Thresholds

PLAN (no plan survives first contact with the enemy)

basics

first moment does

first moment doesn't → coupon collection, "gap"

too many examples

KK Conj → Talagrand

KK Conj ↔? Friedgut

"containers" (BMS/ST)

Shamir

Gen'l $2^X = \{\text{subsets of } X\}$ a.a. $\boxed{\text{FINITE}}$

$\mathcal{F} \subseteq 2^X$ ("family," "property")

(usually) increasing: $B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$

μ_p prod. prob. meas. on 2^X:

$$\mu_p(A) = p^{|A|}(1-p)^{|X \setminus A|} \qquad A \subseteq X$$

notation:

$A \sim \mu_p \equiv A = X_p$

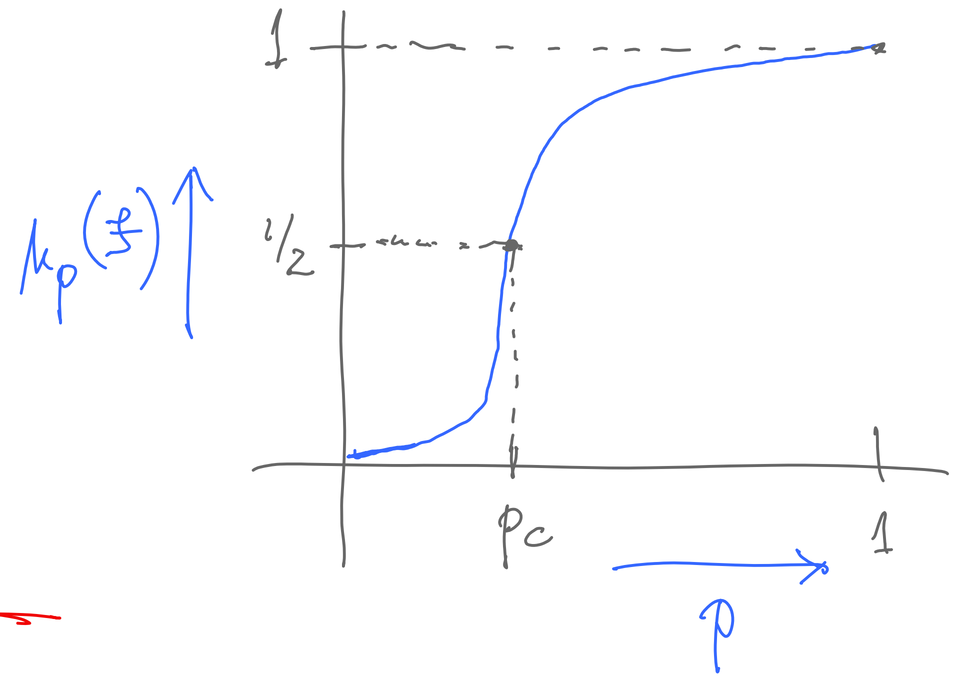
EX (obvious...)

$\mathcal{F}$ incr. $\Rightarrow$ $\mu_p(\mathcal{F})$ incr. in p

threshold:

$p_c = p_c(\mathcal{F})$ does

$$\mu_{p_c}(\mathcal{F}) = 1/2$$



- usually hope for $\Theta(p_c)$ (order of mag.)
  (sometimes do better)

- also: how sharp?

E.g. **k-SAT**

X = k-clauses from $\{x_1, \dots, x_n\}$

$x_3 \vee \bar{x}_5 \vee x_{22}$ $\quad (k=3)$

$Y \subseteq X$ is satisfiable if ...

$(x_1 \vee \bar{x}_3 \vee x_4) \wedge (x_2 \vee x_3 \vee x_5) \wedge (\bar{x}_1 \vee \bar{x}_4 \vee \bar{x}_5)$:

YES: $(1,0,0,0,1)$

$\mathcal{F} = \{$ unsatisfiables $\}$ (incr.)

