Rigidity of the two-dimensional one-component Coulomb plasma

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One-component Coulomb plasma (OCP)

- N positive charges in the two-dimensional plane: $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$.
- Confining potential $V : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ with sufficient growth at $+\infty$.
- Energy:

$$H_{N,V}(z) = \sum_{j \neq k} \log \frac{1}{|z_j - z_k|} + N \sum_j V(z_j).$$

- Probability measure:

$$P_{N,V,\beta}(dz) = \frac{1}{Z_{N,V,\beta}} \mathrm{e}^{-\beta H_{N,V}(z)}\, m^{\otimes N}(dz),$$

  where m is the Lebesgue measure on $\mathbb{C}$ and $Z_{N,V,\beta}$ the partition function.
- More generally, we could consider the 3d Coulomb plasma.

Potential theory

$$I_V(\mu) = \iint \log \frac{1}{|z-w|} \mu(dz)\, \mu(dw) + \int V(z)\, \mu(dz)$$

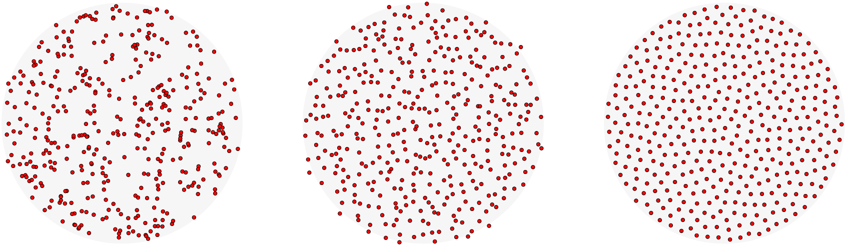
Theorem (Frostman)

- Unique probability measure μ_V minimizing I_V (equilibrium measure).
- Its support $S_V = \operatorname{supp} \mu_V$ is compact.
- Let $U^\mu(z) = \int \log \frac{1}{|z-w|} \mu(dw)$. Characterized by Euler–Lagrange equation:

$$U^{\mu_V} + \tfrac{1}{2} V = c \quad \text{q.e. in } S_V \quad \text{and}$$
$$U^{\mu_V} + \tfrac{1}{2} V \geqslant c \quad \text{q.e. in } \mathbb{C}.$$

- $\Delta \mu_V = \frac{1}{4\pi} (\Delta V) \mathbf{1}_{S_V}$. Main difficulty is to determine support S_V.
- Example: if $V = |z|^2$ then $\mu_V = \frac{1}{\pi} \mathbf{1}_{\{|z| \leqslant 1\}}$.

$N = 400,\ V = |z|^2$



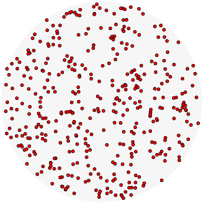
$\beta \to 0$ independent particles | $\beta = 1$ Ginibre ensemble | $\beta = 200$

The Coulomb plasma looks much more rigid than independent particles.

Linear statistics

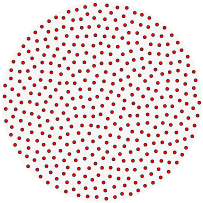
Let $f : \mathbb{C} \to \mathbb{R}$ be macroscopically smooth. How large are the fluctuations of

$$\sum_j f(z_j)?$$



For independent particles on the disk, $f(z_j)$ are i.i.d random variables. The CLT implies

$$\sum_j f(z_j) - N \int_{\mathbb{D}} f \, \frac{dz}{\pi} \approx N^{1/2}.$$



For particles in a crystalline state, on the other hand

$$\sum_j f(z_j) - N \int_{\mathbb{D}} f \, \frac{dz}{\pi} \approx 1.$$

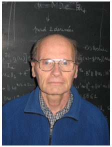
Some motivations

- Laughlin's guess for wave functions at fractional fillings of type $\frac{1}{2s+1}$ (fractional quantum Hall effect):

$$\phi_s(z_1,\dots,z_N)=\prod_{i<j}(z_i-z_j)^{2s+1}e^{-\sum|z_i|^2}$$

- Special case: Ginibre ensemble (eigenvalues of complex Gaussian matrix)

$$\beta=1 \quad \text{and} \quad V(z)=|z|^2.$$



More generally, for $\beta=1$ eigenvalues of random normal matrices (we will come back to it).

