On Magneto-optical traps seen as Coulomb gases with non conservative interactions

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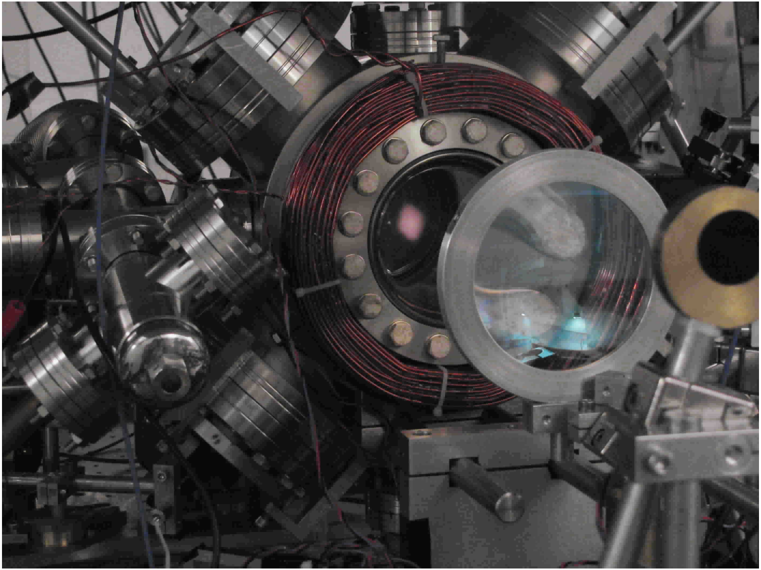
A magneto optical trap



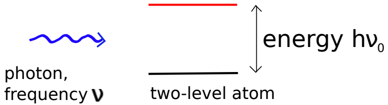
Outline

1. Magneto-optical trap modelling: why do they -partially- fit into this conference's topic?
   Dalibard 1988, Walker-Sesko-Wieman 1990.
2. Analogy with non-neutral plasmas: cloud's shape, cloud's dynamics, variational structures.
3. More realistic modelling: all the variational structure is lost! What can we do?

Basic principle of a Magneto-Optical trap, 1

Trapping and cooling atoms with lasers: techniques developed during the 70s and 80s.

• Classical (ie not quantum) description for atoms; quantum description for light
• Basic mechanism: interactions between photons and atoms

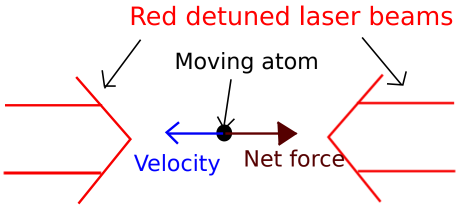


Atom at rest: absorption probability max. when $\nu = \nu_0$

Momentum conservation $\rightarrow$ effective force on the atoms.
Timescale considered for the atomic dynamics: many cycles absorption/spontaneous emission $\rightarrow$ description by an averaged force

Basic principle of a Magneto-Optical trap, 2

- Trapping in velocity space

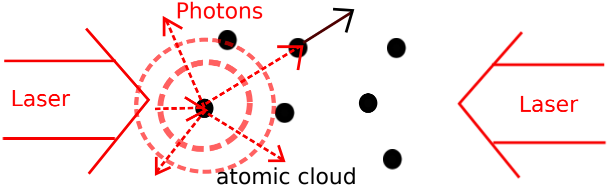


The laser frequency is slightly below an atomic resonance:
$\rightarrow$ Doppler cooling

- Trapping in space: similar idea, uses a magnetic field gradient

$\rightarrow$ an (over)simplified vision: linear friction ($-\kappa\vec{v}$) + velocity diffusion + external trapping potential ($\sim$ quadratic, anisotropic)

Effective long range repulsion in a Magneto-Optical trap



Multiple diffusion and effective force

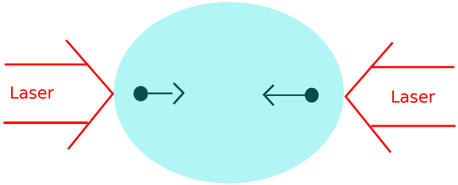
$$\rightarrow \quad \vec{F}_i \propto \sum_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

The $1/r^2$ dependence of the force comes from the solid angle in 3D.