Notes 1 "Easy":

$$p_c = \Theta\left(n^{-(k-1)}\right) \longleftrightarrow \underline{\Theta(n) \text{ clauses}}$$

[Here usually consider not p but
M = <u>number</u> of clauses (but unimp.)]

2 note "hidden parameter" n:

$$X = X_n, \quad \mathcal{F} = \mathcal{F}_n, \quad p = p_n \quad \ldots$$

we do this frequently (usually)

original <u>Erdős-Renyi def</u>*:

(incr)

$p^* = p^*_n$ is a <u>threshold fn</u> for $\mathcal{F} = (\mathcal{F}_n)$ if

$$\mu_p(\mathcal{F}_n) \longrightarrow \begin{cases} 0 & \text{if } p = o(p^*) \\ 1 & \text{if } p = \omega(p^*) \end{cases}$$

$(p = p_n)$

[th. fn unique <u>only up to constant</u>;

nonetheless "the" threshold]

* for <u>random graphs</u>

$\Rightarrow$ Thm (Bollobás–Thomason '87)

Every increasing $\mathcal{F}$ has a th. fn.

Surprising but in retrospect fairly obvious

— essentially Kruskal–Katona

⌈ $p^* = p^*_n$ is a <u>threshold fn</u> for $\mathcal{F} = (\mathcal{F}_n)$ if

$$\mu_p(\mathcal{F}_n) \longrightarrow \begin{cases} 0 & \text{if } p = o(p^*) \\ 1 & \text{if } p = \omega(p^*) \end{cases}$$ ⌋

<u>EX</u> (JŁR* P1.23):

$p^*_n = p_c(\mathcal{F}_n)$ is always a th. fn.

* Janson-Łuczak-Ruciński

E.g. **Random graph:** ["binomial RG"

$G = G_{n,p}$: $V = [n] = \{1, \dots, n\}$

$\Pr(xy \in E) = p$ indeptly.

"Random graphs are dead"

— JK, early 90's*

* according to P.M. Kayll, RU '94

$G_{n,p} \longleftrightarrow \quad X = \binom{V}{2}$

$2^X = \{\text{graphs on } V\}$

$\mathcal{F}$: incr'g property of gphs (on V)

isomorphism-invar

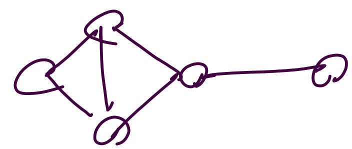
[else why "graph" ??

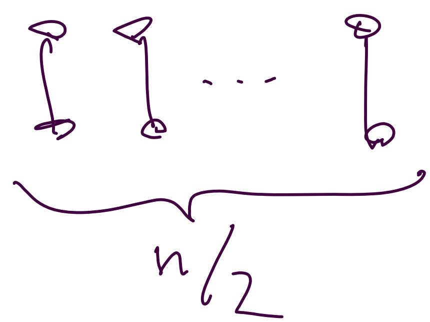
Examples (of incr. gph. properties)

$\mathcal{F}_H = \{ G \text{ contains } H \}$

a copy of

"target" graph

e.g. $H =$ 

e.g. $H =$ 

"perfect matching"

(examples)

- connected
- $\chi(G) > k$

  chromatic #
- **Ramsey:** $\forall\ E(G) = R \cup B$

  $\exists$ monochromatic $\triangle$

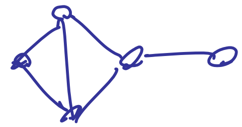
WARNING: LANGUAGE:

"when does G have $\mathcal{F}$?"

probably

for what p $(= p_n)$

($G_{n,p}$ for now $\longrightarrow$)

E.g. 1. $H =$ 

(copy of)

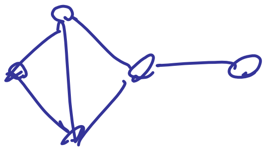
$\mathcal{F} = \mathcal{F}_H := \{\text{contain } H\}$

1st try: $\mathbb{E}[\#\text{ of } H\text{'s}] \overset{*}{=} \Theta(n^5 p^6)$

$\longrightarrow \infty$ if $p \gg n^{-5/6}$

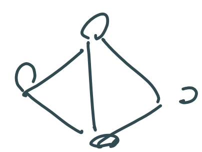
(* linearity of $\mathbb{E}$)

$\longrightarrow$

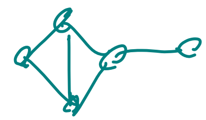
[$H =$  ; $\mathbb{E}[\#\text{ of } H\text{'s}] = \Theta(n^5 p^6)$

is $n^5 p^6$ a th. fn. for $\mathcal{F}_H$?]