- Major question: phase transition for $\beta>\beta_c\approx 142$? Understanding small discrepancy is a small step towards such phenomena.
Alastuey and Jancovici (1980): *It is very likely that the model has a solid-fluid phase transition.*

Convergence to equilibrium measure

Empirical measure $\hat{\mu}$ and equilibrium measure μ_V:

$$\hat{\mu} = \frac{1}{N} \sum_j \delta_{z_j}, \qquad \mu_V = \operatorname{argmin} I_V.$$

Then $\hat{\mu} \to \mu_V$ weakly for $\beta > 0$ and reasonable V. More precise results:

- (Ben Arous–Zeitouni) LDP for $\hat{\mu}$ with rate function I_V at speed N^2.
  - Local density on macroscopic scale 1.
- (Leblé–Serfaty) LDP for certain tagged point process at speed N.
  - Essentially corresponds to partition function estimate

$$-\frac{1}{\beta} \log Z_{N,V,\beta} = N^2 I_V(\mu_V) - \frac{1}{2} N \log N + N \left(\frac{1}{\beta} - \frac{1}{2}\right) \left(\int \mu_V(z) \log \mu_V(z) \, dm\right) + F_\beta N + o(N).$$

  - Local density down to mesoscopic scales down to $N^{-1/4}$ near any fixed point in $\mathbb{C}$.
  - Previous results of Sandier–Serfaty, Rougerie–Serfaty, and others.
- (Lieb, unpublished) Points in minimizers of $H_{N,V}$ separated by $cN^{-1/2}$.

Determinantal case $\beta = 1$

- For $\beta = 1$ the correlation functions are determinantal:

$$p_N^{(n)}(z_1, \dots, z_n) = \det(K_N(z_i, z_j))_{i,j=1}^n$$

with $K_N(z, w) = \sum_{k=0}^{N-1} q_k(z)\overline{q_k(w)} \mathrm{e}^{-NV(z)/2}\mathrm{e}^{-NV(w)/2}$, and q_k are orthogonal polynomials (OP) with respect to $L^2(\mathrm{e}^{-NV})$.

- For Ginibre ensemble $V = |z|^2$ the OP are given by $q_k = z^k/\sqrt{\pi k!}$.
- Very precise results known using determinantal structure. Example: convergence of linear statistics to Gaussian free field (Rider–Virag $V = |z|^2$; Ameur–Hedenmalm–Makarov V smooth):

$$\sum_j f(z_j) - N \int f \, d\mu_V \overset{N\to\infty}{\longrightarrow} \mathrm{Normal}\left(0, \frac{1}{4\pi} \int |\nabla f^S|^2 \, dm\right),$$

for smooth f, where f^S is the bounded harmonic extension of $f|_S$ to $\mathbb{C}$.

- Exercise: Fluctuations of number of particles in a domain Ω are $\sim (N^{1/2}|\partial\Omega|)^{1/2}$ (unpublished). What if the boundary has no finite length (unknown)?

Main result

Theorem (Bauerschmidt–B–Nikula–Yau)

Let $s \in (0, \frac{1}{2})$, z_0 be in the interior of the support of μ_V, and $f : \mathbb{C} \to \mathbb{R}$ have support in the disk of radius N^{-s} centred at z_0.

Then for any sufficiently small $\varepsilon > 0$ and any $\beta > 0$, we have

$$\sum_{j=1}^{N} f(z_j) - N \int f(z)\, \mu_V(dz) = O(N^{\varepsilon}) \left(\sum_{l=1}^{4} N^{-ls} \|\nabla^l f\|_\infty \right),$$

with probability at least $1 - \mathrm{e}^{-\beta N^{\varepsilon}}$ for sufficiently large N.

- Optimal scale N^{-s} for all $s \in (0, \frac{1}{2})$ and applies to all $\beta > 0$.
- Rigidity: fluctuations are $N^{o(1)}$ compared to $N^{\frac{1}{2}-s}$ for i.i.d. particles.
- The dominant fluctuation term is $N^{\varepsilon} O(\int |\nabla f|^2)$.
- Simultaneous result (Leblé): Fluctuations bounded by $N^{\frac{3}{4}-\frac{s}{2}}$.

Comparision with 1D

Pair interaction for particles on real line

- Coulomb interaction: $\sum_{j,k} -|x_j - x_k|$
- Logarithmic interaction: $\sum_{j,k} -\log|x_j - x_k|$

Interactions are convex on simplex $\{x_1 < x_2 < \cdots < x_N\}$.

$1D$-Coulomb gas crystallizes:

- (Kunz) 1-point function is nontrivially periodic for most β;
- (Brascamp–Lieb) 1-point function is nontrivially periodic for all β large;
- (Aizenman–Martin) translational symmetry broken for all β.

