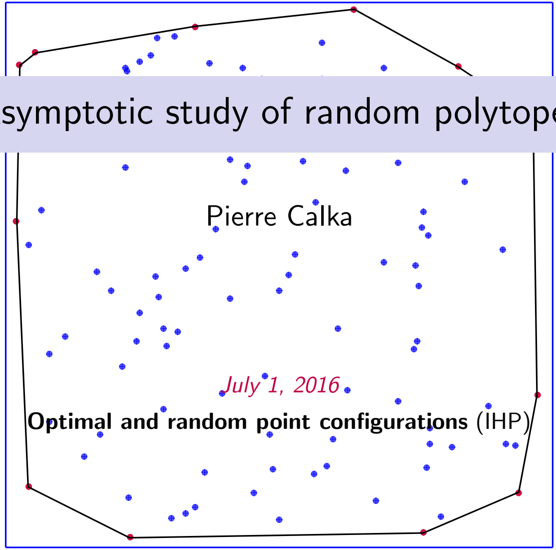
Asymptotic study of random polytopes



Pierre Calka

July 1, 2016

Optimal and random point configurations (IHP)

Outline

Random polytopes: an overview

Main results: variance asymptotics

Case of the ball: sketch of proof and scaling limit

Case of a simple polytope: sketch of proof and scaling limit

Joint works with **Joseph Yukich** (Lehigh University, USA) & **Tomasz Schreiber** (Toruń University, Poland)

Outline

Random polytopes: an overview
- Uniform case
- Gaussian case
- Expectation asymptotics

Main results: variance asymptotics

Case of the ball: sketch of proof and scaling limit

Case of a simple polytope: sketch of proof and scaling limit

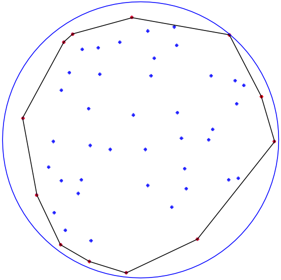
Uniform case

Binomial model

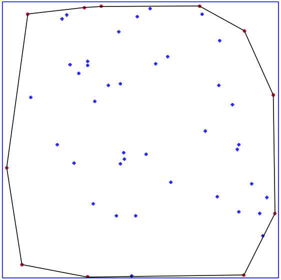
K := convex body of $\mathbb{R}^d$

$(X_k, k \in \mathbb{N}^*)$:= independent and uniformly distributed in K

$\overline{K}_n := \mathrm{Conv}(X_1, \cdots, X_n)$, $n \geq 1$



$\overline{K}_{50}$, K ball



$\overline{K}_{50}$, K square

Uniform case

Binomial model

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$(X_{k}, {\scriptstyle k \in \mathbb{N}^*}) :=$ independent and uniformly distributed in K

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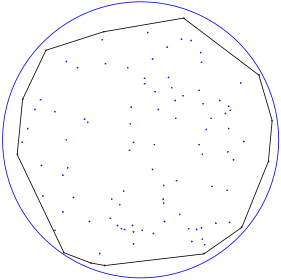
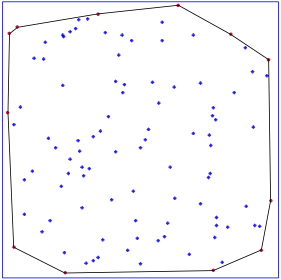


$\overline{K}_{100}$, K ball



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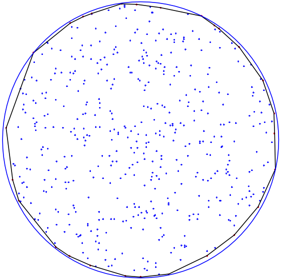
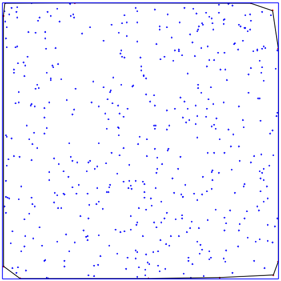