NO: if $n^{-5/6} \ll p \ll n$

then $\mu_p(G \supseteq$  $) \approx 0$

TRUTH: $p_c(\mathcal{F}) = \Theta(n^{-4/5})$

Pf sketch ($H =$  : $p_c(\mathcal{F}_H) = \Theta(n^{-4/5})$)

$H_1, \dots, H_m$ copies of H in K_n . . .

$X_i = X_i^{(n)} = \mathbb{1}_{\{G \supseteq H_i\}}$

$X = \sum X_i = \#$ of H's in G

want:

$p = \omega(n^{-4/5}) \Longrightarrow X \neq 0$ a.s. . . .

"second moment"

<u>Chebyshev</u>: (any X:)

$$\Pr(|X - \mathbb{E}X| > \lambda) < \sigma_X^2 / \lambda^2$$

<u>weak</u> for "nice" X

(e.g. $X \sim \left\{ \begin{array}{c} \text{binom.} \\ N(0,1) \end{array} \right\} \to \Pr \approx e^{-\lambda^2/2\sigma^2}$)

<u>but</u>
- always true
- "easy" to use
- surprisingly powerful

$$\left[\quad X = \sum X_i = \#\text{ of } H\text{'s in } G \right.$$

$$\left. \Pr(|X - \mathbb{E}X| \geq \lambda) \leq \sigma_X^2 / \lambda^2 \quad \right]$$

ETS $\sigma_X^2 = o(\mathbb{E}^2 X)$

EQUIV $\mathbb{E}X^2 \sim \mathbb{E}^2 X$ $\quad [\sigma_X^2 = \mathbb{E}X^2 - \mathbb{E}^2 X]$

...

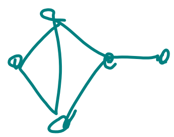
$\mathbb{E}X^2 = \sum\sum$

only pairwise interactions

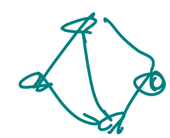
vs. $\mathbb{E}^2 X = \sum\sum$

$$\mathbb{E}X^2 = \sum\sum \mathbb{E}X_iX_j \overset{?}{\sim} \mathbb{E}^2X = \sum\sum \mathbb{E}X_i\,\mathbb{E}X_j$$

☺ most pairs $\boxed{\text{indept}} \to \mathbb{E}X_iX_j = \mathbb{E}X_i\,\mathbb{E}X_j$

☹ ( :)

$\sum\sum \mathbb{E}X_iX_j$ includes term like

$n^6p^7 \longleftrightarrow$ pairs sharing 

$\longrightarrow$ need $n^6p^7 \ll n^{10}p^{12}$ (etc.)

Erdős - Rényi '60 }
& Bollobás '81 } $\underline{\text{same}}$ $\forall$ $\underline{\text{fixed}}$ H

"hardest" subgraph gives $p_c(\mathcal{F}_H)$

(here we know $\underline{\text{much}}$ more ...)

[digressing: more 2nd m.m. & another where 1st moment tells the truth.]

