Large sample behaviour of high dimensional autocovariance matrices

Joint work with Monika Bhattacharjee

Random matrices and their applications
The University of Hong-Kong
January 6-9, 2015

Time series/Linear process

Linear process

$$X_{t,p}^{(n)} = \sum_{j=0}^{q} A_{j,p}^{(n)} \varepsilon_{t-j,p} \; t, n \geq 1 \quad \text{(almost surely).} \tag{1}$$

$\varepsilon_{t,p}$ (unobservable) and $X_{t,p}^{(n)}$ (observable) are p-dimensional vectors.

$\varepsilon_{t,p}$'s are *i.i.d.* with mean 0 and variance-covariance matrix I_p.

$A_{j,p}^{(n)}, j \geq 0$ are $p \times p$ matrices (non-random) *coefficient matrices*.

$p = p(n) \to \infty$ such that $\frac{p}{n} \to y \in [0, \infty)$.

q finite but will comment on this later.

Autocovariance matrix sequence

$$\hat{\Gamma}_i(\varepsilon) = \frac{1}{n} \sum_{t=i+1}^{n} \varepsilon_{t,p} \varepsilon'_{(t-i),p}$$

is the *i*th order “sample autocovariance” matrix of $\{\varepsilon_t\}$.

The sample autocovariance matrix of order i of $\{X_t\}$ equals

$$\hat{\Gamma}_{i,p} := \frac{1}{n} \sum_{t=i+1}^{n} X_{t,p} X'_{(t-i),p} = \sum_{j=0}^{\infty} \sum_{j'=0}^{\infty} A_j \hat{\Gamma}_{j'-j+i}(\varepsilon) A'_{j'},$$

for all i=1,2,3,. . . (n-1).

They are non-symmetric except $\hat{\Gamma}_{0,p}$.

Main questions

Symmetrized Autocovariances: $\hat{\Gamma}_0, \quad \hat{\Gamma}_i\hat{\Gamma}_i', \quad \hat{\Gamma}_i + \hat{\Gamma}_i'$ *for all* $i \geq 1$.

1. Is there a limiting spectral distribution of the above symmetric matrices? Using Stieltjes transform method, answers are known in very specific cases in two articles.

2. How can the limits be described?

3. Can we establish joint convergence of polynomials (convergence of non-commutative probabilty spaces)? Linked to Question 2.

4. Is there a limiting spectral distribution of the non-symmetric $\hat{\Gamma}_i$?

ESD and LSD

R_n: an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. The *Empirical Spectral Distribution* (ESD) of R_n is the measure

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}.$$

If this ESD converges weakly (almost surely or in probability) to a (non-degenerate) probability distribution, then the limit is called the *limiting spectral distribution* (LSD).

Since our models may allow triangular sequences, our convergences will be in probability.

Stieltjes/Cauchy transform

For any measure μ on the real line its Stieltjes/Cauchy transform is the function

$$m_\mu(z) = \pm \int \frac{1}{x - z} \mu(dx), \qquad z \in \mathbb{C}^+, \tag{2}$$

where $\mathbb{C}^+ := \{x + iy : x \in \mathbb{R}, y > 0\}$.

Pointwise convergence of S/C transform to an S/C transform implies the convergence of the corresponding distributions.

Marčenko-Pastur law and Wigner law

1. $y \neq 0$. Let $X_t = \varepsilon_t$, i.i.d. Under suitable assumptions on $\{\varepsilon_t\}$, the LSD of $\hat{\Gamma}_0 = \frac{1}{n}XX'$ is the Marčenko-Pastur law with parameter y whose moment sequence and Stieltjes transform are given by

$$\beta_h = \sum_{k=1}^{h} \frac{1}{k}\binom{h-1}{k-1}\binom{h}{k} y^{k-1}, \quad yzm^2(z)+(1-y-z)m(z)+1 = 0. \quad (3)$$

