CLT for eigenvalue counting function of orthogonal and symplectic matrix models

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CLT for different random matrix ensembles

Let M_n be an ensemble of random $n \times n$ Hermitian or real symmetric matrices (Wigner matrices, band matrices, sample covariance matrices, sparse matrices, etc.). Consider the eigenvalues $\{\lambda_j\}_{j=1}^n$ of M_n

Linear eigenvalue statistics (LES) of the test function h and the counting function of eigenvalues

$$\mathcal{N}_n[h] = \sum_{i=1}^{n} h(\lambda_i), \quad N_n[a] = \sharp\{\lambda_i \le a\} = \mathcal{N}_n[1_{(-\infty,a]}]$$

The most important question of global regime

- $\lim_{n\to\infty} E\{n^{-1}\mathcal{N}_n[h]\} = \int h(\lambda)d\mu(\lambda)$,
  $Var\{\mathcal{N}_n[h]\} \sim d_n << n^2 \Rightarrow Var\{n^{-1}\mathcal{N}_n[h]\} \to 0$;
- CLT: $v_n := d_n^{-1/2}\Big(\mathcal{N}_n[h] - E\{\mathcal{N}_n[h]\}\Big) \to$ Gaussian random variable

Example of CLT for smooth test function

$M_n = n^{-1}XX^*$, $\{X_{ij}\}_{i=1,\dots n, j=1,\dots,m}$ – i.i.d., $E\{X_{jk}\} = 0$, $E\{|X_{jk}|^2\} = 1$

$$E\{X_{jk}^4\} = X_4, \quad E\{|X_{jk}|^{4+\varepsilon_1}\} = X_{4+\varepsilon_1} < \infty, \ \varepsilon_1 > 0.$$

Introduce the characteristic functional

$$\Phi_n[x,h] := E\{e^{ix(\mathcal{N}_n[h] - E\{\mathcal{N}_n[h]\})}\}$$

CLT for SCM, [BS:04],[LP:09],[S:11]

Let h have 2 derivatives. Then $\Phi_n[x,h] \to e^{-x^2/2V_{SC}[h]}$, as $m, n \to \infty$, $m/n \to c \geq 1$.

$$V_{SC}[h] = \frac{1}{2\pi^2}\int_{a_-}^{a_+}\int_{a_-}^{a_+}\left(\frac{\Delta h}{\Delta \lambda}\right)^2 \frac{(4c - (\lambda_1 - a_m)(\lambda_2 - a_m))d\lambda_1 d\lambda_2}{\sqrt{4c - (\lambda_1 - a_m)^2}\sqrt{4c - (\lambda_2 - a_m)^2}} + \frac{\kappa_4}{4c\pi^2}\Big(\int_{a_-}^{a_+}\varphi(\mu)\frac{\mu - a_m}{\sqrt{4c - (\mu - a_m)^2}}d\mu\Big)^2,$$

where $\frac{\Delta h}{\Delta \lambda} = \frac{h(\lambda_1) - h(\lambda_2)}{\lambda_1 - \lambda_2}$, $\kappa_4 = X_4 - 3$, $a_\pm = (1 \pm \sqrt{c})^2$, $a_m = \frac{1}{2}(a_+ + a_-)$.

β matrix models

Distributions in $\mathbb{R}^n$, depending on the function V and $\beta > 0$

$$p_{n,\beta}(\lambda_1, ..., \lambda_n) = Z_{n,\beta}^{-1}[V] \prod_{j=1}^{n} e^{-\beta V(\lambda_j)/2} \prod_{j<k} |\lambda_j - \lambda_k|^{\beta},$$

where $Z_{n,\beta}[V]$ (partition function) is the normalizing constant

$$V(\lambda) > (1+\varepsilon)\log(1+\lambda^2).$$

For $\beta = 1, 2, 4$ it is a joint eigenvalues distribution of real symmetric, hermitian and symplectic matrix models respectively.

$$E\Big\{\Phi(\lambda_1, ..., \lambda_n)\Big\} := \int \Phi(\lambda_1, ..., \lambda_n) p_{n,\beta}(\lambda_1, ..., \lambda_n) d\lambda_1 \dots d\lambda_n$$

Characteristic functional

$$\Phi_{n,\beta}[x,h] = E\left\{e^{x(\mathcal{N}_n[h]-E\{\mathcal{N}_n[h]\})}\right\}.$$

Theorem [Jo:98, SK:10] CLT for LES in the one cut case

V is real analytic, $\sigma = [a,b]$, and ρ is "generic". Then for any $h : \mathbb{R} \to \mathbb{R}$ with $||h^{(4)}||_\infty, ||h'||_\infty \le \log n$

$$\Phi_{n,\beta}[h] = \exp\left\{\frac{x^2}{2\beta}(\overline{D}_\sigma h, h)\right\}\Big(1 + n^{-1}O\big(||h^{(4)}||_\infty^3\big)\Big)$$

where the "variance operator" $\overline{D}_\sigma$ and the measure ν have the form

$$(\overline{D}_\sigma h, h) = \int_\sigma \frac{h(\lambda)d\lambda}{\pi^2 X_\sigma^{1/2}(\lambda)} \int_\sigma \frac{h'(\mu)X_\sigma^{1/2}(\mu)d\mu}{\lambda - \mu},$$

$$X_\sigma(\lambda) = (b-\lambda)(\lambda - a)$$

CLT for indicators in the case $\beta = 2$

Characteristic functional for indicators ($\beta = 2$)