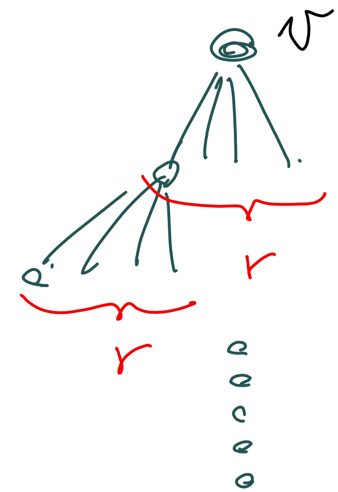
G infinite (loc. fin.), $v \in V(G)$ "root"

$\Theta(p) = \Pr_p(\exists \text{ inf. path from } v)$

$p_c = p_c(G) = \sup\{p : \Theta(p) = 0\}$

<u>E.g.</u> r-branching tree:

$p_c = 1/r$



[G infinite (loc. fin.), $v \in V(G)$ "root"]

$\Pi \subseteq V$ __cutset__ if "__separates v from ∞__"

(i.e. cpt. of v in $G - \Pi$ finite)

$\overset{}{\underset{\text{EX}}{\Longleftrightarrow}}$ Π meets all inf. paths from v

[$\theta(p) = \Pr_p(\exists \text{ inf. path from } v)$

$p_c = p_c(G) = \sup\{p : \theta(p) = 0\}$]

$\forall \Pi, p: \quad \theta(p) \le \Pr_p(C_v \cap \Pi \neq \phi) \le \mathbb{E}_p |C_v \cap \Pi|$

C_v: cpt (cluster)

$\longrightarrow$

$$p_{cut}(G) := \sup\{p : \inf_{\Pi} \mathbb{E}_p |C_v \cap \Pi| = 0\}$$

◎ sup. of p's for which gives $\theta(p) = 0$

◎ TRIVIAL L.B.: $p_{cut} \le p_c$ $(\forall G)$

$\left[\; p_{cut}(G) \quad \sup\{p : \inf_{\Pi} \mathbb{E}_p |C_v \cap \Pi| = 0\} \right.$

$\left. \text{TRIV} : \underline{p_{cut} \le p_c} \right]$

Thm (Russ Lyons 89)

$G = \underline{\text{tree}} \implies p_{cut}(G) = p_c(G)$

Pf sketch

SHOW: $p > p_{cut} \implies$

$\Pr_p(C_v \cap \Pi \neq \emptyset) > \varepsilon = \varepsilon_p \quad \underline{\forall \Pi}$

$$\left[\, p > p_{cut} \overset{?}{\Longrightarrow} \Pr_p(C_v \cap \Pi \neq \phi) > \varepsilon \;(= \varepsilon_p) \,\right]$$

weights: $\alpha : \Pi \longrightarrow \mathbb{R}^+$ (TBA)

$$X = \sum_{\omega \in \Pi} \alpha_\omega \underbrace{\mathbb{1}_{\{v \leftrightarrow \omega\}}}_{X_\omega}$$

want: $\Pr_p(X \neq 0) \overset{?}{>} \varepsilon$

2nd m.m. ? note this $\not\longrightarrow 1$

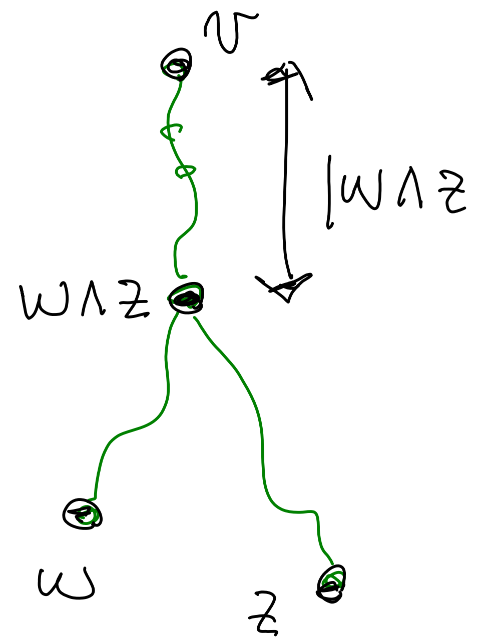
$\rightsquigarrow$?

$$\left[\; X = \sum_{\omega \in \Pi} \boxed{X_\omega} = \alpha_\omega \mathbb{1}_{\{v \leftrightarrow \omega\}} \qquad \underline{\text{want}} \quad \underline{Pr_p(X \neq 0) > \varepsilon} \;\right]$$