2. $y = 0$. Let $X_t = \varepsilon_t \ \ \forall t$, i.i.d. Then again under suitable assumptions on ε_t, the LSD of $\sqrt{\frac{n}{p}}\left(\frac{1}{n}XX' - I\right)$ is the Wigner law (semi-circle law) with moments and Stieltjes transform given by

$$\beta_{2h}(s) = \frac{1}{h+1}\binom{2h}{h}, \quad m^2(z) + zm(z) + 1 = 0.$$

Jin, Wang, Bai, Nair and Harding (2014), $y \neq 0$

Suppose $X_t = \varepsilon_t$ i.i.d. standardized, $E\varepsilon_{t,i}^4 < \infty$.

Then LSD of $\frac{1}{2}(\hat{\Gamma}_i + \hat{\Gamma}_i^*)$ exists and the Stieltjes transform satisfies

$$(1 - y^2 m^2(z))(yzm(z) + y - 1)^2 = 1 \tag{4}$$

Limit does not depend on i.

Liu, Aue and Paul (2013), $y \neq 0$, assumptions

(a) $\{\varepsilon_{t,i} : t, i = 1, 2, 3, \ldots\}$ i.i.d. standardized, $E|\varepsilon_{t,i}|^{4+\delta} < \infty$.

(b) A_j simultaneously diagonalizable Hermitian, with additional norm boundedness conditions.

(c) there exists unitary $p \times p$ matrix U such that for all $j \in \mathbb{Z}$, $U^* A_j U = \operatorname{diag}\left(f_j(\lambda_1), f_j(\lambda_2), \ldots, f_j(\lambda_p)\right)$ where $f_l(\cdot)$ are continuous, can also depend on p as long as they uniformly converge to continuous functions when $p \to \infty$.

(d) Let F_p^A be the distribution with mass $\frac{1}{p}$ on each $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p$. F_p^A converges weakly to a probability distribution function F^A as $p \to \infty$.

Liu, Aue and Paul (2013), result, $y \neq 0$

Then the LSD of $\frac{1}{2}(\hat{\Gamma}_j + \hat{\Gamma}_j^*)$ exists with Stieltjes transform

$$m_j(z) = -\int \frac{dF^A(\lambda)}{z - \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(jv)h(\lambda,v)}{1+y\cos(jv)K_j(z,v)}dv} \quad \forall z \in \mathbb{C}^+, \tag{5}$$

where,

$$K_j(z,v) = -\int \frac{h(\lambda,v)dF^A(\lambda)}{z - \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(jv')h(\lambda,v')}{1+y\cos(jv')K_j(z,v')}dv'} \tag{6}$$

and

$$h(\lambda, v) = |\sum_{l=0}^{\infty} e^{ilv} f_l(\lambda)|^2. \tag{7}$$

Basic idea: embedding, $y \neq 0$

Let $X_t = \varepsilon_t$, i.id. In this case $\hat{\Gamma}_0$ is the sample covariance matrix and can be embedded as

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} W \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} W \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} n\hat{\Gamma}_0 & 0 \\ O & 0 \end{pmatrix}_{(n+p)\times(n+p)},$$

for some Wigner matrix W.

- Non-zero eigenvalues of $\begin{pmatrix} n\hat{\Gamma}_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $n\hat{\Gamma}_0$ are same.

- LSD of W is the semi-circle law.

- W and the deterministic matrices $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ jointly converge and are *asymptotically free*.

Using this, can “easily” show that the LSD of $\hat{\Gamma}_0$ is the MP law.

Embedding, $y \neq 0$

For any matrix B of order p, let $\bar{B}$ of order $(n + p)$ equal

$$\bar{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}. \tag{8}$$

We will state our main result in terms of these types of matrices and get back to the orginal matrices in our corollaries.

