Large complex correlated Wishart matrices: Fluctuations and asymptotic independence at the edges

Joint work with W. Hachem and J. Najim.

Adrien Hardy

Royal Institute of Technology KTH, Stockholm

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Plan

1. Introduction and statement of the results
2. More precisions
3. Beyond universality

1) The matrix model

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Complex correlated Wishart matrix:

$$\mathbf{M}_N = \frac{1}{N} \mathbf{X}_N \mathbf{\Sigma}_N \mathbf{X}_N^*$$

where

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Let $x_1 \leqslant \cdots \leqslant x_N$ be the eigenvalues of $\mathbf{M}_N$ (main characters),
and $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ be the eigenvalues of $\mathbf{\Sigma}_N$ (parameters).

Asymptotic regime:

$$N, n \to \infty, \qquad \tfrac{n}{N} \to \gamma \in (0, \infty),$$

$$\nu_N := \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j} \xrightarrow[N\to\infty]{*} \nu \quad \text{with compact support.}$$

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Global behavior (Marčenko-Pastur,67):

There exists $\mu_{(\nu,\gamma)}$ only depending on ν,γ such that

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{x_j} \xrightarrow[N\to\infty]{*} \mu_{(\nu,\gamma)} \quad a.s.$$

[The Stieltjes transform of $\mu_{(\nu,\gamma)}$ satisfies a fixed-point equation]

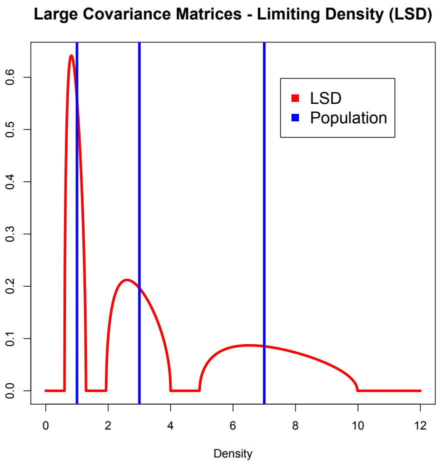
Remark (to keep in mind for later):

At fixed finite N,

a good approximation for the distribution of $x_1, \ldots, x_N$

is the deterministic equivalent $\mu_{(\nu_N, \frac{n}{N})}$.

Examples: $\nu = \frac{1}{3}(\delta_1 + \delta_3 + \delta_7)$ and $\gamma = \frac{1}{10}$



1) Local behavior

Question: Fluctuations of the extremal eigenvalues at each edge ?

More precisely,

- Can we identify the extremal eigenvalues ?
- Law of the fluctuations ?
- Given several extremal eigenvalues,
  asymptotic independence of the fluctuations ?

1) Local behavior, $\mathbf{\Sigma}_N = I_N$

Example: The non-correlated case $\mathbf{\Sigma}_N = I_N$ $\quad \Longrightarrow \quad \nu = \delta_1$

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- Limiting support (ignoring the Dirac mass at zero):

$$\mathrm{Supp}\ \mu_{(\delta_1,\gamma)} = [\mathfrak{a}, \mathfrak{b}], \qquad \begin{cases} \mathfrak{a} & = (1-\sqrt{\gamma})^2 \\ \mathfrak{b} & = (1+\sqrt{\gamma})^2 \end{cases}$$

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- **(Geman, 80/Bai-Yin,93)**

$$x_{\min} \xrightarrow[N\to\infty]{a.s.} \mathfrak{a}, \qquad x_{\max} \xrightarrow[N\to\infty]{a.s.} \mathfrak{b}$$

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Then, for some bounded sequence (σ_N) [*varying from line to line*],

- **(Johansson,00)**,

$$N^{2/3}\sigma_N\big(x_{\max} - \mathfrak{b}_N\big) \xrightarrow[N\to\infty]{\mathcal{L}} \text{Tracy-Widom}$$

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- **(Borodin-Forrester,03)**,

$$\text{If } \gamma \neq 1, \qquad N^{2/3}\sigma_N\big(\mathfrak{a}_N - x_{\min}\big) \xrightarrow[N\to\infty]{\mathcal{L}} \text{Tracy-Widom}$$