Improved Cheb:

$$\mathbb{E} X_\omega X_z = \mathbb{E} X_\omega \mathbb{E} X_z \, p^{-|\omega \wedge z|}$$

$\mathbb{E} = \mathbb{E}_p$



for $\mathbb{E} X^2$ "small" want

α "spread": not too much

wt on $\{\omega, z\}$'s $\overline{\omega}$ $|\omega \wedge z|$ large

⌈ goal: α "spread" ⌋

use:

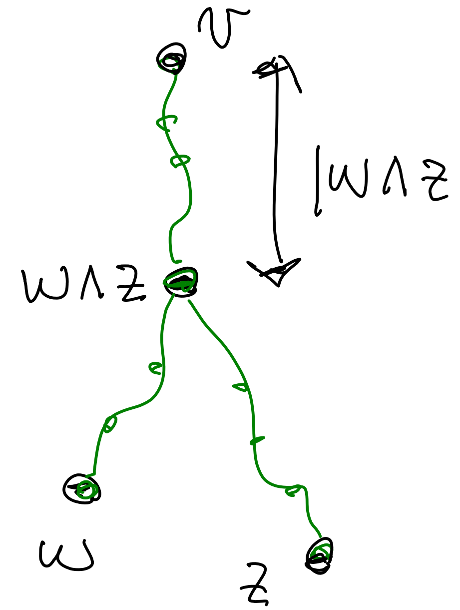
① $p >$ [illegible] $> p_{cut}$

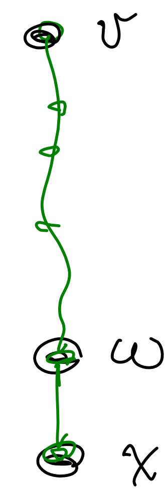
$\left(\mathbb{E}_{[illegible]} |c_v \cap \pi'| > \delta > 0 \ \forall \pi'\right)$

② mfmc

legal flow through $wx \propto$ [illegible]$^{|x|}$

α_w: flow into w ($\forall \underline{=} w$)





[<u>Lyons</u>: $p_{cut}(T) = p_c(T)$ $\forall T$ = <u>tree</u>]

R.L. conj'd same <u>$\forall G$</u>, but <u>FALSE</u>

OBV. IDEA: connections in Π $\not\longrightarrow$

$\mathbb{E}\left[\, |C_v \cap \Pi| \;\middle|\; \text{reach } \Pi \right]$ large $\not\longrightarrow$

$\mathbb{E}|C_v \cap \Pi|$ large but $\Pr(\text{reach } \Pi)$ small

$\longrightarrow$ <u>possible substitute</u>:

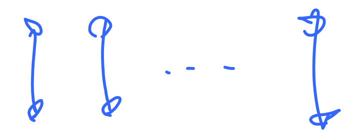
$A(\omega, \Pi) := \{ \exists$ open (v, Π)-path P with $P \cap \Pi = \{\omega\} \}$

[reach ω avoiding $\Pi \setminus \omega$]

$$p'_{cut}(G) = \sup \{ p : \inf_{\Pi} \underbrace{\sum_{\omega \in \Pi} \Pr_p(A(\omega, \Pi))}_{\leq \mathbb{E}_p |C_v \cap \Pi|} = 0 \}$$

TRIV: $p_{cut} \leq p'_{cut} \leq p_c$

Conj (??) $p'_{cut}(G) = p_c(G) \quad \forall G$

E.g. 2 $H =$  (perfect matching)

$$\mathbb{E}[\#\text{ of } H\text{'s}] = \ldots \approx \left(\frac{np}{e}\right)^{n/2}$$

large if (say) $p > 3/n$

GAP

but TRUTH: $p_c(\mathcal{F}_H) \approx \frac{\ln n}{n}$

(E-R)

(why?)

$$\left[H = | \; | \cdots | \longrightarrow p_c(\mathcal{F}_H) \approx \frac{\ln n}{n} \right]$$

obstruction: isolated vertices:*

need $\approx$ $n \log n$ edges $\left(\leftrightarrow p \approx \frac{\log n}{n} \right)$