Again, this is an oversimplification; more or less a "standard model" (Sesko, Walker, Wieman 1990).

Effective long range attraction in a Magneto-Optical trap

- Shadow effect:



Laser intensities decrease while propagating into the cloud
→ effective force towards the center
Weak absorption approximation: $\nabla \cdot \mathbf{F}_{Shadow} \propto -\rho$ (Dalibard 1988)
→ Just like gravitation... but it does not derive from a potential!

In most experimental situation, the repulsive force is stronger

$$\nabla \cdot \mathbf{F}_{\mathrm{Coulomb}} = c_1 \rho \ , \ \nabla \cdot \mathbf{F}_{\mathrm{shadow}} = -c_2 \rho \ , \ \text{with } 0 < c_2 < c_1$$

→ standard approximation: replace c_1 by $C_{\mathrm{Coulomb}} = c_1 - c_2$.

Part 2: First model = dissipative non neutral plasma

- Effective 2-body Coulomb force + external trapping potential + linear friction + velocity diffusion (no magnetic force).

- In typical MOTs, interaction can be "strong" (with respect to temperature) for large clouds, but **correlations remain weak**

→ Basic equation = Vlasov-Poisson-Fokker-Planck, for the **one particle density** $f(\mathbf{x}, \mathbf{v}, t)$

$$\partial_t f + \mathbf{v} \cdot \nabla_x f - \left(\nabla_x \Phi_{\rm ext} + \nabla_x \Phi_{\rm Coulomb}\right) \cdot \nabla_v f = \nabla_v \cdot \left(\mathbf{v} f + \theta \nabla_v f\right)$$

$$\Delta_x \Phi_{\rm Coulomb} = -C_{\rm Coulomb}\rho, \qquad \rho(t, \mathbf{x}) = \int f(t, \mathbf{x}, \mathbf{v})\, \mathrm{d}v$$

- Lyapunov functional, $d\mathcal{F}/dt \leq 0$

$$\mathcal{F}[f] = \theta \int f \ln f d\mathbf{x} d\mathbf{v} + \int \frac{\mathbf{v}^2}{2} f d\mathbf{x} d\mathbf{v} + \int f \Phi_{\rm ext} d\mathbf{x} d\mathbf{v} + \frac{1}{2} \int f \Phi_{\rm Coulomb} d\mathbf{x} d\mathbf{v}$$

Cloud's shape

Stationary solution of VFP equation separable:

$$f_{\mathrm{stat}}(\mathbf{x},\mathbf{v}) = \rho_{\mathrm{stat}}(\mathbf{x})\,(2\pi\theta)^{-d/2}\,e^{-\frac{\mathbf{v}^2}{2\theta}}$$

Equilibrium density $\rho_{\mathrm{stat}}(\mathbf{x})$, normalized as $\int \rho_{\mathrm{stat}}(\mathbf{x})d\mathbf{x} = 1$, given by the minimization problem:

$$\min\left\{\theta\int \rho\ln\rho d\mathbf{x} + \int \Phi_{\mathrm{ext}}(\mathbf{x})\rho(\mathbf{x})d\mathbf{x} + \frac{1}{2}\int \Phi_{\mathrm{Coulomb}}(\mathbf{x})\rho(\mathbf{x})d\mathbf{x}\right\}$$

with $\Delta\Phi_{\mathrm{Coulomb}} = -C_{\mathrm{Coulomb}}\rho$, and $C_{\mathrm{Coulomb}} \propto N$.

Small N $\rightarrow$ negligible interactions $\rho_{\mathrm{stat}} \propto e^{-\frac{\Phi_{\mathrm{ext}}(\mathbf{x})}{2\theta}}$
Large N $\rightarrow$ strong interactions; classical problem...

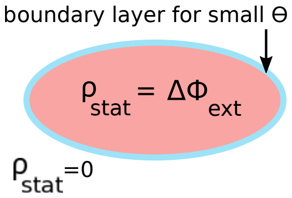
Cloud's shape - strong interaction regime

- Leading order: balance interaction / external potential

$$\min \left\{ \int \Phi_{\text{ext}}(\mathbf{x})\rho(\mathbf{x})d\mathbf{x} + \frac{1}{2}\int \Phi_{\text{Coulomb}}(\mathbf{x})\rho(\mathbf{x})d\mathbf{x} \right\}$$

Typically: the optimal ρ_{stat} has compact support $\mathcal{K}$; on $\mathcal{K}$, $\rho_{\text{stat}} = \Delta\Phi_{\text{ext}}$. $\mathcal{K}$ is not always easy to find...