Assume now $n = N + \alpha$ **with** $\alpha \in \mathbb{N}$ **fixed.**

$$\text{Thus} \quad \frac{n}{N} \to \gamma = 1, \qquad x_{\min} \xrightarrow[N\to\infty]{a.s.} \mathfrak{a} = 0 \qquad \text{(hard edge)}$$

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- **(Forrester,93)**,

$$N^2 \sigma_N \, x_{\min} \xrightarrow[N\to\infty]{\mathcal{L}} \text{Bessel}(\alpha)$$

Finite rank perturbation (Baik-Ben Arous-Péché,05):

$$\mathbf{\Sigma}_N = \mathrm{diag}(\underbrace{1+\varepsilon, \ldots, 1+\varepsilon}_{k}, 1, \ldots, 1), \qquad k \text{ fixed.}$$

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- If $\varepsilon > \varepsilon_c$,

$$x_{\max} \xrightarrow[N\to\infty]{a.s.} \mathfrak{b}_{jump} > \mathfrak{b}, \qquad \text{GUE(k) behavior}$$

Conclusion: Local behaviors are sensitive to the convergence

$$\nu_N = \frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j} \xrightarrow[N\to\infty]{*} \nu$$

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- **Fact:** The limiting support $\mathrm{Supp}\,\mu_{(\nu,\gamma)}$ is compact, but not necessarily connected.

- For an edge $\mathfrak{b}$ of $\mathrm{Supp}\,\mu_{(\nu,\gamma)}$, we introduce a **regularity condition**.

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Theorem (Right edges)

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- *(Existence of the extremal eigenvalue)*

  There exists a deterministic sequence $(\Phi(N))$ *such that*

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- *(Tracy-Widom fluctuations)*

  There exists a right edge $\mathfrak{b}_N$ *of* $\mu_{(\nu_N, \frac{n}{N})}$ *such that* $\mathfrak{b}_N \to \mathfrak{b}$ *and*

$$N^{2/3}\sigma_N\big(x_{\Phi(N)} - \mathfrak{b}_N\big) \xrightarrow[N\to\infty]{\mathcal{L}} \text{Tracy-Widom},$$

  for some explicit bounded sequence (σ_N).

When $\mathfrak{b}$ is the rightmost edge and there is no outliers, the Tracy-Widom fluctuations have already been obtained **(El Karoui,07)** when $\gamma \leqslant 1$, and then extended to general $\gamma \in (0, \infty)$ **(Onatski,08)**

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Theorem (Asymptotic independence)

Given two finite families of positive **regular** *left edges* $(\mathfrak{a}_i)_{i\in I}$ *and* **regular** *right edges* $(\mathfrak{b}_j)_{j\in J}$, *the associated fluctuations are asymptotically independent.*

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Corollary (Study of the Condition number)

We obtain convergence and fluctuations for

$$\kappa_N = \frac{x_{\mathrm{max}}}{x_{\mathrm{min}}}$$

in different regimes.

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Recall

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Local law (Knowles-Yin,14): One can drop the Gaussian assumption and still have independent Tracy-Widom fluctuations.

2) More precisions

2) Regularity condition

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and which analytically extends to

$$\boxed{\mathcal{D}om = \left\{ m \in \mathbb{R} : \quad m \neq 0, \quad \frac{1}{m} \notin \mathrm{Supp}(\nu) \right\}}$$

(and takes real values there).

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Characterization of $\mathrm{Supp}\,\mu_{(\nu,\gamma)}$ **(Silverstein-Choi,95)**:

Consider every (maximal) intervals $I \subset \mathcal{D}om$ where g decreases, and delete the $g(I)$'s from $\mathbb{R}$, what is left is $\mathrm{Supp}\,\mu_{(\nu,\gamma)}$ (but zero).