Recall autocovariance matrix sequence

$$\hat{\Gamma}_i(\varepsilon) = \frac{1}{n}\sum_{t=i+1}^{n} \varepsilon_{t,p}\varepsilon'_{(t-i),p}.$$

$$\hat{\Gamma}_i = \hat{\Gamma}_{i,p} = \frac{1}{n}\sum_{t=i+1}^{n} X_{t,p}X'_{(t-i),p} = \sum_{j=0}^{\infty}\sum_{j'=0}^{\infty} A_j\hat{\Gamma}_{j'-j+i}(\varepsilon)A'_{j'}.$$

They are non-symmetric except $\hat{\Gamma}_{0,p}$.

Non-commutative probability space (NCP), $y \neq 0$

Consider the sequence of non-commutative $*$-probability spaces $(\mathcal{A}_n, \varphi_n)$ of $(n+p) \times (n+p)$ matrices

$\mathcal{A}_n =$ *-algebra generated by$\{n(n+p)^{-1}\bar{\hat{\Gamma}}_i, n(n+p)^{-1}\bar{\hat{\Gamma}}_i^* : i \geq 0\}$

and state

$$\varphi_n = \frac{1}{n+p} E\,\mathrm{Tr}(\cdot).$$

Limit NCP

Let $(\mathcal{A}, \varphi)$ be a non-commutative $*$-probability space such that

$\mathcal{A} =$ *-algebra generated by $s, \{c_i, c_i^* : c_0 = c_0^*, i \geq 0\}, \{a_i, a_i^* : i \geq 0\}$.

Limit NCP (continued)

s is semi-circular (from the Wigner matrix), $\varphi(s^{2k}) = \beta_{2k}(s)$.

Suppose A_j are uniformly norm bounded and jointly converge. For any monomial m, let

$$\varphi(m(a_i, a_i^*)) = \frac{y}{1+y} \lim \left\{ \frac{1}{p} \operatorname{Tr} \Big(m \Big(A_j, A_j^* \ : \ j = 0, 1, 2, \ldots \Big) \Big) \right\}.$$

The lagged variables give rise to matrices P_i whose ith diagonal equals one. They converge jointly and the limit are captured by c_i. For any monomial m, let

$$\varphi(m(c_i, c_i^*)) = \begin{cases} \frac{1}{1+y}, & \text{if there is same number of } c_i \text{ and } c_i^* \\ 0, & \text{otherwise,} \end{cases}$$

Limit NCP (continued)

s, $\mathrm{Span}\{c_i, c_i^* : c_0 = c_0^*, i \geq 0\}$ and $\mathrm{Span}\{a_i, a_i^* : i \geq 0\}$ are *freely independent.*

Quick explanation: Joint limit of A_1, A_2, WB_1W, WB_2W is same as when $\{A_1, A_2\}$, $\{B_1, B_2\}$ and W are free.

Now recall the building blocks $A_j \hat{\Gamma}_k(\varepsilon) A_j^*$....

Limit NCP (continued)

Let

$$\gamma_{iq} = \sum_{j=0}^{q}\sum_{j'=0}^{q} a_j sc_{j'-j+i} sa_{j'}^*, \qquad \gamma_{iq}^* = \sum_{j=0}^{q}\sum_{j'=0}^{q} a_{j'} sc_{j'-j+i}^* sa_j^*.$$

For all $q \geq 0$, consider the $*$-probability spaces $(\mathcal{B}_q, \varphi)$ where

$$\mathcal{B}_q = \text{Span}\{\gamma_{iq}, \gamma_{iq}^* : i \geq 0\} \subset \mathcal{A}. \tag{9}$$

Theorem

Theorem

Under suitable assumptions, $(\mathcal{A}_n, \varphi_n)$ *converges to* $(\mathcal{B}_q, \varphi)$ *and* $(n(n+p)^{-1}\bar{\hat{\Gamma}}_i, n(n+p)^{-1}\bar{\hat{\Gamma}}_i^*)$ *are asymptotically distributed as* $(\gamma_{iq}, \gamma_{iq}^*)$.

Independence assumptions.

Identical distribution assumption.

Moment assumptions.

Joint convergence of A_j and norm boundedness.