$\overline{K}_{500}$, K ball



$\overline{K}_{500}$, K square

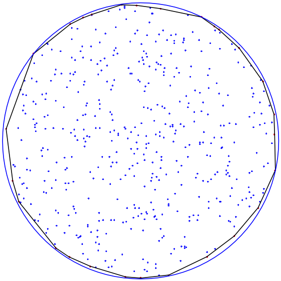
Uniform case

Poisson model

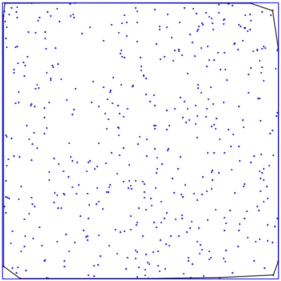
K := convex body of $\mathbb{R}^d$

$\mathcal{P}_\lambda$, $\lambda > 0$:= Poisson point process of intensity measure $\lambda \mathrm{d}x$

$K_\lambda := \mathrm{Conv}(\mathcal{P}_\lambda \cap K)$



$\overline{K}_{500}$, K ball



$\overline{K}_{500}$, K square

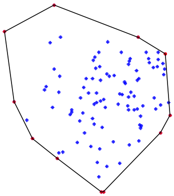
Gaussian case

Poisson model

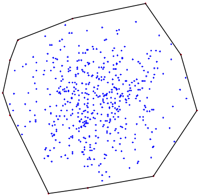
$\varphi_d(x) := \frac{1}{(2\pi)^{d/2}} e^{-\|x\|^2/2}$, $x \in \mathbb{R}^d$, $d \geq 2$

$\mathcal{P}_\lambda$, $\lambda > 0$:= Poisson point process of intensity measure $\lambda \varphi_d(x) \mathrm{d}x$

$K_\lambda := \mathrm{Conv}(\mathcal{P}_\lambda)$



K_{100}



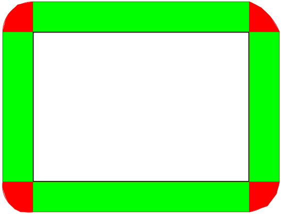
K_{500}

Considered functionals

- $f_k(\cdot)$: number of k-dimensional faces, $1 \leq k \leq d$
- $\mathrm{Vol}(\cdot)$: volume, $V_{d-1}(\cdot)$: half-area of the boundary
- $V_k(\cdot)$: k-th intrinsic volume, $1 \leq k \leq d$

The functionals V_k are defined through Steiner formula:

$$\mathrm{Vol}(K+B(0,r)) = \sum_{k=0}^{d} r^{d-k} \kappa_{d-k} V_k(K), \quad \text{where } \kappa_d := \mathrm{Vol}(\mathbb{B}^d)$$



$d = 2$: $A(K + B(0,r)) = A(K) + P(K)r + \pi r^2$

Expectation asymptotics

B. Efron's relation (1965) $\mathbb{E}f_0(\overline{K}_n) = n\left(1 - \frac{\mathbb{E}\mathrm{Vol}(\overline{K}_{n-1})}{\mathrm{Vol}(K)}\right)$

Uniform case, K smooth $\mathbb{E}[f_k(K_\lambda)] \underset{\lambda\to\infty}{\sim} c_{d,k} \int_{\partial K} \kappa_s^{\frac{1}{d+1}} \mathrm{d}s \; \lambda^{\frac{d-1}{d+1}}$

$\kappa_s :=$ Gaussian curvature of ∂K

Uniform case, K polytope $\mathbb{E}[f_k(K_\lambda)] \underset{\lambda\to\infty}{\sim} c'_{d,k} F(K) \log^{d-1}(\lambda)$

$F(K) :=$ number of flags of K

Gaussian polytope $\mathbb{E}[f_k(K_\lambda)] \underset{\lambda\to\infty}{\sim} c''_{d,k} \log^{\frac{d-1}{2}}(\lambda)$

A. Rényi & R. Sulanke (1963), **H. Raynaud** (1970), **R. Schneider & J. Wieacker** (1978), **F. Affentranger & R. Schneider** (1992)