Consider any smooth (having $2 + \epsilon$ derivatives) potential $V >> \log|\lambda^2 + 1|$, take $\Delta_a = [-\infty, a]$ and set $x_n = x\pi/\log^{1/2} n$. Then it is known that

$$\hat{\Phi}_{n,\beta}(x) = \Phi_{n,\beta}[x_n, 1_{\Delta_a}] = \det\Big\{1 + (e^{x_n} - 1)K_n[a]\Big\}e^{-x_n E\{\mathcal{N}_n[1_{\Delta_a}]\}}$$

where

$$K_n(\lambda, \mu) = \sum_{k=0}^{n-1} \psi_k^{(n)}(\lambda)\psi_k^{(n)}(\mu)$$

$$\psi_k^{(n)}(\lambda) = P_k^{(n)}(\lambda)e^{-nV(\lambda)/2}, \quad \int_{\Delta_a} P_k^{(n)}(\lambda)P_m^{(n)}(\lambda)e^{-nV(\lambda)}d\lambda = \delta_{km}$$

$$(K_n[a]f)(\lambda) := \int_{\Delta_a} K_n(\lambda, \mu)f(\mu)d\mu,$$

$$K_n[a](\lambda, \mu) := 1_{\Delta_a}(\lambda)K_n(\lambda, \mu)1_{\Delta_a}(\mu)$$

Theorem (CLT for $\beta = 2$) [So:00]

$$\lim_{n\to\infty} \log \hat{\Phi}_{n,\beta}(x) = \frac{x^2}{2}$$

Proof.

$$\begin{aligned} F(x_n) :=& \log \hat{\Phi}_{n,\beta}(x) = \mathrm{Tr}\ \log(1 + (e^{x_n} - 1)K_n[a]) - nx_n E\{1_{\Delta_a}\} \\ =& \frac{x_n^2}{2} \mathrm{Tr}\ K_n[a](1 - K_n[a]) + \frac{x_n^3}{6} \mathrm{Tr}\ K_n[a](1 - K_n[a]) R(K_n[a]) \\ & C_1 \le R(t) \le C_2, \quad t \in [0,1] \end{aligned}$$

It is easy to check that

$$\mathrm{Tr}\ K_n[a](1 - K_n[a]) = \int_{\Delta_a} d\lambda \int_{\bar{\Delta}_a} K_n^2(\lambda, \mu) d\mu = \pi^{-2} \log n(1 + o(1))$$

CLT for indicators in the case $\beta = 1$, n-even

Characteristic functional for indicators ($\beta = 1$)

$$\hat{\Phi}_n(x) = \Phi_{n,\beta}[x_n 1_{\le a}] = \det{}^{1/2}\Big\{1 + (e^{x_n} - 1)\hat{K}_n[a]\Big\} e^{-x_n E\{\mathcal{N}_n[1_{\Delta_a}]\}}$$

where

$$\hat{K}_n[a](\lambda,\mu) := 1_{\Delta_a}(\lambda)\hat{K}_n(\lambda,\mu)1_{\Delta_a}(\mu)$$

$$\hat{K}_n(\lambda,\mu) = \begin{pmatrix} S_n(\lambda,\mu) & D_n(\lambda,\mu) \\ \mathcal{I}_n(\lambda,\mu) - \epsilon(\lambda-\mu) & S_n^T(\lambda,\mu) \end{pmatrix}$$

Kernels S_n, $\mathcal{I}_n$, D_n, and S_n^T

$$S_n(\lambda,\mu) = -\sum_{j,k=0}^{n-1} \psi_j^{(n)}(\lambda)(M^{(n)})^{-1}_{jk}(\epsilon\psi_k^{(n)})(\mu), \quad n - \text{even}$$

$$\mathcal{I}_n = (\epsilon S)(\lambda,\mu), \quad D_n(\lambda,\mu) = -\frac{\partial}{\partial\mu} S_n(\lambda,\mu), \quad S_n^T(\lambda,\mu) = S_n(\mu,\lambda)$$

$$\epsilon(\lambda-\mu) := \frac{1}{2}\text{sign}(\lambda-\mu), \quad \epsilon f(\lambda) := \int_{\Delta_a} \epsilon(\lambda-\mu) f(\mu)\, d\mu$$