Related results for log gas in $d = 1$

β-ensemble has been studied extensively in $d = 1$. In particular:

- (Johansson) Linear statistics converge to Gaussian field with covariance proportional to $(-\Delta)^{1/2}$ for all $\beta > 0$;
- (Deift et al., Bleher–Its, Pastur–Shcherbina, ...) Universality of local correlations for $\beta = 1, 2, 4$;
- (Dumitriu–Edelman) Representation as eigenvalues of tridiagonal matrix for $V = \lambda^2$ and all $\beta > 0$;
- (Valko–Virag) Explicit characterization of the point process for $V = \lambda^2$ and all $\beta > 0$;
- (Borot–Guionnet) $1/N$ expansion of the partition function;
- (B-Erdős-Yau) Rigidity and universality of local correlations for all $\beta > 0$;
- (Shcherbina), (Bekerman–Figalli–Guionnet) alternative proofs of the universality for all $\beta > 0$;

The proofs do not apply in $d = 2$. For non-Hermitian matrices with iid entries, similar rigidity by B-Yau-Yin.

Strategy

Step 1 Multiscale iteration to show that μ_V provides local density on all scales N^{-s} with $s \in (0, \frac{1}{2})$:

- Use mean-field bounds and potential theory in each step.
- Optimal scale but bound on order of fluctuations is not optimal.

Step 2 Use Loop Equation to obtain optimal order for smooth linear statistics:

- The loop equation is singular in two dimensions.
- Singularity controlled using Step 1.

For f with support in $B(z_0, N^{-s})$:

(Step 1) $$\frac{1}{N}\sum_{j=1}^{N} f(z_j) - \int f(z)\,\mu_V(dz) = O(N^{-\frac{1}{2}}\log N)\left(\sum_{l=1}^{2} N^{-ls}\|\nabla^l f\|_\infty\right)$$

(Step 2) $$\frac{1}{N}\sum_{j=1}^{N} f(z_j) - \int f(z)\,\mu_V(dz) = O(N^{-1+\varepsilon})\left(\sum_{l=1}^{4} N^{-ls}\|\nabla^l f\|_\infty\right)$$

Initial estimate

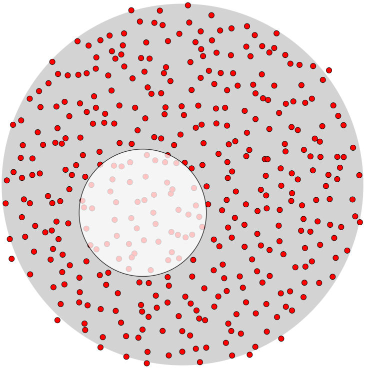
Simple mean-field estimate controls scales $\gg N^{-1/4}$.

- Let
$$Z_{N,V,\beta} = \int e^{-\beta H_{N,V}(z)}\, m^{\otimes N}(dz).$$
- Newton's electrostatic theorem $-\log \geqslant -\log * \rho$ for radial probability ρ:
$$Z_{N,V,\beta} \leqslant \mathrm{e}^{-N^2 I_V(\mu_V)+O(N\log N)}$$
- Jensen inequality:
$$Z_{N,V,\beta} \geqslant \mathrm{e}^{-N^2 I_V(\mu_V)-O(N\log N)}$$
- Applying this with $V \to V + \frac{1}{\beta N} f$ gives
$$\mathbb{E}_{N,V,\beta}(\mathrm{e}^{\sum_j f(z_j)}) \leqslant e^{N\int f\, d\mu_V + \frac{1}{8\pi}(f,-\Delta f)+O(N\log N)}.$$

  This gives control on scales $\gg N^{-1/4}$.

Multiscale iteration

- Condition on particles outside a small disk of radius $\approx N^{-1/4}$.
- Conditional measure (inside small disk) is again a Coulomb gas, but with only $\approx N^{1/2}$ particles.
- If initial mean-field estimate can be applied to conditional system would get an estimate at scale $\gg (N^{1/2})^{-1/4}$
- Difficulty: the conditional system has a singular potential given by the external charges.

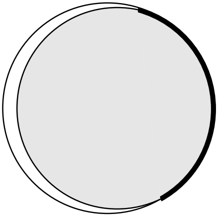


Control of conditional measure

For the equilibrium measure of the conditioned system with high probability:

- The support contains most of the disk conditioned on.
- The boundary charge (which exists since $V = +\infty$ outside the disk) has uniformly bounded density.

