[**Robinson–Wormald** (92, 94):

for each fixed d

$G_{n,d}$ is a.s. Hamiltonian]

notes 1. true for $d \not\Rightarrow$ true for $d+1$

(as far as we know)

2. $d \to \infty$? Yes, but took a while

(lose "$\Pr(G^* \text{ simple}) = \Omega(1)$" ...)

[fix d:] $G_{n,d}$ a.s. Ham (~~briefly~~)

$Y = \#$ H.c.'s

want $\Pr(Y > 0) \to 1$

2nd m.m.? $\mathrm{Var}\, Y = \Theta(\mathbb{E}^2 Y)$

$\longrightarrow$??

▶ "small subgraph conditioning method":

$\mathrm{Var}\, Y$ ess all due to

fluctuations in small cycle counts

[$Y = \#$ H.c.'s (want $\Pr(Y>0) \to 1$]

$X_i = \#$ i-cycles $i = 1, \ldots$

(note config model)

$X = (X_1, \ldots, X_k)$ ← TBA

$$\underbrace{\mathrm{Var}(Y)}_{A} = \underbrace{\mathrm{Var}(\mathbb{E}[Y \mid X])}_{B} + \mathbb{E}\,\mathrm{Var}(Y \mid X)$$

P: SHOW (not easy): $A \overset{k \to \infty}{\sim} B$

for typ. X: $\mathrm{Var}(Y \mid X) \ll \mathbb{E}^2[Y \mid X]$

$$\xrightarrow{\text{Cheb}} \Pr(Y = 0 \mid X) \to 0$$

(2)

OPEN:

① (e.g.) $\underline{d} = (\overbrace{3, \ldots, 3}^{n/2}, \overbrace{4, \ldots, 4}^{n/2}) \longrightarrow$

$G_{n, \underline{d}}$ Hamiltonian a.s.

② (maximum digression)

G random cubic $\Longrightarrow$

$\boxed{A_G}$ <u>nonsingular</u> a.s.

adjacency matrix

[(finally) back to: "coupons" are vertices]

"tree conj":

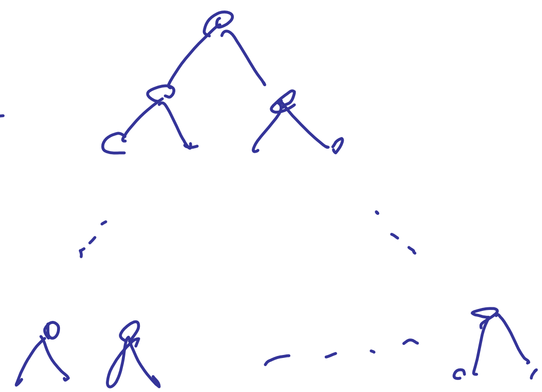
$H = T = n$-vertex tree, max deg $\leq C$

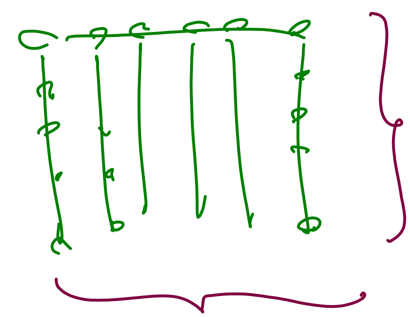
e.g. 3

$\xrightarrow{\text{(conj)}}$ $p_c(\mathcal{F}_T) \approx \frac{\log n}{n}$

[$\xleftrightarrow{\text{approx}}$ covering vertices is enough]

- e.g.  (Ham. path)

- e.g. $T =$  is easy

- e.g. "comb conj" $\longleftrightarrow$ 

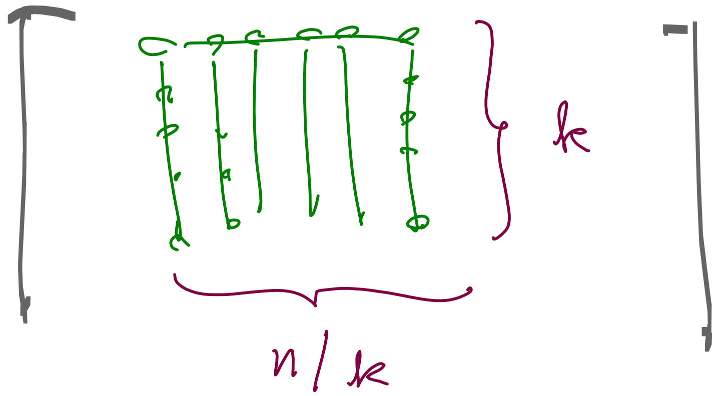
  k

  n/k

KLW (never?) TRUE.

