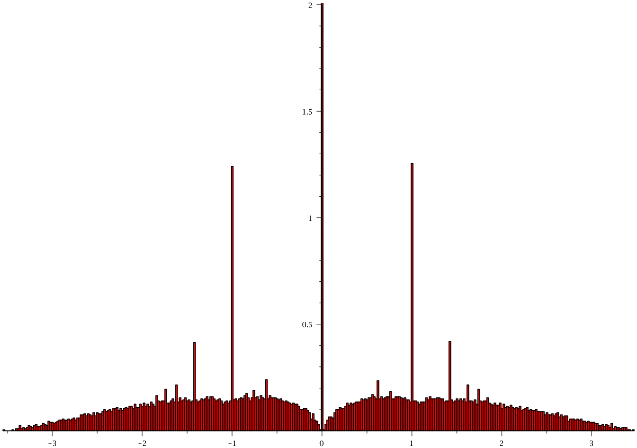
Atoms in the limiting spectrum of sparse graphs

Justin Salez (LPMA)



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Question: How does μ_G **typically** look when G is large ?

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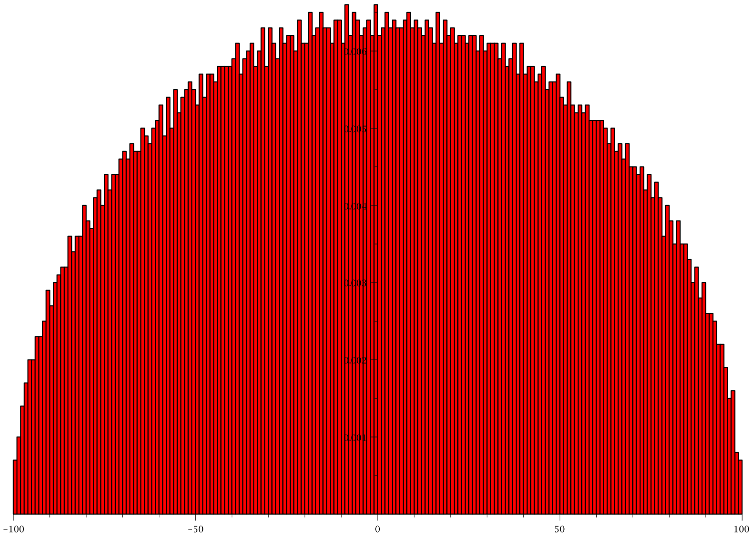

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- What about **sparse graphs**: $|E| \asymp |V|$?

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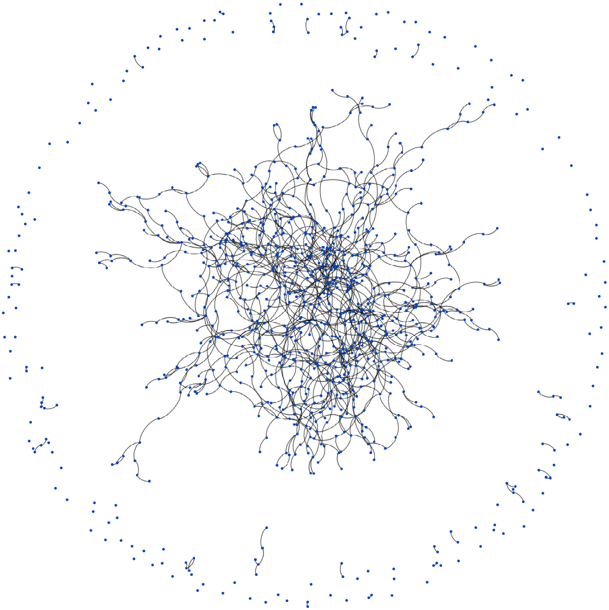