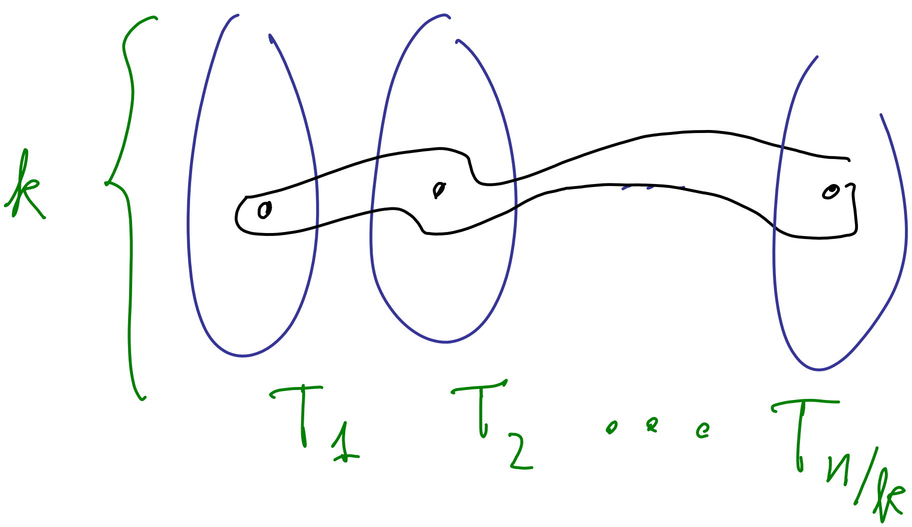
to cover vertices $\leftrightarrow$

---

* N.B. only shows $p_c \gtrsim \frac{\log n}{n}$

Coupon collector (more or less) as a set property (a.k.a. "dual tribes")

Ben-Or & Linial



e.g. $k \approx \log n$

f: meet every T_i ("tribe")

gap = $\Theta(\log n)$ (for reasonable k)

Coupon collector (more or less) as a

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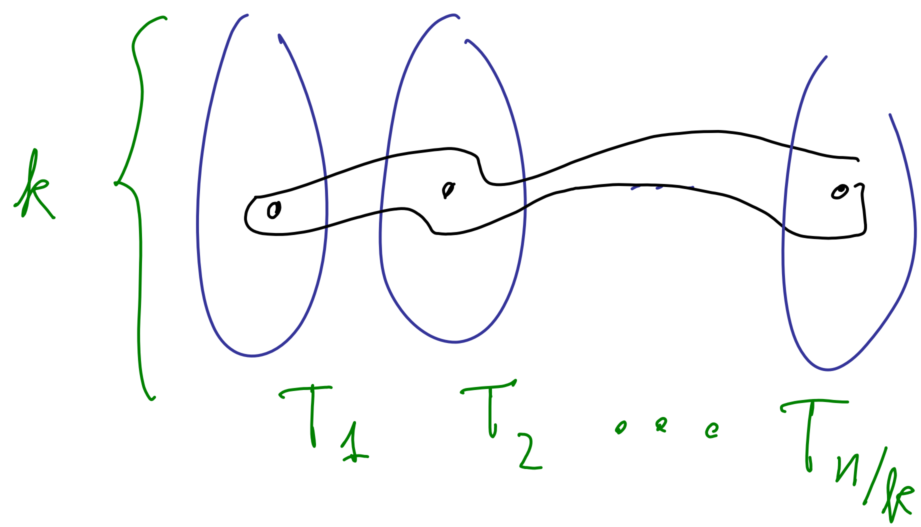


e.g. $k \approx \log n$

f: meet every T_i ("tribe")

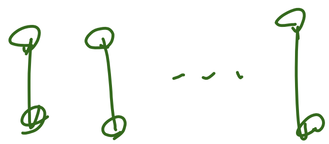
gap $= \Theta(\log n)$ (for reasonable k)