Lubetzky, Wormald

proof mumbles $\longrightarrow$

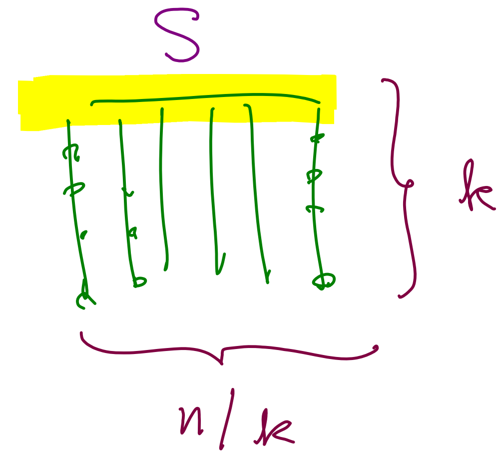


two different arguments:

- $k < k_0 = \Theta(\log n)$ [elementary]
- $k > k_0$: back to random cubic

$\longrightarrow$

$\left[k > k_0 = \Theta(\log n) \right]$

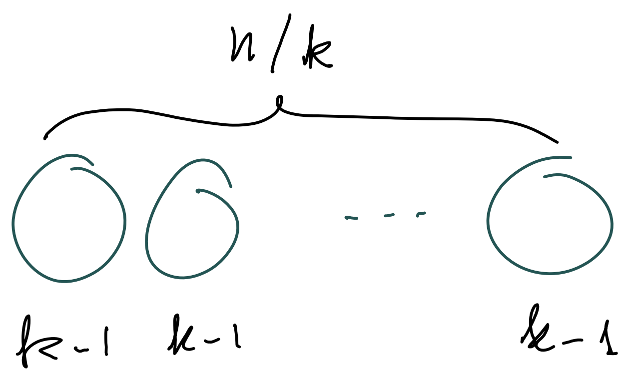


① Cn edges → "spine"

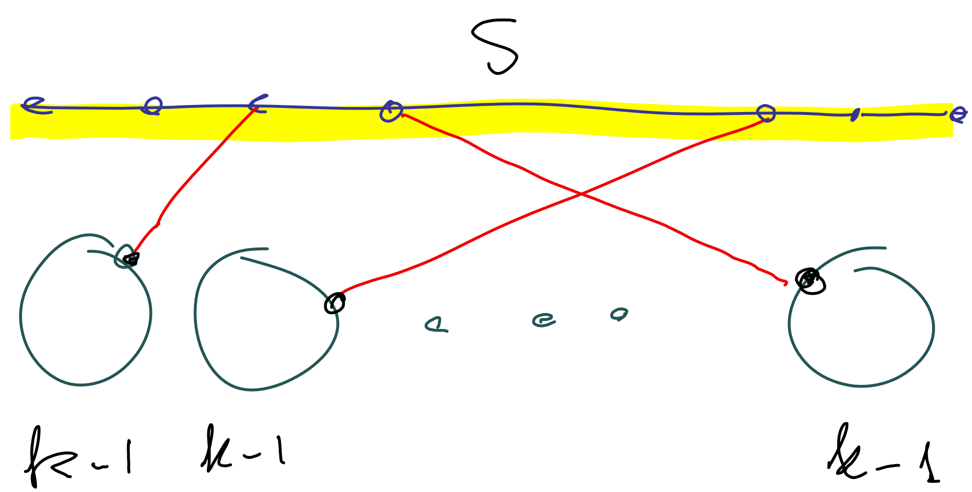
② $Cn \log n$ (new) edges in $V \setminus S$