These conditions give enough regularity to repeat the mean-field bound.



Their proof is achieved in the obstacle problem formulation of the equilibrium measure by construction of dominating potentials.

Obstacle problem

- Potential of equilibrium measure characterized by obstacle problem:

$$u_V(z) = \sup \Big\{ v(z) \colon v \text{ subharmonic on } \mathbb{C},\ v \leqslant \tfrac{1}{2}V \text{ on } \mathbb{C}, \limsup_{|z|\to\infty} \big(v(z) - \log|z| \big) < \infty \Big\}$$

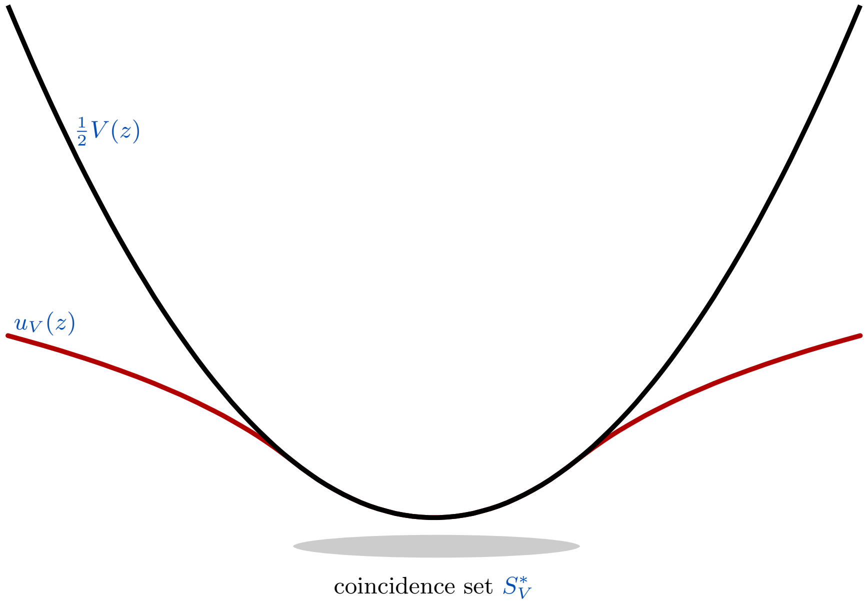
- v subharmonic and $\limsup_{|z|\to\infty} \big(v(z) - \log|z| \big) < \infty$ imply $v = c - U^{\nu}$ where U^{ν} is some potential of positive measure ν with mass $\leqslant 1$.
- Coincidence set: $S_V^* = \{ u_V(z) = \frac{1}{2}V \}$.

Theorem

Let μ_V be the equilibrium measure (minimizer of I_V). Then (essentially)

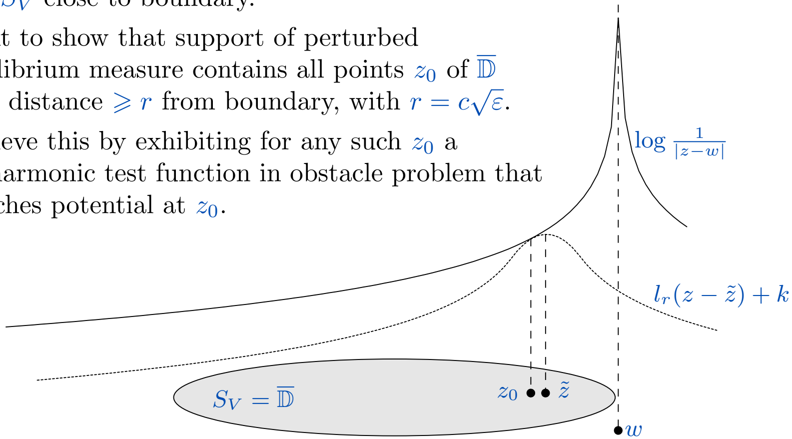
$$u_V(z) = c - U^{\mu_V}(z), \qquad S_V = S_V^*.$$

Obstacle problem



Example

- Assume support of equilibrium measure is unit disk $S_V = \overline{\mathbb{D}}$.
- Perturb external potential by a single charge ε at $w \notin S_V$ close to boundary.
- Want to show that support of perturbed equilibrium measure contains all points z_0 of $\overline{\mathbb{D}}$ with distance $\geqslant r$ from boundary, with $r = c\sqrt{\varepsilon}$.
- Achieve this by exhibiting for any such z_0 a subharmonic test function in obstacle problem that matches potential at z_0.