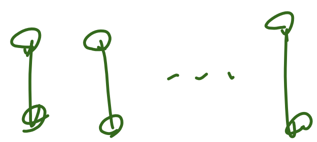
Back to "gap" :=

E

RECAP: $\mathfrak{J} = \mathfrak{J}_H$:

- H fixed: no gap

- H =  : gap $= \Theta(\log n)$

- H fixed: no gap
- H =  : gap = $\Theta(\log n)$

▶ Conj (TBA) there's nothing worse

▶ any $\exists$ (not just $\exists_H$, gph. prop's)

Fantasy Conj:

gap $\Longrightarrow$ underlying CC.

unhelpful (?) remark

Every $\mathcal{F}$ is a "CC problem":

$\mathcal{G} \subseteq 2^X \to$ blocker of $\mathcal{G}$ is

$$b(\mathcal{G}) = \{A \subseteq X : A \cap B \neq \emptyset \ \forall B \in \mathcal{G}\}$$

$$\mathcal{F} \text{ incr} \xRightarrow{\text{EX}} b(b(\mathcal{F})) = \mathcal{F}$$

Next: (too) many CCish e.g.'s

"local-global"

(many examples)

first group : "coupons" are vertices

① Hamilton cycle (H = H.c. , $\mathcal{F} = \mathcal{F}_H$)

Hamilton cycle $\left(H = \text{H.c.},\ \mathcal{F} = \mathcal{F}_H\right)$

similar* to p.m.: $p_c \approx \dfrac{\ln n}{n}$

STOP when $\underline{\text{min deg} = 2}$

$(\text{gap} \approx \log n)$

* except difficulty: Posa 76

Komlós - Szemerédi
Bollobás $\Big\}$ 83

— major OPEN from ER'60

long Ham. digression

deg ≥ 2 the issue $\longrightarrow$ consider

models where $\delta \geq 2$ is built in

min degree

<u>E.g. 1</u> "<u>k-out</u>": each vertex chooses

k neighbors (ignore repeats)

. . . long story . . .

<u>Bohman-Frieze</u> (09):

<u>3-out is (a.s.) Hamiltonian</u>

[unfair question: why not 2-out?]

E.g. 2 G random cubic $\longrightarrow$ 3-regular.

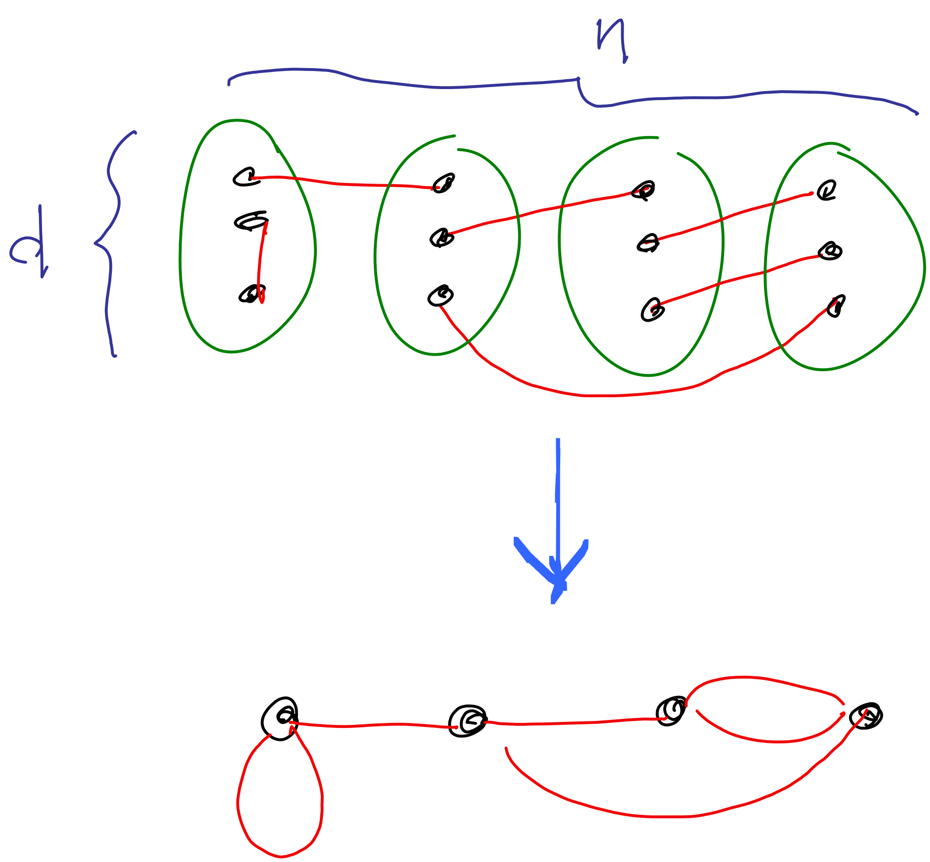
(or random d-reg., $d \geq 3$)