Outline

Random polytopes: an overview

Main results: variance asymptotics

- Uniform case, K smooth
- Gaussian polytopes
- Uniform case, K simple polytope

Case of the ball: sketch of proof and scaling limit

Case of a simple polytope: sketch of proof and scaling limit

Uniform case, K smooth: state of the art

- *Identities relating higher moments*

C. Buchta (2005):

$$c\lambda^{\frac{d-1}{d+1}} \leq \mathrm{Var}[f_0(K_\lambda)]$$

- *Number of faces and volume*

M. Reitzner (2005):

$$c\lambda^{\frac{d-1}{d+1}} \leq \mathrm{Var}[f_k(K_\lambda)] \leq C\lambda^{\frac{d-1}{d+1}}$$

- *Intrinsic volumes*

I. Bárány, F. Fodor & V. Vigh (2009):

$$c\lambda^{-\frac{d+3}{d+1}} \leq \mathrm{Var}[V_k(K_\lambda)] \leq C\lambda^{-\frac{d+3}{d+1}}$$

- *Central limit theorems*

M. Reitzner (2005):

$$\mathbb{P}\left[\frac{f_k(K_\lambda) - \mathbb{E}[f_k(K_\lambda)]}{\sqrt{\mathrm{Var}[f_k(K_\lambda)]}} \leq t\right] \underset{\lambda\to\infty}{\to} \int_{-\infty}^{t} e^{-x^2/2}\frac{\mathrm{d}x}{\sqrt{2\pi}}.$$

Uniform case, K smooth: limiting variances

$K :=$ convex body of $\mathbb{R}^d$ with volume 1 and with a $\mathcal{C}^3$ boundary

$\kappa :=$ Gaussian curvature of ∂K

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d+1)} \mathrm{Var}[f_k(K_\lambda)] = c_{k,d} \int_{\partial K} \kappa(z)^{1/(d+1)} dz$$

$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)} \mathrm{Var}\left[\mathrm{Vol}(K_\lambda)\right] = c'_d \int_{\partial K} \kappa(z)^{1/(d+1)} dz$$

($c_{k,d}, c'_d$ explicit positive constants)

Remarks.

- Similar results for the binomial model
- Case of the ball: similar results for $V_k(K_\lambda)$, functional central limit theorem for the defect volume

Gaussian polytopes: state of the art

- *Number of faces*

  D. Hug & M. Reitzner (2005), **I. Bárány & V. Vu** (2007):

$$c \log^{\frac{d-1}{2}}(n) \leq \mathrm{Var}[f_k(K_n)] \leq C \log^{\frac{d-1}{2}}(n)$$

- *Volume*

  I. Bárány & V. Vu (2007):

$$c \log^{\frac{d-3}{2}}(n) \leq \mathrm{Var}[\mathrm{Vol}(K_n)] \leq C \log^{\frac{d-3}{2}}(n)$$

- *Central limit theorems*

  I. Bárány & V. Vu (2007)

- *Intrinsic volumes*

  D. Hug & M. Reitzner (2005):

$$\mathrm{Var}[V_k(K_n)] \leq C \log^{\frac{k-3}{2}}(n)$$

Gaussian polytopes: limiting variances

$$\lim_{n\to\infty} \log^{-\frac{d-1}{2}}(\lambda)\mathrm{Var}[f_k(K_\lambda)] = c_{d,k} \in (0,\infty)$$

$$\lim_{n\to\infty} \log^{-k+\frac{d+3}{2}}(\lambda)\mathrm{Var}[V_k(K_\lambda)] = c'_{d,k} \in (0,\infty)$$

$$c''_{d,k}{}^{-1} \log^{-k/2}(\lambda)\mathbb{E}[V_k(K_\lambda)] \underset{\lambda\to\infty}{=} 1 - \frac{k\log(\log\lambda)}{4\log\lambda} + O((\log^{-1}(\lambda))$$

Remarks.