Corollaries: LSD results for any symmetric matrix polynomial. Can remove high moment assumptions from these LSD results by truncation arguments.

Consequences

1. MP law: consider $\hat{\Gamma}_0$ with $X_t = \varepsilon_t$.

2. Free Bessel: $X_t = \varepsilon_t$. Then for every i, LSD of $\left(\frac{n}{p}\right)^2 \hat{\Gamma}_i \hat{\Gamma}_i^*$ is the free Bessel$(2, y^{-1})$ law.

3. JWBN (2014) and LAP (2013) results follow.

4. Let $X_t = \varepsilon_t$ standardized i.i.d. with finite fourth moment and Σ be a positive definite and symmetric matrix with compactly supported LSD F_Σ. Then the Stieltjes transform of the LSD of $\Sigma^{1/2}\hat{\Gamma}_0\Sigma^{1/2}$ is given by

$$m(z) = \int \frac{dF_\Sigma(t)}{z - t(1 - y - yzm(z))}. \tag{10}$$

For a more general result see Bai and Zou (2008).

More consequences/comments

5. Results for additive and multiplicative symmetrisation can be proved for $q = \infty$ under additional conditions.

6. The LSD of the additive and multiplicative symmetrisations depend on the order q. They remain same once $i > q$. This can be used for order determination, at least in an exploratory way.

7. No closed form expression for moments of the LSD. However, using free porbability, Kreweras complement etc, moments of successively higher order can be obtained recursively.

8. Can handle more than one sequence of independent linear processes.

9. What happens to the LSD of the non-symmetric Γ_i? Open.

Existing results, $y = 0$

Ongoing work.

Results (1)–(4) given below used the Stieltjes transformation method to obtain LSDs.

Let $X_t = \varepsilon_t$, standardized i.i.d. with finite fourth moment.

(1). Bai and Yin (1988): the LSD of $\sqrt{\frac{n}{p}}\left(\frac{1}{n}XX' - I\right)$ is the Wigner law (semi-circle law) with moments and Stieltjes transform given by

$$\beta_{2h}(s) = \frac{1}{h+1}\binom{2h}{h}, \quad m^2(z) + zm(z) + 1 = 0.$$

Existing results, $y = 0$

Let $X_t = \varepsilon_t$, standardized i.i.d. with finite fourth moment. Let A be a non-negative definite matrix whose LSD exists.

(2). Pan and Gao (2009), Bao (2012): established the LSD of $\sqrt{np^{-1}}(n^{-1}A^{1/2}XX^*A^{1/2} - A)$.

(3). Bai and Zhang (2010) established the existence of LSD of $p^{-1/2}A^{1/2}WA^{1/2}$. LSD same as that in (2).

(4). Wang and Paul (2014): derived the LSD of $\sqrt{np^{-1}}(n^{-1}A^{1/2}XBX^*A^{1/2} - n^{-1}\mathrm{Tr}(B)A)$. It coincides with the LSD of $\sqrt{\lim n^{-1}\mathrm{Tr}(B^2)}p^{-1/2}A^{1/2}WA^{1/2}$.

Assumptions, $y = 0$

$Z_u = ((\varepsilon_{u,i,j}))_{p\times n},\ \forall u = 1, 2, \ldots U$ be $p \times n$ matrices such that

1. $\{\varepsilon_{u,i,j}\}$ are standardized, independent, $\sup_{u,i,j} E|\varepsilon_{u,i,j}|^4 < \infty$.
2. For some $\eta > 0,\ 0 < \delta \leq 2$, $P(|\varepsilon_{u,i,j}| < \eta p^{\frac{1}{2+\delta}}) = 1$.
3. $\{A_{l,2i-1} : i = 1, 2, \ldots, k_l + 1, l = 1, 2, \ldots, r\}$ are l $p \times p$ matrices, compactly supported and jointly converge.
4. $\{A_{l,2i} : i = 1, 2, \ldots, k_l,\ l = 1, 2, \ldots, r\}$ are $n \times n$ matrices with bounded spectral norms. Moreover,

$$\lim_{n\to\infty} n^{-1}\mathrm{Tr}(A_{l,2i}) < \infty, \quad \lim_{n\to\infty} n^{-1}\mathrm{Tr}(A_{l,2i}A_{l',2i'}) < \infty. \tag{11}$$

5. $p, n(p) \to \infty$ such that $pn^{-1} \to 0$.