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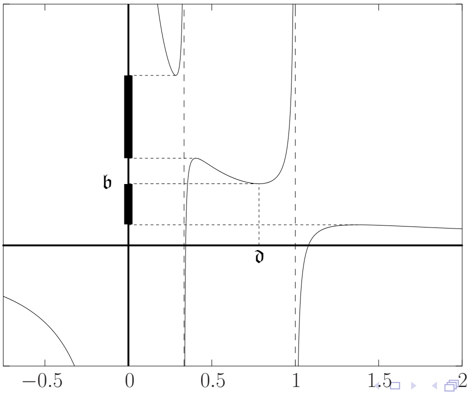
Example: $\nu = \frac{7}{10}\delta_1 + \frac{3}{10}\delta_3$ and $\gamma = \frac{1}{10}$



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Definition

We say $\mathfrak{b}$ is **regular** if

$$\liminf_{N\to\infty} \min_{j=1}^{n} \left|\mathfrak{d} - \frac{1}{\lambda_j}\right| > 0.$$

Remark: If $\mathfrak{b} = g(\mathfrak{d})$ is **regular**, then necessarily $\mathfrak{d} \notin \partial(\mathcal{D}om)$.

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- **Complex analysis (Montel, Rouché,...)**

  $\Longrightarrow$ Existence of an edge $\mathfrak{b}_N$ of $\mu_{(\nu_N, \frac{n}{N})}$ such that $\mathfrak{b}_N \to \mathfrak{b}$

2) Tracy-Widom fluctuations

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- **Determinantal structure (Johansson/BBP,05)**
  $\Longrightarrow$ Repartition function $\simeq$ Fredholm determinant, i.e.

$$\mathbb{P}\Big[N^{2/3}\sigma_N\big(x_{\Phi(N)} - \mathfrak{b}_N\big) \leqslant s\Big] = \det\big(I - \mathrm{K}_N\big)_{L^2(s,\varepsilon N^{2/3})} + o(1)$$

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- **Complex integral representation for K_N (idem):**

$$\mathrm{K}_N(x,y) = \frac{1}{(2i\pi)^2}\oint_\Gamma \mathrm{d}z \oint_\Theta \mathrm{d}w \;\; \mathrm{F}_N(x,y;z,w),$$

where F_N is explicit.

2) Tracy-Widom fluctuations

Asymptotic analysis as $N \to \infty$ for

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- **Local analysis around $\mathfrak{d} \Rightarrow$ Airy kernel**
  - Saddle point of order two, almost routine computation

- **The remaining of the integral is negligible**
  - Clever analytic deformation of Γ and Θ,
  this is the HARD part

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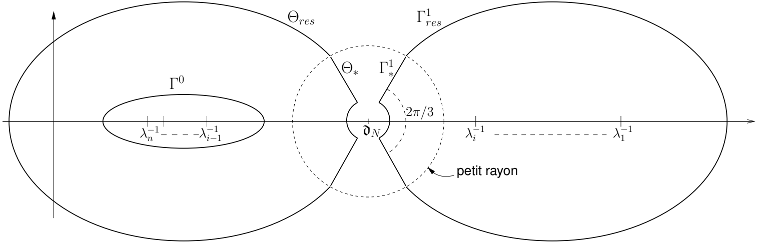

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Existence of the steepest descent contours ?