$\underset{\text{ess. a.s.}}{\supseteq} G_{m,3}$ $\quad (m = n - n/k)$

③ $G_{m,3} \underset{\text{a.s.}}{\supseteq}$

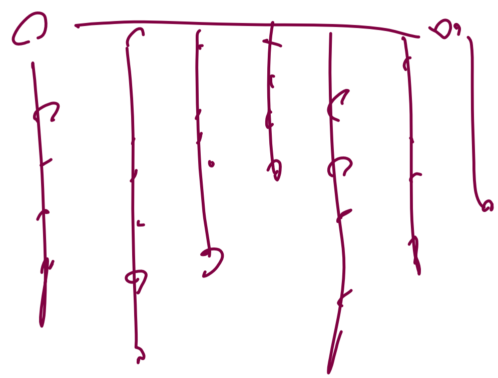


4 cn edges in $\nabla(S, V \setminus S)$ $\xrightarrow{\text{Hall}}$ join to cycles



☺

CTCAA (e.g.):



$k_1 \quad k_2 \quad \ldots \quad k_m \longrightarrow k \qquad ?$

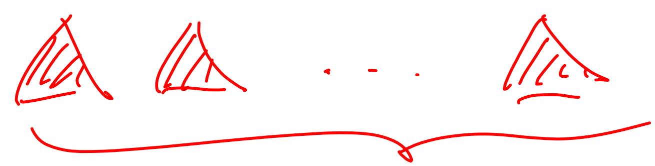
③ "Shamir's problem" ($\sim$ 1980) (non-graphic)

$$X = \binom{V}{3}, \quad |V| = n \qquad (3 \mid n)$$

or k

$$\Big[\; X_p =: \mathcal{H}_k(n,p) \text{ random } [\text{illegible}] \;\Big]$$

H = p.m. :



$n/3$

$$\longrightarrow \boxed{p_c(\mathcal{F}_H)} \; ?$$

$$\left[\; H = \underbrace{\triangle\ \triangle\ \cdots\ \triangle}_{n/3} \longrightarrow \boxed{p_c(\mathcal{J}_H)}\ ? \;\right]$$

<u>natural guesses</u>:

(A) $p_c = \Theta\left(\frac{\log n}{n^2}\right)$

(B) <u>STOP</u> when <u>no iso's</u> . . .

. . . more stories . . .

(A): TRUE: <u>JKV</u> '08 (we'll "do")

Johansson, Vu

(B) ??

<u>Similar</u> (Erdős 92, Ruciński 92):

threshold for a $\triangle$-factor in $G_{n,p}$?

$$\underbrace{\triangle \ \triangle \ \cdots \ \triangle}_{n/3}$$

also:

- H-factors
- hypergraphs

[redacted]

STOPPING TIME??

why is $\ldots$ (or Ham ...)

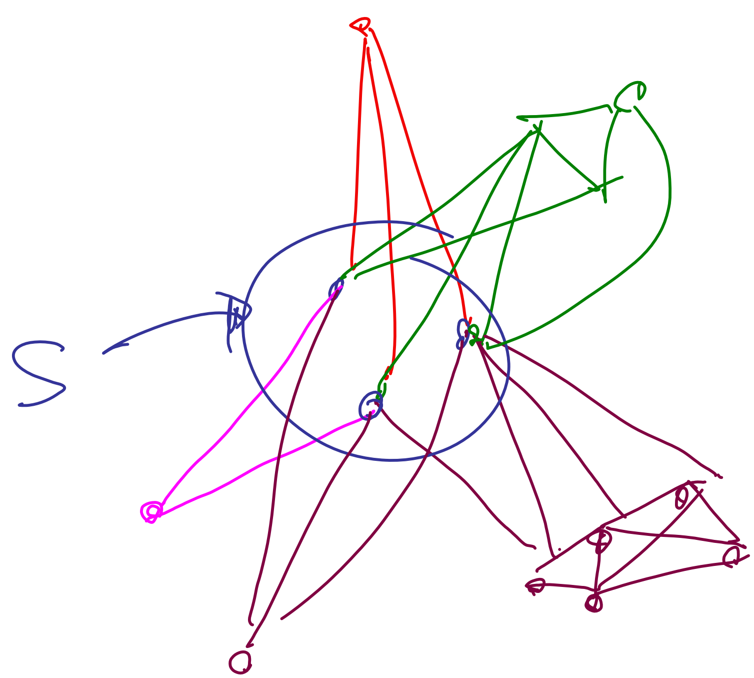
harder than $\ldots$?