- $+$ boundary layers where θ (temperature) is important to match high and low density regions.



Cloud's shape - harmonic Φ_{ext}

$$\Phi_{\mathrm{ext}} = \frac{1}{2}\left(\frac{x_1^2}{L_1^2} + \frac{x_2^2}{L_2^2} + \frac{x_3^2}{L_3^2}\right) \ ; \ \Delta\Phi_{\mathrm{ext}} = \mathrm{cst.}$$

Small miracle: an ellipsoid with constant charge density creates a quadratic Coulomb potential inside the ellipsoid:

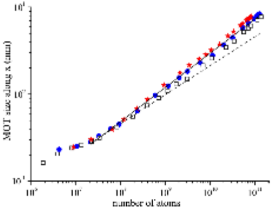
$$\mathcal{E} = \left\{\frac{x_1^2}{a_1^1} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leq 1\right\} \ ; \ \Delta\Gamma = -\delta$$

Then $(\Gamma \star \mathbb{I}_{\mathcal{E}})(\mathbf{x})$ is quadratic inside $\mathcal{E}$ (non trivial coefficients).

$\rightarrow$ exact "step-like" solution for all harmonic Φ_{ext}
density = constant on a certain ellipsoid, almost zero outside.

Experimental measurements

The "Coulomb model" explains qualitatively some experimental observations



Cloud's size as a function of N (number of atoms). Dashed line = "Coulomb theory". Experiment Camara et al. (INLN, Nice).

• Orders of magnitude (Rb experiments at INLN, U. of Nice): Number of atoms$\sim 10^6 - 10^{11}$. Cloud's size$\sim$ 0.2-10mm. Temperature$\sim$1mK

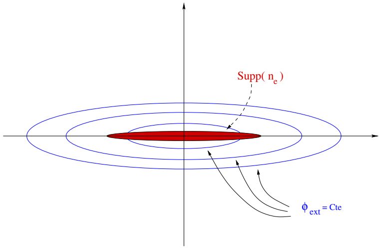
Cloud's shape - pancake traps

- Take a "pancake-shaped" trap

$$\Phi_{\mathrm{ext}} = \frac{1}{2}\left(\frac{x_1^2}{L_1^2} + \frac{x_2^2}{L_2^2} + \frac{x_3^2}{L_3^2}\right)$$

with $L_1 = L_2 > L_3$.

- Small interaction limit $\rightarrow$ cloud $=$ gaussian shape, same aspect ratio as the trap.
- Strong interaction limit $\rightarrow$ cloud $=$ homogeneous ellipsoid with axes $a_1 = a_2 > a_3$.



Side view: some level sets of Φ_{ext} (blue) and the support of ρ_{stat} (red).

Cloud's shape - pancake traps

- Take a "pancake-shaped" trap

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with $L_1 = L_2 > L_3$.

- Small interaction limit $\to$ cloud $=$ gaussian shape, same aspect ratio as the trap.
- Strong interaction limit $\to$ cloud $=$ homogeneous ellipsoid with axes $a_1 = a_2 > a_3$.

Explicit computations:

$$\frac{a_1}{a_3} \sim \frac{\pi}{4}\left(\frac{L_1}{L_3}\right)^2 , \text{ when } (L_1/L_3) \to \infty.$$

$\to$ strongly enhanced anisotropy!

Cloud's shape - cigar traps

- Take a "cigar-shaped" trap

$$\Phi_{\text{ext}} = \frac{1}{2}\left(\frac{x_1^2}{L_1^2} + \frac{x_2^2}{L_2^2} + \frac{x_3^2}{L_3^2}\right)$$

with $L_1 > L_2 = L_3$.