$G_{n,d}$: unif d-reg gph on $[n]$ "$\{1, \ldots, n\}$"

[illegible]

[$G_{n,d}$: unif d-reg gph on $[n]$]

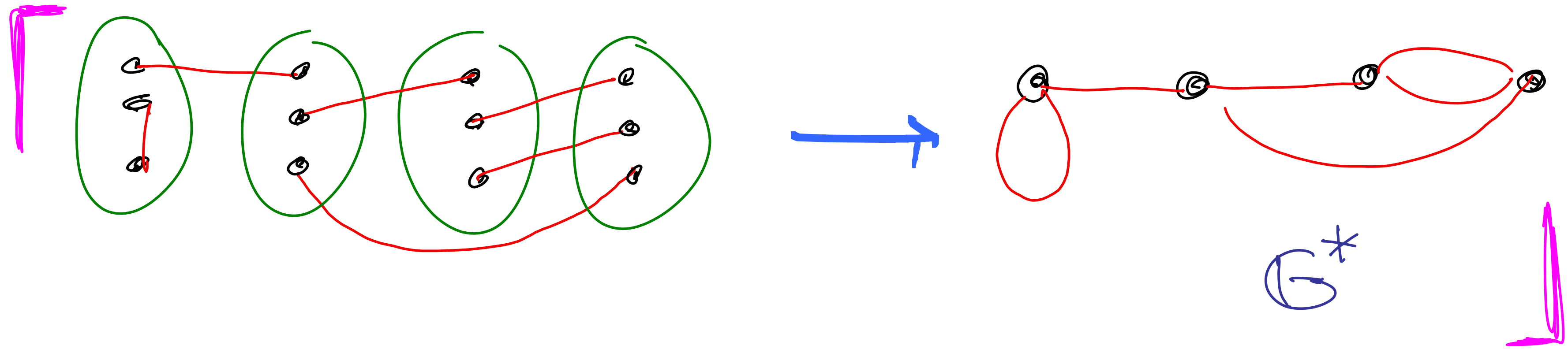
⇒ <u>configuration model</u> :



$\leftarrow$ [$\{$ ∘—∘ 's $\}$: unif. p.m.

$=: G^*$

(random <u>multigph</u>)



- law of $\mathbb{G}^*$ conditioned on $\{$simple$\}$

  $=$ law of $G_{n,d}$

- d fixed $\Longrightarrow \Pr(\mathbb{G}^* \text{ simple}) = \Omega(1)$

$\longrightarrow$ for a.s. statements about $G_{n,d}$ we can work with $\mathbb{G}^*$

. . . another long story . . .

<u>Robinson - Wormald</u> (92, 94):

for each <u>fixed</u> d

$G_{n,d}$ is a.s. Hamiltonian

[**Thm** Robinson - Wormald (92, 94):

for each <u>fixed</u> d

$G_{n,d}$ is a.s. Hamiltonian]

<u>notes</u> ① true for $d \not\Rightarrow$ true for $d+1$

(as far as we know)

② $d \to \infty$? Yes, but took a while

(lose "$\Pr(G^* \text{ simple}) = \Omega(1)$" ...)