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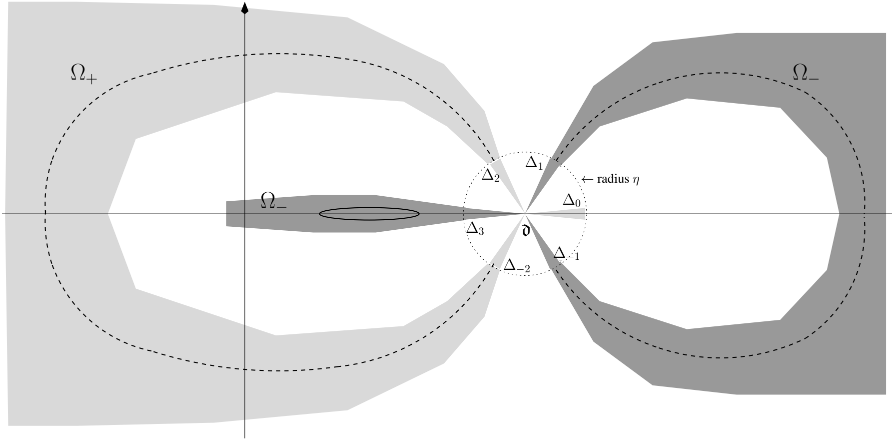
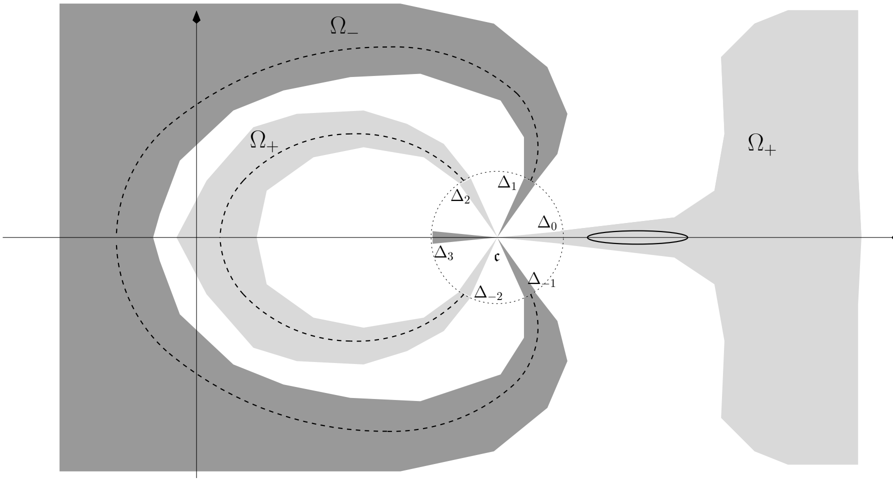
Non-constructive proof using the

maximum principle for subharmonic functions

"Right edge" analytic landscape:



"Left edge (generic)" analytic landscape:



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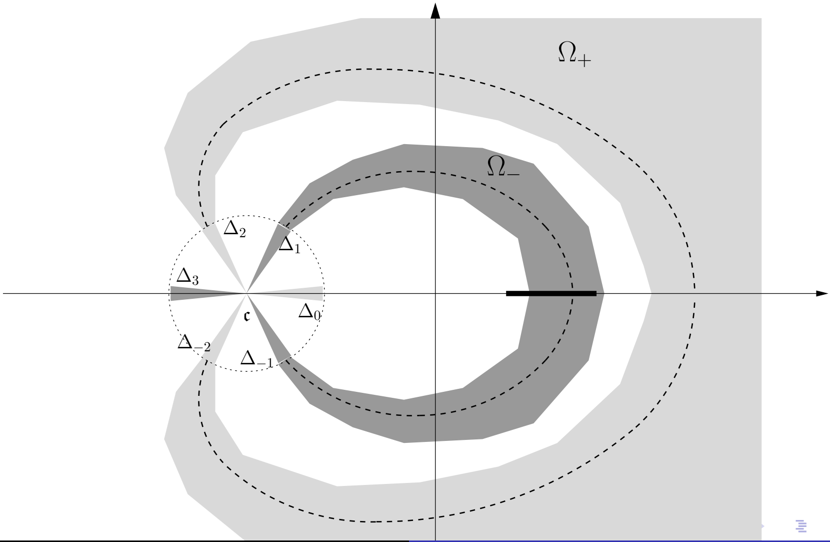

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The prove the asymptotic independence:

- **(Bornemann,10)** Asymptotic independence $\Longleftrightarrow$ Some matrix valued operator becomes asymptotically diagonal in the trace-class norm.
- Use regularized Fredholm determinant $\Longrightarrow$ "Trace-class norm" $\rightarrow$ "Hilbert-Schmidt norm"
- Use the steepest descent paths from the TW analysis.

The prove the asymptotic independence:

- **(Bornemann,10)** Asymptotic independence $\iff$
  Some matrix valued operator becomes asymptotically diagonal in the trace-class norm.
- Use regularized Fredholm determinant $\Longrightarrow$
  "Trace-class norm" $\rightarrow$ "Hilbert-Schmidt norm"
- Use the steepest descent paths from the TW analysis.
- Use Bleher-Kuijlaars representation for $\mathrm{K}_N(x, y)$

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- Asymptotic analysis (now easy)

3) Beyond Universality

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Similar situations:

- Additive perturbations of Wigner matrices **(Capitaine-Péché,14)**
- Random paterns on the Gelfand-Tsetlin cone **(Duse-Metcalfe,14)**

Thank you for your attention !