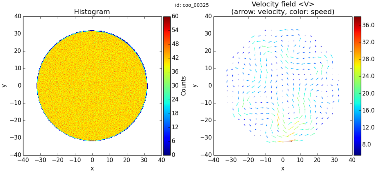
- Small interaction limit $\rightarrow$ cloud $=$ gaussian shape, same aspect ratio as the trap.
- Strong interaction limit $\rightarrow$ cloud $=$ homogeneous ellipsoid with axes $a_1 > a_2 = a_3$. Explicit computations:

$$\frac{a_1}{a_3} \sim \sqrt{2}\frac{L_1}{L_3}\sqrt{\ln\frac{L_1}{L_3}}\ ,\ \text{when } (L_1/L_3) \rightarrow \infty.$$

$\rightarrow$ anisotropy only very weakly enhanced.

Beyond the one-particle density: dynamics

• Experiments: sometimes with external forces, or instabilities $\rightarrow$ dynamical questions

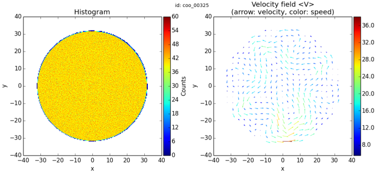


Particles simulation

$\rightarrow$ calls for a fluid description inside the domain $\mathcal{K}$.

Beyond the one-particle density: dynamics

• Experiments: sometimes with external forces, or instabilities → dynamical questions



Particles simulation

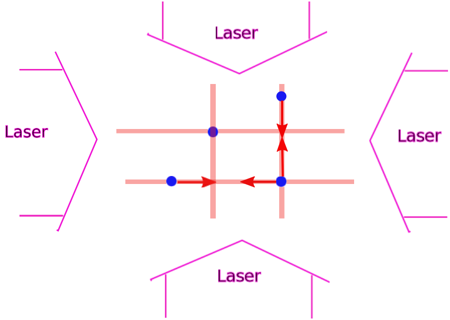
Theorem (B., Chiron, Goudon, Masmoudi): solutions of VP-FP tends in the strong interaction-small temperature limit towards solution of the incompressible Euler equation.

Beyond the one-particle density: correlations

- Strong Coulomb forces: liquid, or even crystal structure.
- Typical Magneto Optical Traps regimes: "weak" interactions, far from liquid or solid.
- Can one see the correlations anyway? Is there something like a"Debye length" in MOTs?
  $\rightarrow$ more stringent test for the "Coulomb picture".
  Current experiments at Institut Non Linéaire de Nice (G. Labeyrie, D. Métivier).

Part 3: Taking really into account the shadow effect

- The "shadow" force $F_{\text{shadow}}(\mathbf{x}) = \int \mathbf{K}(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\mathbf{dy}$



$$K_1(\mathbf{x}-\mathbf{y}) = -\frac{c_2}{6} sgn(x_1-y_1)\delta(x_2-y_2)\delta(x_3-y_3)$$
$$K_2(\mathbf{x}-\mathbf{y}) = -\frac{c_2}{6} sgn(x_2-y_2)\delta(x_1-y_1)\delta(x_3-y_3)$$
$$K_3(\mathbf{x}-\mathbf{y}) = -\frac{c_2}{6} sgn(x_3-y_3)\delta(x_2-y_2)\delta(x_1-y_1)$$

With the shadow effect. What do we loose?

What do we keep?

- Write a Vlasov-Fokker-Planck equation: still OK, at least formally.
- Properties that depend only on the divergence $\nabla \cdot \mathbf{F}_{\text{shadow}}$.

What do we loose? not a reversible process any more.

- Back to particles: no explicit expression for the N-particles stationary distribution (cf Gibbs distribution $\propto \exp(-\beta H_N)$ with potential forces)
- No obvious Lyapunov functional for the Vlasov Fokker-Planck equation $\rightarrow$ variational structure lost.
- Stationary solution of Vlasov-Fokker-Planck: never separable in space/velocity. No easy way to compute them!

Somewhat simpler: large friction limit

• Overdamped limit: usually rather reasonable for experiments.
$\rightarrow$ Write an equation for ρ alone:

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0 \ , \ \mathbf{J} = (-\nabla \Phi_{\mathrm{ext}} - \nabla \Phi_{\mathrm{Coulomb}} + \mathbf{F}_{\mathrm{shadow}})\rho - D\nabla \rho.$$

$\mathbf{J} =$ particles current.

• No stationary solution with $\mathbf{J} = 0 \rightarrow$ presence of stationary currents.
Existence, uniqueness, shape of stationary solutions not easy.

• Even simpler: investigate a small temperature regime, where $D\nabla\rho$ "negligible".