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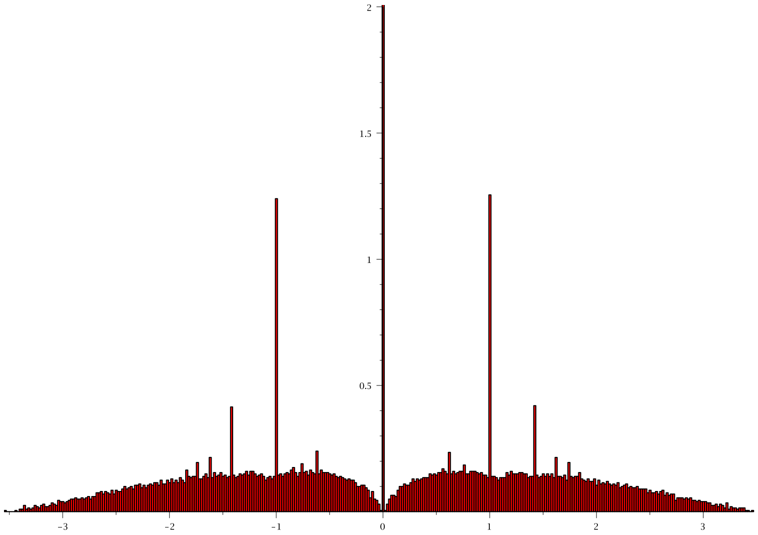

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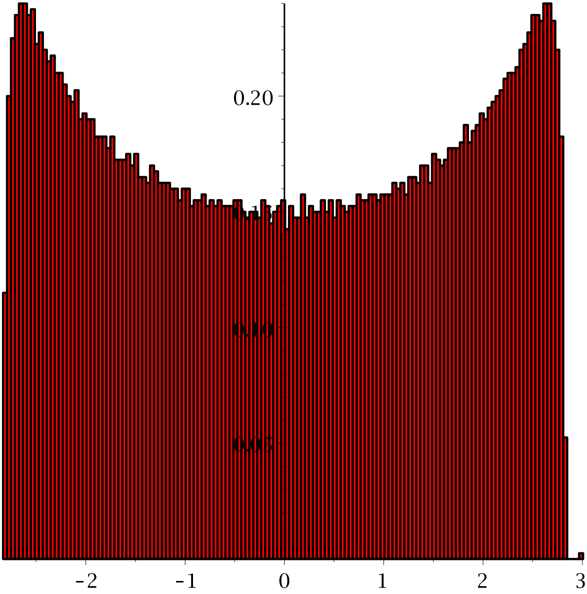

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Actually, this phenomenon is just one of the many consequences of the fact that the **underlying local geometry** converges.

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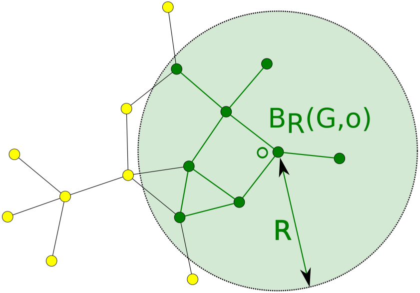

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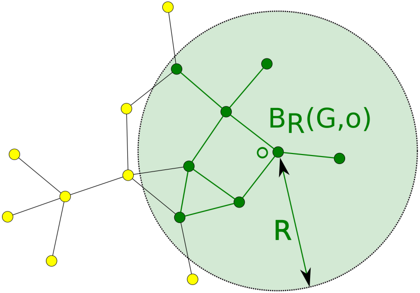

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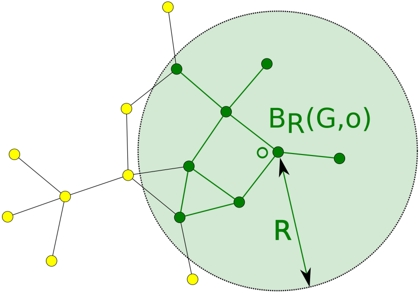

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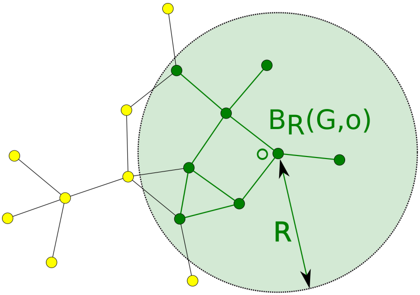

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▷ $\mathcal{L}$ describes the local geometry of G_n around a random node.