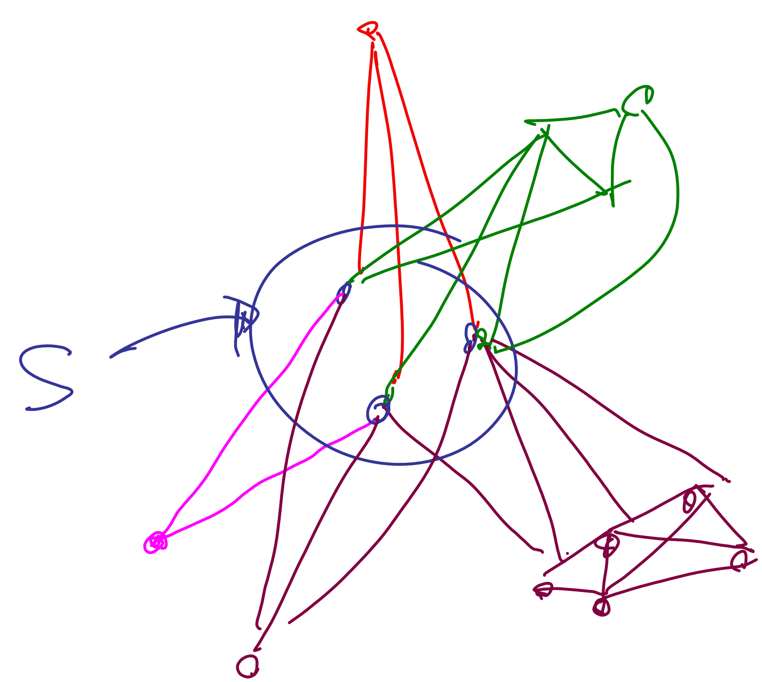
<u>one answer</u>: no duality (we think)

<u>Tutte</u>: $\exists$ p.m.

$\Updownarrow$

$\nexists$ one of these





[**Tutte:** $\exists$ p.m.

$\Updownarrow$

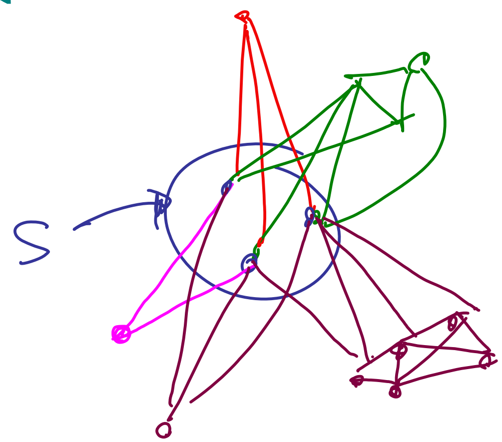
$\nexists$ one of these]

E.g. $\exists$ iso $\longrightarrow$ take $S = \emptyset$

EX: threshold ($\Theta\left(\frac{\ln n}{n}\right)$) from Tutte

[$\sum \{ \Pr(\Theta) : \Theta \text{ obstruction} \} = o(1)$

with some care..]



EX: th. from Tutte

$\sum \{ \Pr(\mathcal{O}) : \mathcal{O} \text{ obst} \} = o(1)$

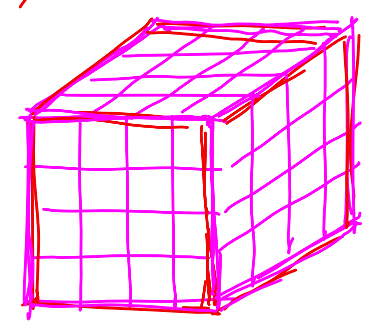
- easier EX: th. for connectivity
- **Remk** In a sense more elementary than $\mathcal{F}_H$ (e.g. $H = \Delta$):
  - "union bd" over obstructions vs. 2^{nd} m.m.
  - obstructions for $\{\text{contain } \Delta\}$??