Small temperature regime

- At leading order, look for force balance:

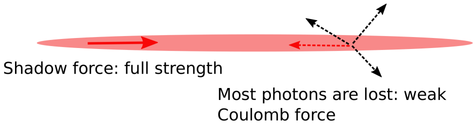
$$-\nabla\Phi_{\mathrm{ext}} - \nabla\Phi_{\mathrm{Coulomb}} + \mathbf{F}_{\mathrm{shadow}} = 0$$

- Further small miracle: for some harmonic external potentials, one can still construct a solution (Sesko et al. 90, Verkerk et al. 2013) Assume a constant density on an ellipsoid $\mathcal{K}$ (axes x, y, z); then $\mathbf{F}_{\mathrm{shadow}}$ is linear inside $\mathcal{K}$!
$\rightarrow$ all forces are linear, one can find a solution ρ_{stat}=constant on $\mathcal{K}$, $\rho_{\mathrm{stat}} = 0$ outside.

- Boundary layer where temperature matters: structure not clear...

Anisotropic traps: the shadow force can win

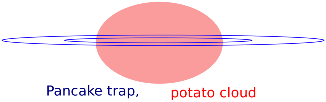
Hand waving argument:



Cleaner argument: from the exact solution. Take

$$\nabla\cdot\mathbf{F}_{\text{Coulomb}} = c_1\rho\,,\ \nabla\cdot\mathbf{F}_{\text{shadow}} = -c_2\rho = -\gamma c_1\rho\,,\ \text{with}\ 0<\gamma<1.$$

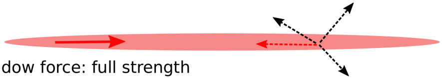
Then:



→ very different from the pure Coulomb gas!

Anisotropic traps: the shadow force can win

Hand waving argument:



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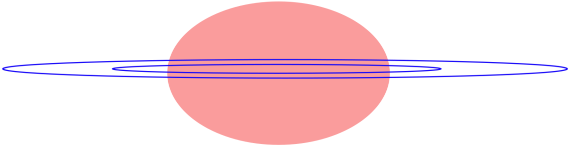
Then:

Numerical illustration 1



Pancake trap, potato cloud

Numerical illustration 2

Pure Coulomb: Pancake trap, "crèpe" (from Brittany) cloud

Pseudo self-gravitating regime

Use a very anisotropic (cigar or pancake) trap.
• Weak interaction (small N) $\rightarrow$ gaussian anisotropic cloud
• Strong interaction (large N) $\rightarrow$ step-like $\sim$ "potato" cloud, Coulomb dominated
• In between: regime with negligible repulsive interaction and strong attraction

$\rightarrow$ pseudo (Brownian) self gravitating systems in the lab!

Experimentally: set-up very anisotropic traps is not very easy, but possible.

"Cigar" $\sim$ 1D gravity: explicitly solvable; no phase transition
"Pancake" $\sim$ 2D gravity: finite time singularity for diffusion D small enough; but what happens with the shadow force?

Pseudo self-gravitating regime, 2

- Cigar traps: experiments (M. Chalony and D. Wilkowski) qualitatively confirm the analysis; difficult to get quantitative agreement...
- Pancake traps: work in progress
  Simplest model (2D; integrated over the thin direction)

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0 \ , \ \mathbf{J} = (-\nabla \Phi_{\rm ext} + \mathbf{F}_{\rm shadow})\rho - D \nabla \rho .$$

  - Global existence for D large enough: OK (math. work in progress with T. Goudon and D. Crisan)
  - Finite-time singularity for D small enough??

Simulations in the pseudo self gravitating regime (2D)