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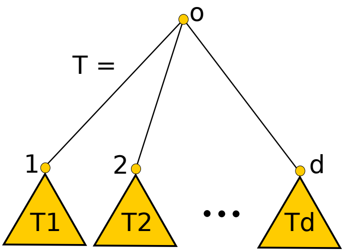
Fact: $\boxed{G_n \xrightarrow[n\to\infty]{loc.} \mathcal{L} \qquad \Longrightarrow \qquad \mu_{G_n} \xrightarrow[n\to\infty]{} \mu_{\mathcal{L}}}$

RECURSION IN THE CASE OF TREES



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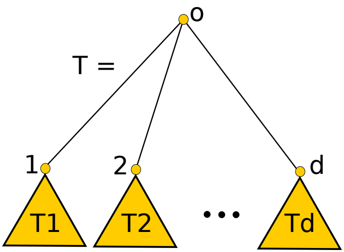

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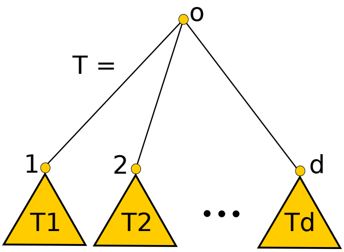

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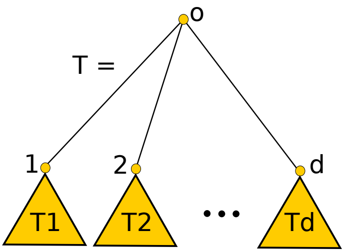

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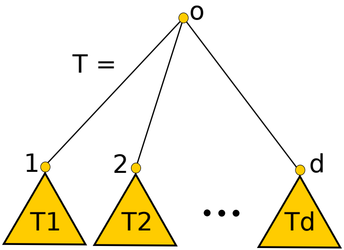

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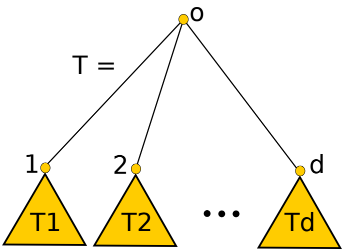

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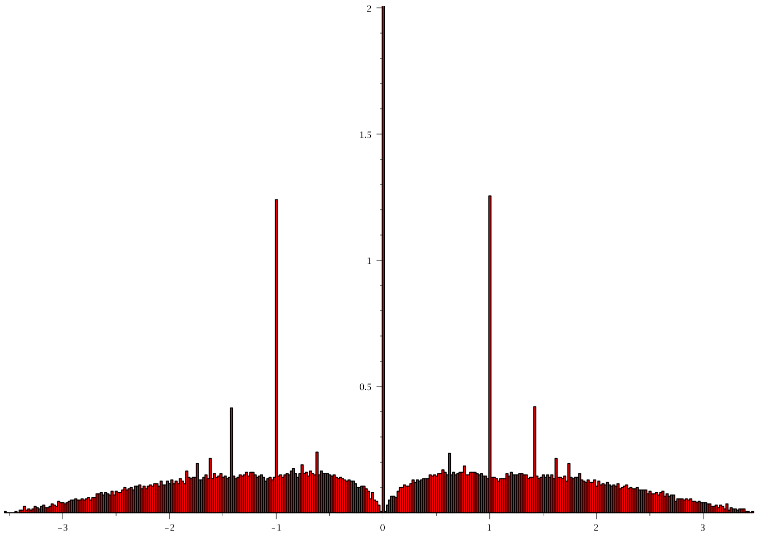
- Example: computation of $\mu_{\mathcal{L}}(\{0\})$ (Bordenave-Lelarge-S. '11)