<u>digression</u> : one more related to Shamir

a_{ijk} $\quad i,j,k \in [n]$ indept $\exp(1)$

<u>3-d assignment</u> : n cells (I)

<u>meeting each plane</u> (once)

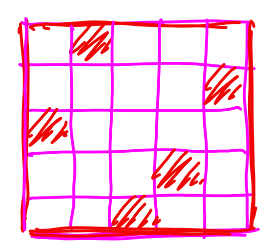


<u>weight</u> : $\sum \{ a_{ijk} : (i,j,k) \in I \}$

Frieze-Sorkin :

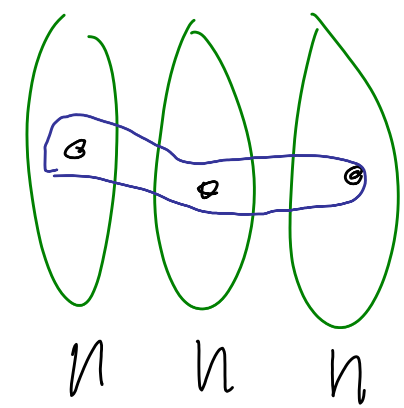
2-d : very well understood :

$\mathbb{E} \to \pi^2/6$, Parisi Conj ...



[Frieze-Sorkin:]

<u>note</u> assignment $\longleftrightarrow$ p.m. in 3-partite $\mathcal{H}$ $\longrightarrow$



$\circledast$ JKV $\longrightarrow$ $\mathbb{E} = O\left(\frac{\log n}{n}\right)$

($Cn \log n$ weights below $\frac{C \log n}{n^2}$)

<u>Obvious Guess</u>: $\Theta(1/n)$

(Ex: $\Omega(1/n)$)

2nd group : "coupons" are edges

Remk : a recent theme :

understand when various combinatorial thms hold in a random setting

① <u>Ramsey</u>:

<u>notation</u>: $G \to (H)_q$ if

⑥ Ramsey's thm: $\exists n$ $K_n \to (H)_q$

◎ We stick to $\underline{H = K_r}$

Q: th. for $\{ \underline{G} \to (K_r)_q \}$?

$G_{n,p}$ $\quad$ "K"

Th ([redacted]): $p_c = \underline{\underline{\Theta}}(p_0)$ $\quad$ depends on q.

$p_0 = n^{-\frac{2}{r+1}}$

why? (?) $\quad p \gg p_0$: most edges in K's

$p \ll p_0$: few —— ——

* Rödl, Ruciński

[$p \gg p_0$: **most** edges in K's

$p \ll p_0$: **few** —— ——]

$$\mathbb{E}[\#\text{ of pre-}K\text{'s on }\{x,y\}] \approx$$

$$n^{r-2} p^{\binom{r}{2}-1} = \left(n p^{\frac{r+1}{2}}\right)^{r-2}$$

$Z \in \binom{V}{r}$: $x, y \in Z$

$\binom{Z}{2} \setminus \{xy\} \subseteq G$ $(= G_{n,p})$

["$p \ll p_0 \longrightarrow$ few edges in K's"]

$\xrightarrow{??}$ $p_c \gtrsim p_0$

@ *should* be true:

edge in ≤ 1 K can't cause trouble

⊖ *is* true, but (atypically) *not so easy*:

▫ RR: 29 pp.

▫ easier: Nenadov - Steger (recent)

② Turán + Szemerédi

$$t_r(G) := \max\{|H| : H \subseteq G,\ K_r\text{-free}\}$$

$$b_r(G) := \max\{|H| : H \subseteq G,\ (r-1)\text{-partite}\}$$

TRIV: $t_r(G) \geq b_r(G)$

Mantel (1907): $t_3(K_n) = b_3(\quad)$ $\left(= \lfloor n/2 \rfloor \lceil n/2 \rceil\right)$

Turán (1941) $t_r(K_n) = b_r(\quad)$ $\sim \left(1 - \frac{1}{r-1}\right)\binom{n}{2}$

"Erdős-Simonovits stability thm" (~66)

$\forall\, r, \delta \;\; \exists\, \varepsilon \;\; \ni$

$\boxed{\delta, \varepsilon > 0 \text{ always}}$

$H \subseteq K_n$, K_r-free, $|H| > \left(1 - \frac{1}{r-1}\right)\binom{n}{2}$

$\Longrightarrow$ H δ-close to $(r-1)$-partite

i.e.