Local density

By iteration of mean-field bound we show that μ_V provides local density.

Theorem

Let $s \in (0, \frac{1}{2})$. For any z_0 in the interior of the support of μ_V, and for any $f \in C_c^2(\mathbb{C})$ with support in the disk of radius N^{-s} centred at z_0, we have

$$\frac{1}{N}\sum_{j=1}^{N} f(z_j) - \int f(z)\,\mu_V(dz) = O(\log N)\left(N^{-1-2s}\|\Delta f\|_\infty + N^{-\frac{1}{2}-s}\|\nabla f\|_2\right),$$

with probability at least $1 - \mathrm{e}^{-(1+\beta)N^{1-2s}}$ for sufficiently large N.

- RHS is $N^{-\frac{1}{2}-s+o(1)}$ for smooth f on scale N^{-s} (similar to i.i.d. particles).
- Rigidity: RHS is actually $N^{-1+o(1)}$ for such f.

Rigidity

Cumulant generating function for linear statistics:

$$F_{N,V,\beta}(f) = \log \mathbb{E}_{N,V,\beta}(\mathrm{e}^{X_f}),$$

with

$$X_f = \sum_j f(z_j) - N \int f \, d\mu_V = N \int f \, d\tilde{\mu}_V$$

where

$$\hat{\mu} = \frac{1}{N} \sum_j \delta_{z_j} \quad \text{and} \quad \tilde{\mu}_V = \hat{\mu} - \mu_V.$$

- Rigidity follows from estimate $F_{N,V,\beta}(f) = O(\beta N^{\varepsilon})$.
- Difficult to see using direct potential theory.
- It would suffice to bound $\frac{\partial}{\partial t} F_{N,V,\beta}(tf)$ since $F_{N,V,\beta}(0) = 0$.

Loop Equation

For any reasonable function h, we have the loop equation:

$$\mathbb{E}_{N,V,\beta}\left(\frac{1}{2}\sum_{j\neq k}\frac{h(z_j)-h(z_k)}{z_j-z_k}+\frac{1}{\beta}\sum_j \partial h(z_j)-N\sum_j h(z_j)\partial V(z_j)\right)=0.$$

Proof.

By integration by parts:

$$\begin{aligned}\mathbb{E}_{N,V,\beta}\left(\partial h(z_j)\right) &= \beta\mathbb{E}_{N,V,\beta}(h(z_j)\partial_{z_j}H(z)) \\ &= \beta\mathbb{E}_{N,V,\beta}\left(h(z_j)\left(\sum_{k:k\neq j}\frac{-1}{z_j-z_k}+N\partial V(z_j)\right)\right).\end{aligned}$$

Loop equation follows immediately by summation over j. □

Loop Equation also has an interpretation as Schwinger–Dyson equation or Conformal Ward Identity (Wiegmann–Zabrodin, Makarov et al.).

Loop Equation and Euler–Lagrange equation for equilibrium measure give (here $h = \bar\partial f/\Delta V$):

$$\frac{\partial}{\partial t} F_{N,V,\beta}(tf) = \mathbb{E}_{N,V-tf/(\beta N),\beta}\bigg(\frac{1}{\beta}\int \partial h\, d\hat\mu + \frac{t}{\beta}\int h\partial f\, d\hat\mu + \frac{N}{2}\iint \frac{h(z)-h(w)}{z-w}\mathbf{1}_{\{z\neq w\}}\,\tilde\mu_V(dz)\,\tilde\mu_V(dw)\bigg).$$

- First two terms on RHS are linear statistics: could be estimated by standard estimates for macroscopic f, by local density for mesoscopic f.
- Difficulty is the third term on RHS:

$$\frac{h(z)-h(w)}{z-w} = \partial h(z) + \bar\partial h(z)\frac{\bar z - \bar w}{z-w} + O(|z-w|),$$

  and the second term on the right-hand side is not smooth on the diagonal.
- Use multiscale decomposition and local density to control singularity. The Fefferman/de la Llave trick: for any compactly supported $\varphi : [0,\infty) \to \mathbb{R}$ we have

$$\frac{h(z)-h(w)}{z-w} = C\int_0^\infty \int_{\mathbb{C}} \varphi(|z-\zeta|/t)\varphi(|w-\zeta|/t)(\bar z - \bar w)(h(z)-h(w))\, m(d\zeta)\,\frac{dt}{t^5}.$$