Let

$$\mathcal{A} \quad = \quad \mathrm{Span}\{a_{l,2i+1}, a^*_{l,2i+1} \; : \; i = 0,1,2,\ldots,k_l, \; l = 1,2,\ldots r\}$$

with state φ such that for any monomial m,

$$\lim p^{-1}\mathrm{Tr}\left(m(A_{l,2i-1}, A^*_{l,2i-1} \; : \; \forall i,l)\right) = \varphi\left(m(a_{l,2i-1}, a^*_{l,2i-1} \; : \; \forall i,l)\right). \tag{12}$$

Consider the *semicircle families*

$$\mathcal{T}_u = \{s_{l,j,u} : j = 1,2,\ldots,k_l, l = 1,2,\ldots,r\}, \tag{13}$$

which are freely independent over u and with $\mathcal{A}$ such that

$$\varphi(s_{l_1,j_1,t}s_{l_2,j_2,t}) = \lim n^{-1}\mathrm{Tr}(A_{l_1,2j_1}A_{l_2,2j_2}), \; \forall j_1,j_2,l_1,l_2. \tag{14}$$

Define

$$a_{l,-j} = (\prod_{\substack{i=1\\ i\neq j}}^{k_l} \lim n^{-1}\mathrm{Tr}(A_{l.2i}))(\prod_{i=0}^{j-1} a_{l,2i+1}), \qquad C_{l,-j} = \prod_{i=j}^{k_l} a_{l,2i+1}.$$

Theorem

The non-commutative $*$-probability space

$$\mathrm{Sp}\left(\sqrt{\frac{n}{p}}\left[\left(\prod_{i=1}^{2k_l-1}\frac{A_{l,i}Z_{u_{l,i}}A_{l,i+1}Z_{u_{l,i}}^{*}}{n}\right)A_{l,2k_l+1}-\left(\prod_{i=1}^{k_l}n^{-1}\mathrm{Tr}\left(A_{l,2i}\right)\right)\prod_{i=0}^{k_l}A_{l,2i+1}\right]:\right),$$

where $1 \le i \le r$, $u_{l,i} \in \{1, 2, \ldots, U\}$, with state $p^{-1}\mathrm{Tr}$ converge to

$$\left(\mathrm{Span}\left(\sum_{j=1}^{k_l} a_{l,-j}s_{l,j,u_{l,j}}c_{l,-j}\ :\ l = 1, 2, \ldots, r\right), \varphi\right),\quad u_{l,i} \in \{1, 2, \ldots, U\}. \quad (15)$$

Results 1–4 are special cases.

Corollary

Let A be a $p \times p$ non-negative definite compactly supported Hermitian matrix whose LSD exists. Suppose $(\mathrm{Span}\{A\}, p^{-1}\mathrm{Tr}) \to (\mathrm{Span}\{a\}, \varphi)$. Also let B be a $n \times n$ Hermitian matrix with bounded spectral norm and $\lim n^{-1}\mathrm{Tr}(B^k) < \infty, \ \forall k = 1, 2$. Under suitable assumptions,

$$\left(\mathrm{Sp}\{\sqrt{\frac{n}{p}}\frac{A^{1/2}ZBZ^{*}A^{1/2}}{n} - \frac{\mathrm{Tr}(B)}{n}A)\}, p^{-1}\mathrm{Tr}\right) \to \left(\mathrm{Sp}\{a^{1/2}sa^{1/2}\}, \varphi\right) \tag{16}$$

where a and s are freely independent and $\varphi(s^2) = \lim n^{-1}\mathrm{Tr}(B^2)$.