$\exists$ part'n $V = V_1 \cup \dots \cup V_{r-1} \ni$

$|H \setminus \nabla(V_1, \dots, V_{r-1})| < \delta n^2$

(Fix r; $G = G_{n,p}$)

Thm D* $\forall \varepsilon\ \exists C \ni p > Cn^{-\frac{2}{r+1}} \Longrightarrow$

a.s. $t_r(G) < \left(1 - \frac{1}{r-1} + \varepsilon\right) \boxed{|G|} \approx \frac{n^2 p}{2}$

Thm S* $\forall \delta\ \exists \varepsilon, C : p > Cn^{-\frac{2}{r+1}} \Longrightarrow$

a.s. $H \subseteq G$ K_r-free, $|H| > \left(1 - \frac{1}{r-1} - \varepsilon\right)|G|$

$\Longrightarrow H$ $\boxed{\delta\text{-close to } (r-1)\text{-partite}} \leftrightarrow [\delta n^2 p$ edges

* "density" (Conlon-Gowers, Schacht)

* "stability" (C-G)

Thm D $\forall \varepsilon\ \exists C \ni\ p > Cn^{-\frac{2}{r+1}} \Rightarrow$

a.s. $t_r(G) < \left(1 - \frac{1}{r-1} + \varepsilon\right)|G|$

Thm S $\forall \delta\ \exists \varepsilon, C:\ p > Cn^{-\frac{2}{r+1}} \Rightarrow$

a.s. $H \subseteq G$ K_r-free, $|H| > \left(1 - \frac{1}{r-1} - \varepsilon\right)|G|$

$\Longrightarrow$ H δ-close to $(r-1)$-partite

both conj'd: Kohayakawa-Łuczak-Rödl 97

Thm D_3: Frankl-Rödl 86

Thm S_3: KŁR

Thm D $\forall \varepsilon \; \exists C \ni \; p > Cn^{-\frac{2}{r+1}} \Longrightarrow$

a.s. $t_r(G) < \left(1 - \frac{1}{r-1} + \varepsilon\right)|G|$

Thm S $\forall \delta \; \exists \varepsilon, C : \; p > Cn^{-\frac{2}{r+1}} \Longrightarrow$

a.s. $H \subseteq G$ K_r-free, $|H| > \left(1 - \frac{1}{r-1} - \varepsilon\right)|G|$

$\Longrightarrow$ H δ-close to $(r-1)$-partite

Remk Thm S $\Longrightarrow$ Thm D

Thm D $\forall \varepsilon\ \exists C \ni \ p > Cn^{-\frac{2}{r+1}} \Longrightarrow$

a.s. $t_r(G) < \left(1 - \frac{1}{r-1}\right)|G|$

Thm S $\forall \delta\ \exists \varepsilon, C:\ p > Cn^{-\frac{2}{r+1}} \Longrightarrow$

a.s. $H \subseteq G$ K_r-free, $|H| > \left(1 - \frac{1}{r-1} - \varepsilon\right)|G|$

$\Longrightarrow$ H δ-close to $(r-1)$-partite

Remk: best possible (except C):

$p < n^{-\frac{2}{r+1}} \longrightarrow$ n [redacted] s

(e.g.) [redacted]

Thm D $\forall \varepsilon\ \exists C \ni\ p > Cn^{-\frac{2}{r+1}} \Rightarrow$

a.s. $t_r(G) < \left(1 - \frac{1}{r-1} + \varepsilon\right)|G|$

Thm S $\forall \delta\ \exists \varepsilon, C:\ p > Cn^{-\frac{2}{r+1}} \Rightarrow$

a.s. $H \subseteq G$ K_r-free, $|H| > \left(1 - \frac{1}{r-1} - \varepsilon\right)|G|$

$\Longrightarrow$ H δ-close to $(r-1)$-partite

Remk these are <u>not incr. properties</u>

(more on this below)

<u>Thm Sz</u> (Conlon-Gowers, Schacht)

$\forall\, \delta, k\ \exists\, C \ni$ if $p > C n^{-\frac{1}{k-1}}$

and $S = [n]_p$ then a.s.

$A \subseteq S,\ |A| > \delta |S| \Longrightarrow$

$A \supseteq$ k-term A.P.