2
1.5
1
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−3
−2
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Theorem (Bordenave-Sen-Virag'13): $\mu_{pp}(\mathbb{R}) < 1$ as soon as $c > 1$

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$\triangleright$ all tree eigenvalues are atoms of $\mu_{\mathcal{L}}$ (e.g. $0, 1, \sqrt{3}, 2\cos\frac{2\pi}{5}, \ldots$)

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In particular, $\mu_{\mathcal{L}}(\{\lambda\}) = 0$ unless λ is a **totally real algebraic integer** ($=$ root of some real-rooted monic integer polynomial).

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Corollary: many graph limits have the set of totally real algebraic integers as atomic support. This includes all *Galton-Watson trees* with $\text{supp}(\nu) = \mathbb{N}$, as well as the *Infinite Skeleton Tree*.

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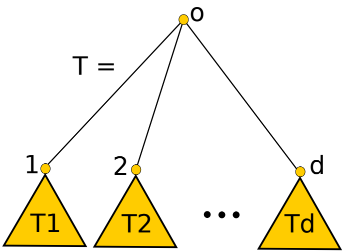

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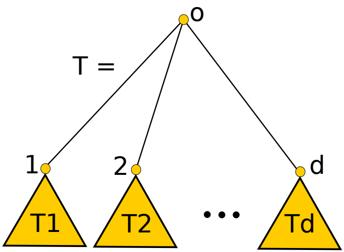

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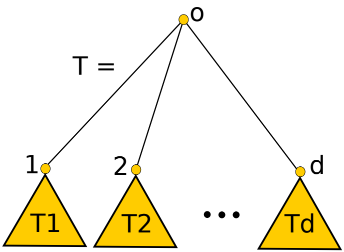

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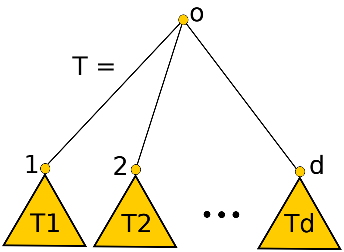

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$\triangleright$ $\lambda \neq 0$ is a tree eigenvalue $\Longleftrightarrow$ 1 can be generated from 0 by repeated applications of $(x_1, \ldots, x_d) \mapsto \frac{1}{\lambda^2}\sum_i \frac{1}{1-x_i}$ $(d \in \mathbb{N})$.

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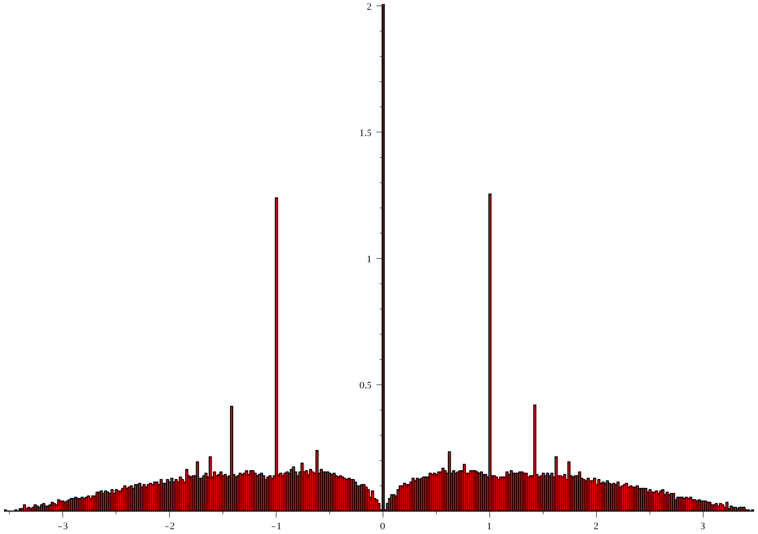
Corollary: λ is a tree eigenvalue !



2
1.5
1
0.5
−3
−2
−1
0
1
2
3

Thank you for your attention !