The $n \times n$ matrix $M^{(n)}$ has the form

$$M^{(n)}_{jk} = (\psi_j^{(n)}, \epsilon\psi_k^{(n)}), \quad j,k = 0,\dots n-1$$

Remark

It is known that in the case $\beta = 1$ the characteristic functional can be written in the form

$$\hat{\Phi}_n(x) = \det{}^{1/2}\Big\{J + (e^{x_n} - 1)\hat{A}_n[a]\Big\}e^{-x_n E\{\mathcal{N}_n[1_{\Delta_a}]\}}$$

where

$$(A_n[a])^* = -A_n[a], \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Eigenvalue problem for $\hat{K}_n[a]$ ($\beta = 1$)

$$\begin{cases} S_n f_{\Delta_a} + D_n g_{\Delta_a} = E f_{\Delta_a} \\ \mathcal{I}_n f_{\Delta_a} - \epsilon f_{\Delta_a} + S_n^T g_{\Delta_a} = E f_{\Delta_a} \end{cases} \qquad (1)$$

$$\Rightarrow \begin{cases} \mathcal{I}_n f_{\Delta_a} + S_n^T g_{\Delta_a} = E\epsilon f_{\Delta_a} + (f_{\Delta_a}, \Psi_a) \\ \mathcal{I}_n f_{\Delta_a} - \epsilon f_{\Delta_a} + S_n^T g_{\Delta_a} = E g_{\Delta_a} \end{cases}$$

$$\Rightarrow E g_{\Delta_a} = (E-1)(\epsilon f_{\Delta_a}) - (f_{\Delta_a}, \Psi_a)$$

Multiply the first line of (1) by E and use the above equation for $E g_{\Delta_a}$. Since

$$D_n(\epsilon f_{\Delta_a})(\lambda) = S_n f_{\Delta_a}(\lambda) - S_n(\lambda, a)(\epsilon f_{\Delta_a})(a)$$

$$\Rightarrow (2E-1) S_n f_{\Delta_a} - E^2 f_{\Delta_a} + E P_1 f_{\Delta_a} - P_2 f_{\Delta_a} = 0$$

Hence the solutions $\{E_k\}$ of (1) are solutions of the equation

$$\mathcal{P}(E) := \det\Big\{ E^2 - (2E-1) S_n - E P_1 + P_2 \Big\} = 0$$

It is evident that $\mathcal{P}(\mathrm{E})$ is the polynomial of 2nth degree, and $\mathrm{E_k}$ are the roots of $\mathcal{P}(\mathrm{E})$. We are interested in

$$\prod_{k=1}^{2n}(1+\delta_n E_k) = \delta_n^{2n}\prod_{k=1}^{2n}(\delta_n^{-1}+E_k) = \delta_n^{2n}\mathcal{P}(-\delta_n^{-1}),$$

where $\delta_n := e^{x_n} - 1$. Thus we obtain

$$\hat{\Phi}_n(x) = e^{-x_n E\{\mathcal{N}_n[1_{\Delta_a}]\}} \det{}^{1/2}\Big\{1 + (2\delta_n + \delta_n^2)S_n + \delta_n P_1 + \delta_n^2 P_2\Big\}$$

But it was proven before (see [S:12]) that

$$S_n(\lambda,\mu) = K_n(\lambda,\mu) + \sum_k d_k Q_n^{(k)}(\lambda,\mu), \quad |d_k| \le Ce^{-kc}$$

$$\mathrm{rank}\ Q_n^{(k)} = 1.$$

Hence, finally we get

$$\begin{aligned}\hat{\Phi}_n(x) &= e^{-x_n E\{\mathcal{N}_n[1_{\Delta_a}]\}} \det{}^{1/2}\Big\{1 + (2\delta_n + \delta_n^2)K_n\Big\}(1 + O(\delta_n)) \\ &= e^{-x_n E\{\mathcal{N}_n[1_{\Delta_a}]\}} \det{}^{1/2}\Big\{1 + (e^{2x_n} - 1)K_n\Big\}(1 + O(\delta_n))\end{aligned}$$

Then

$$\log \hat{\Phi}_n(x) = \log \det{}^{1/2}\Big\{1 + (e^{2x_n} - 1)K_n\Big\} - x_n E\{\mathcal{N}_n[1_{\Delta_a}]\} + o(1)$$
$$= \frac{1}{2}\mathrm{Tr}\log(1 + (e^{2x_n} - 1)K_n) - x_n E\{\mathcal{N}_n[1_{\Delta_a}]\} + o(1)$$

Similarly to the case $\beta = 2$ we have

$$\frac{1}{2}\mathrm{Tr}\ \log(1 + (e^{2x_n} - 1)K_n) = x_n E\{\mathcal{N}_n[1_{\Delta_a}]\} + x_n^2 \mathrm{Tr}\ K_n(1 - K_n) + \frac{(2x_n)^3}{12}\mathrm{Tr}\ K_n(1 - K_n)R(K_n[a])$$

From the case $\beta = 2$, we know that

$$\mathrm{Tr}\ K_n(1 - K_n) = \pi^{-2}\log n\,(1 + o(1))$$

Finally we obtain the theorem

Theorem 1 (CLT for indicators with $\beta = 1$ and n-even)[S:15]

If V is a analytic potential $V(\lambda >> \log(1+\lambda^2)$, then

$$\log \hat{\Phi}_n(x) = x^2 + O(x^3 \log^{-1/2} n)$$

Theorem 2 (CLT for indicators with $\beta = 4$ and n-even)

Assuming the same conditions on V, we have

$$\log \hat{\Phi}_n(x) = x^2/8 + O(x^3 \log^{-1/2} n)$$