- Similar results for the binomial model
- Functional CLT for the defect volume

Uniform case, K polytope: state of the art

- *Number of faces and volume*

  I. Bárány & M. Reitzner (2010)

$$c_{d,k} F(K) \log^{d-1}(\lambda) \leq \mathrm{Var}[f_k(K_\lambda)] \leq c'_{d,k} F(K)^3 \log^{d-1}(\lambda)$$

$$c_{d,k} F(K) \frac{\log^{d-1}(\lambda)}{\lambda^2} \leq \mathrm{Var}[\mathrm{Vol}(K_\lambda)] \leq c'_{d,k} F(K)^3 \frac{\log^{d-1}(\lambda)}{\lambda^2}$$

- *Central limit theorems*

  I. Bárány & M. Reitzner (2010b)

Uniform case, K simple polytope: limiting variances

$K :=$ simple polytope of $\mathbb{R}^d$ with volume 1

$$\lim_{\lambda\to\infty} \log^{-(d-1)}(\lambda)\mathrm{Var}[f_k(K_\lambda)] = c_{d,k} f_0(K)$$

$$\lim_{\lambda\to\infty} \lambda^2 \log^{-(d-1)}(\lambda)\mathrm{Var}[\mathrm{Vol}(K_\lambda)] = c'_{d,k} f_0(K)$$

Outline

Calculation of the expectation of $f_k(K_\lambda)$

- Decomposition:

$$\mathbb{E}[f_k(K_\lambda)] = \mathbb{E}\left[\sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda)\right]$$

$$\xi(x, \mathcal{P}_\lambda) := \begin{cases} \frac{1}{k+1} \# k\text{-face containing } x & \text{if } x \text{ extreme} \\ 0 & \text{if not} \end{cases}$$

- Mecke-Slivnyak formula

$$\mathbb{E}[f_k(K_\lambda)] = \lambda \int_{\mathbb{B}^d} \mathbb{E}[\xi(x, \mathcal{P}_\lambda \cup \{x\})] \mathrm{d}x$$

Calculation of the variance of $f_k(K_\lambda)$

$$
\begin{aligned}
&\mathrm{Var}[f_k(K_\lambda)] \\
&= \mathbb{E}\left[\sum_{x\in\mathcal{P}_\lambda} \xi^2(x,\mathcal{P}_\lambda) + \sum_{x\neq y\in\mathcal{P}_\lambda} \xi(x,\mathcal{P}_\lambda)\xi(y,\mathcal{P}_\lambda)\right] - \left(\mathbb{E}[f_k(K_\lambda)]\right)^2 \\
&= \lambda \int_{\mathbb{B}^d} \mathbb{E}[\xi^2(x,\mathcal{P}_\lambda\cup\{x\})]\mathrm{d}x \\
&\qquad + \lambda^2 \iint_{(\mathbb{B}^d)^2} \mathbb{E}[\xi(x,\mathcal{P}_\lambda\cup\{x,y\})\xi(y,\mathcal{P}_\lambda\cup\{x,y\})]\mathrm{d}x\mathrm{d}y \\
&\qquad - \lambda^2 \iint_{(\mathbb{B}^d)^2} \mathbb{E}[\xi(x,\mathcal{P}_\lambda\cup\{x\})]\mathbb{E}[\xi(y,\mathcal{P}_\lambda\cup\{y\})]\mathrm{d}x\mathrm{d}y \\
&= \lambda \int_{\mathbb{B}^d} \mathbb{E}[\xi^2(x,\mathcal{P}_\lambda\cup\{x\})]\mathrm{d}x \\
&\qquad + \lambda^2 \iint_{(\mathbb{B}^d)^2} \text{"Cov"}(\xi(x,\mathcal{P}_\lambda\cup\{x\}),\xi(y,\mathcal{P}_\lambda\cup\{y\}))\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

Scaling transform

Question: Limits of $\mathbb{E}[\xi(x,\mathcal{P}_\lambda)]$ and "Cov"$(\xi(x,\mathcal{P}_\lambda),\xi(y,\mathcal{P}_\lambda))$?
Answer: definition of limit scores in a new space

- *Scaling transform*:

$$T^\lambda : \begin{cases} \mathbb{B}^{d-1}\setminus\{0\} & \longrightarrow & \mathbb{R}^{d-1}\times\mathbb{R}_+ \\ x & \longmapsto & (\lambda^{\frac{1}{d+1}}\exp_{d-1}^{-1}(x/\|x\|), \lambda^{\frac{2}{d+1}}(1-\|x\|)) \end{cases}$$

$\exp_{d-1} : \mathbb{R}^{d-1} \simeq T_{u_0}\mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ exponential map at $u_0 \in \mathbb{S}^{d-1}$

