliers of Sample covariance matrices by Bai&Yao (2012)

Remark

According to previous studies dealing with finite rank perturbations, universality of fluctuations of outliers of deformed Wigner models is not expected in full generality.

Example



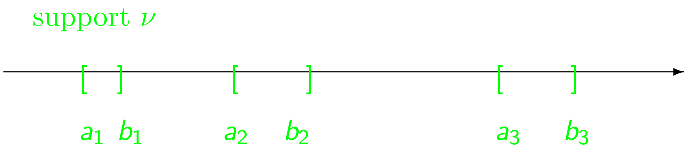
Example

$$\mathbb{R} \setminus \text{support } \mu_{sc} \boxplus \nu = h_{\mu_{sc},\nu}(\mathcal{O}), \;\; h_{\mu_{sc},\nu} : z \mapsto z + \sigma^2 g_\nu(z)$$

$$\mathcal{O} := \{u \in \mathbb{R} \setminus \text{support } \nu, \int \frac{1}{(u-x)^2} d\nu(x) < \frac{1}{\sigma^2}\}$$

$$^c\mathcal{O} := \text{support } \nu \cup \{u \in \mathbb{R} \setminus \text{support } \nu, \int \frac{1}{(u-x)^2} d\nu(x) \geq \frac{1}{\sigma^2}\}$$
$$= \overline{\{u \in \mathbb{R}, \int \frac{1}{(u-x)^2} d\nu(x) > \frac{1}{\sigma^2}\}}$$

Each connected component of $^c\mathcal{O}$ contains at least one connected component of support ν



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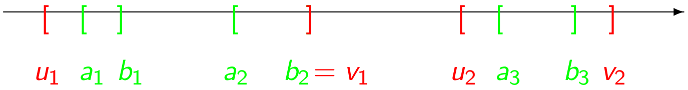
$$= \{u \in \mathbb{R}, \int \frac{1}{(u-x)^2} d\nu(x) > \frac{1}{\sigma^2}\}$$

$^c\mathcal{O}$ $\quad$ support ν



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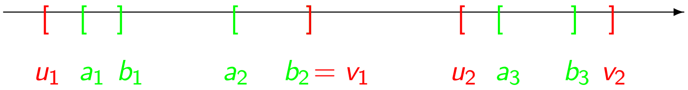
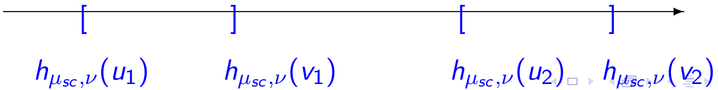
$$= \overline{\{u \in \mathbb{R}, \int \frac{1}{(u-x)^2} d\nu(x) > \frac{1}{\sigma^2}\}}$$

${}^c\mathcal{O}$ support ν



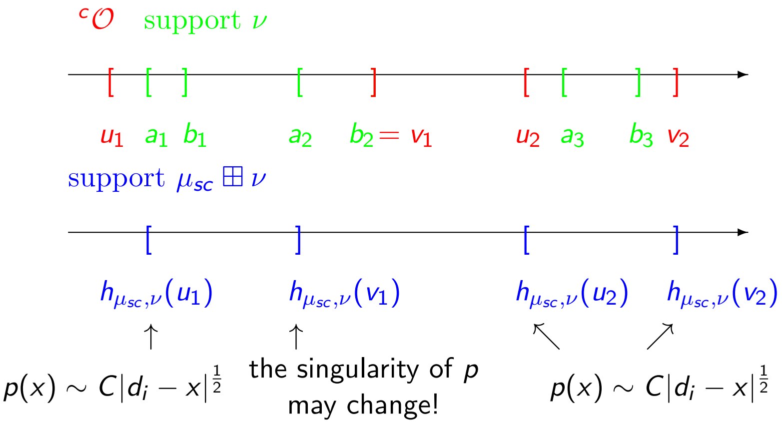
$$h_{\mu_{sc},\nu} : z \mapsto z + \sigma^2 g_\nu(z).$$

support $\mu_{sc} \boxplus \nu$



Example

$\mu_{sc} \boxplus \nu$ is absolutely continuous. p: density of $\mu_{sc} \boxplus \nu$



Example investigated by Lee &Schnelli (2013) :

$$d\nu(x) := Z^{-1}(1+x)^a(1-x)^b f(x) 1_{[-1,1]}(x)dx$$

where $a<1, b>1$, f is a strictly positive $\mathcal{C}^1$-function and Z is a normalization constant.

$$\int \frac{1}{(1-x)^2} d\nu(x) = \frac{1}{\sigma_0^2}$$

$$^c\mathcal{O} = \text{support } \nu \cup \{u \in \mathbb{R} \setminus \text{support } \nu, \int \frac{1}{(u-x)^2} d\nu(x) \geq \frac{1}{\sigma^2}\}$$

$$h_{\mu_{sc},\nu} : z \mapsto z + \sigma^2 g_\nu(z)$$

$\forall \sigma > \sigma_0$, $^c\mathcal{O} = [u_\sigma; v_\sigma]$ with

$u_\sigma < -1 < 1 < v_\sigma$, support $\mu \boxplus \nu = [h_{\mu_{sc},\nu}(u_\sigma); h_{\mu_{sc},\nu}(v_\sigma)]$

$\longrightarrow p(x) \sim C(h_{\mu_{sc},\nu}(v_\sigma) - x)^{\frac{1}{2}}$

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$\forall \sigma \leq \sigma_0$, ${}^c\mathcal{O} = [u_\sigma; 1]$, $u_\sigma < -1$, support $\mu \boxplus \nu = [h_{\mu_{sc},\nu}(u_\sigma); h_{\mu_{sc},\nu}(1)]$

$\longrightarrow p(x) \sim C(h_{\mu_{sc},\nu}(1) - x)^b$

Letting the perturbation A_N be random ...

Lee &Schnelli (2014)

$$\frac{W_N}{\sqrt{N}}+\mathrm{diag}(v_1,\ldots,v_N),\ \ v_i \text{ i.i.d} \sim d\nu(x)=Z^{-1}(1+x)^a(1-x)^b f(x)1_{[-1,1]}(x)dx$$

$a<1, b>1,\ f>0\ \mathcal{C}^1$-function.

σ_0 defined by $\int \frac{1}{(1-x)^2}d\nu(x)=\frac{1}{\sigma_0^2}$, support $\mu_{sc}\boxplus\nu=[d_\sigma^-;d_\sigma^+]$

- $\forall\sigma>\sigma_0,\ p(x)\sim C(d_\sigma^+-x)^{\frac{1}{2}}$.

$d_\sigma^+(N)$: upper right edge of support $\mu_{sc}\boxplus\mu_{A_N}$,

$$N^{2/3}(\lambda_1(M_N)-d_\sigma^+(N))\xrightarrow{\mathcal{D}} TW,$$

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- $\forall \sigma < \sigma_0$, $p(x) \sim C(d_\sigma^+ - x)^b$

$$N^{\frac{1}{b+1}}(\lambda_1(M_N) - d_\sigma^+) \xrightarrow{\mathcal{D}} G_{b+1}(s)$$

as N goes to infinity, where $G_{b+1}(s) = (1 - \exp((\frac{s}{c})^{b+1}))\mathbf{1}_{[0;+\infty[}(s)$
(Weibull distribution with parameters $b+1$ and $c = c(\nu,\sigma)$).

THANK YOU FOR YOUR ATTENTION!