- "Large" D, $D = 0.2$

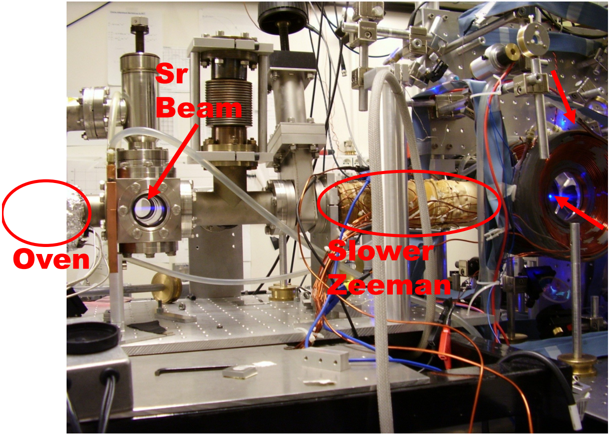
- "Small" D, $D = 0.1$

Conclusion and open questions

- Standard traps, dominated by repulsion: plasma analogy. How far can this be pushed? Can one observe a "Debye length"?
  $\rightarrow$ this would characterize correlations in the cloud
  Work and current experiments by D. Métivier, G. Labeyrie and R. Kaiser.
- Traps dominated by the attraction (very anisotropic for instance). Can one see experimental signatures of the non potential force?
  PhD thesis of V. Mancois, NTU Singapour and Université Pierre et Marie Curie.
- Many mathematical challenges related to the non conservative forces!

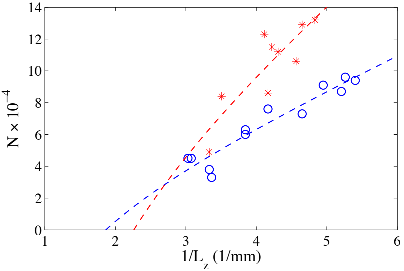
Cigar-shaped traps - experiments

- Experiment: Maryvonne Chalony, David Wilkowski (Institut Non Linéaire de Nice)
- Strontium. Size $\sim$ $500\mu m$; temperature $\sim 2\mu K$; number of atoms $\sim 10^5$.



Experimental signatures

Theory, in the self gravitating limit (trap=negligible): $L \propto 1/N$
L = cloud's size; N = number of atoms



N vs $1/L$. Red: $T \simeq 1.5\mu K$, Blue: $T \simeq 2.1\mu K$; the theory includes the trap.

Other experimental signatures: density profile; oscillation modes $\rightarrow$ qualitative agreement; difficult to be more precise...

Dynamics: incompressible Euler limit (1)

Starting point = scaled Vlasov-Poisson-Fokker-Planck equation.
$\varepsilon \to 0$: strong interaction, strong external force; $\theta \to 0$: small temperature.

$$\partial_t f_\varepsilon + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\varepsilon - \left(\frac{1}{\varepsilon} \nabla_{\mathbf{x}} \Phi_{\rm ext} + \nabla_{\mathbf{x}} \Phi_\varepsilon \right) \cdot \nabla_{\mathbf{v}} f_\varepsilon = \nabla_{\mathbf{v}} \cdot \left(\mathbf{v} f_\varepsilon + \theta \nabla_{\mathbf{v}} f_\varepsilon \right), \quad (1)$$

$$\Delta_x \Phi = -\frac{1}{\varepsilon} \rho_\varepsilon, \qquad \rho_\varepsilon(t,x) = \int f_\varepsilon(t,x,v)\, \mathrm{d}v$$

Question: Solution of (1) $\to$ incompressible fluid dynamics?
More precisely: Take the first moment $J^\varepsilon(\mathbf{x},t) = \int f_\varepsilon d\mathbf{v}$.
Question: $\rho^\varepsilon(x,t) \to \rho_{\rm stat}(x)$, $J^\varepsilon(x,t) \to \rho_{\rm stat}(x) \mathbf{V}(x,t)$?

$$\begin{cases} \partial_t \mathbf{V} + \nabla_x \cdot (\mathbf{V} \otimes \mathbf{V}) + \nabla_x p = -\mathbf{V} \text{ on } \mathcal{K} + \text{no flux on } \partial\mathcal{K} \\ \nabla_x \cdot (\rho_{\rm stat} \mathbf{V}) = 0, \end{cases} \quad (2)$$

Dynamics: incompressible Euler limit (2)

Statement of the Theorem (simplified!):
Main hypothesis:
Well prepared initial conditions: initial density and velocity compatible with the limit equation; small potential energy, initial velocity distribution close to a Dirac
$\rightarrow$ fast "sound wave" oscillations are ruled out.

Conclusions:
i) Density $\rho_\varepsilon \rightarrow \rho_{\text{stat}}$
ii) First velocity moment $J_\varepsilon \rightarrow \rho_{\text{stat}}\mathbf{V}$

Method: The monokinetic approximation propagates in time.

Consequences: A simpler model to study the dynamics in certain regimes.