<u>$k=3$</u>: KŁR 96 (hard)

__Remk__ RR & Thms D, S, Sz:

- major results
- original pfs of RR & D, S, Sz very different (gap ≈ 20 yrs)
- we'll see amazing new development (thm BMSST) that easily gives all (and much more that we a.s. won't get to)

<u>One more</u>: what about (<u>exact</u>) Turán ?

- i.e. "when" do we have

$$t_r(G_{n,p}) = b_r(G_{n,p}) \quad ?$$

$r=3$ $\quad (t_3(G_{n,p}) = b_3(G_{n,p})\ ?)$

Babai-Simonovits-Spencer (90):

1 true if $p > \frac{1}{2}$ (actually $\frac{1}{2} - \delta$)

2 Q: $p > n^{-c}$? (some fixed $c>0$)

Brightwell-Panagiotou-Steger (07/12):

3 YES to 2 ($c = 1/250$)

4 Q: $p > \boxed{n^{-\frac{1}{2}+\varepsilon}}$ now want all edges in Δ's

like CC

DeMarco-K: true if $p > C n^{-\frac{1}{2}} \sqrt{\log n}$

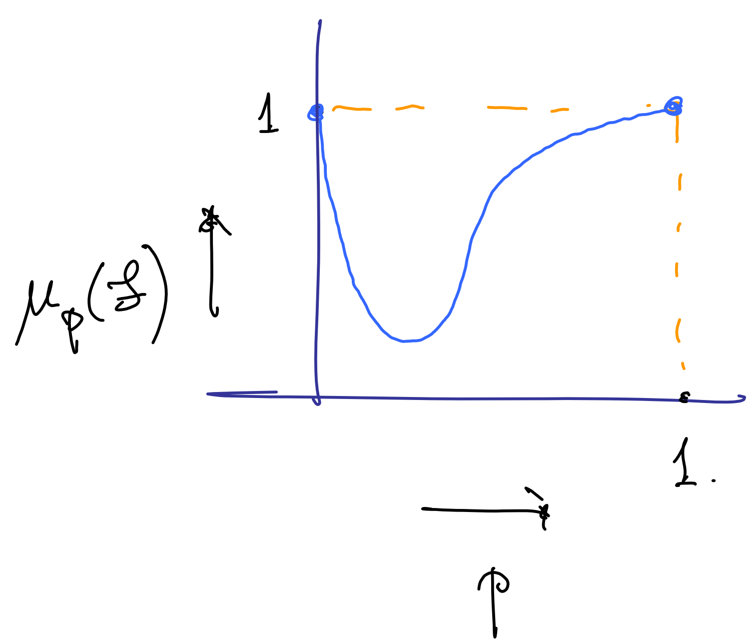
Notes ① "stopping time" version? (NO)

— but guess "cover edges" gives right C

② not a "threshold":

$\mathcal{F} := \{ t_r(G) = b_r(G) \}$ isn't increasing

MAYBE:



??

general r:

BPS $\quad p > n^{-c_r} \implies$

$$\text{a.s.} \quad t_r(G_{n,p}) = b_r(G_{n,p})$$

DK $\quad$ same if

$$p > C_r n^{-\frac{2}{r+1}} \log^{\frac{2}{(r+1)(r-1)}} n$$

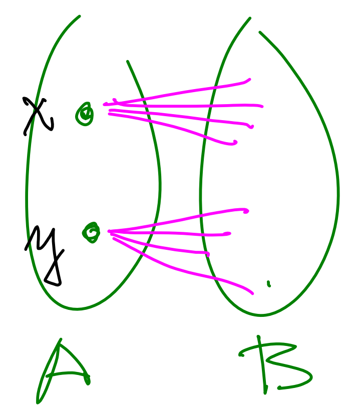
[$\longleftrightarrow$ all edges in K_r's (obvious guess)]

E.g. show (must be true):

$\boxed{r=3}$ a.s. $\nexists$ max cut (A,B) &

$x,y \in A$ $\bar{\omega}$ $d_B(x,y) = 0$

should be $\approx np^2/2$



$\boxed{r=4}$ a.s. $\nexists$ max cut (A,B,C) &

$x,y \in A$ $\bar{\omega}$ $K(x,y,B,C) = 0$

number of these

(should be $\approx n^2p^5/4$)