- *Image of a score*: $\xi^{(\lambda)}(T^\lambda(x), T^\lambda(\mathcal{P}_\lambda)) := \xi(x,\mathcal{P}_\lambda)$
- *Convergence of $\mathcal{P}_\lambda$*: $T^\lambda(\mathcal{P}_\lambda) \xrightarrow{D} \mathcal{P}$ where

$\mathcal{P}$:= homogeneous Poisson point process in $\mathbb{R}^{d-1}\times\mathbb{R}$ of intensity 1

Dual characterization of the extreme points

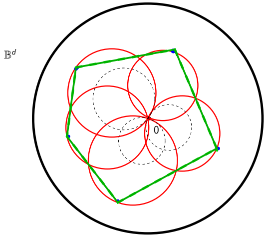
Given K_λ contains the origin,

$x \in \mathcal{P}_\lambda$ extreme

$\iff \exists H$ support hyperplane of K_λ, $x \in H$

$\iff \exists y \in \partial B\left(\frac{x}{2}, \frac{\|x\|}{2}\right)$ s. t. 0 and $\mathcal{P}_\lambda \setminus \{x\}$ on the same side of $(x + y^\perp)$

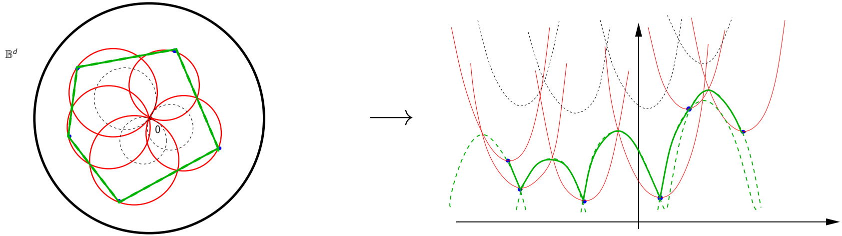
$\iff$ the petal of x, $B\left(\frac{x}{2}, \frac{\|x\|}{2}\right) \not\subset \bigcup_{x' \in \mathcal{P}_\lambda \setminus \{x\}} B\left(\frac{x'}{2}, \frac{\|x'\|}{2}\right)$



Action of the scaling transform

$$\Pi^{\uparrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \geq \frac{\|v\|^2}{2}\},\ \Pi^{\downarrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \leq -\frac{\|v\|^2}{2}\}$$

Half-space	Translate of $\Pi^{\downarrow}$
Boundary of the convex hull	Union of portions of down paraboloids
Petal	Translate of $\partial\Pi^{\uparrow}$
Extreme point	$(x + \Pi^{\uparrow})$ not fully covered



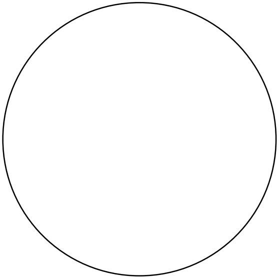
Outline

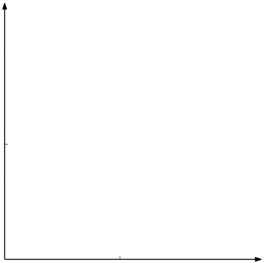
Floating body

$v(x) := \inf\{\mathrm{Vol}(K \cap H^+) : H^+ \text{ half-space containing } x\},\ x \in K$

Floating body : $K(v \geq t) := \{x \in K : v(x) \geq t\}$

$K(v \geq t)$ is a convex body and $K(v \geq 1/\lambda)$ is *close* to K_λ.





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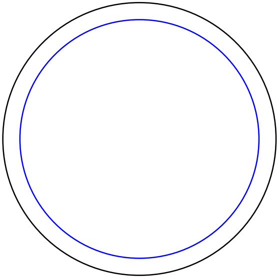
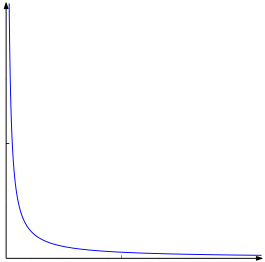


$\mathbb{B}^d(v \geq 1/\lambda) = (1 - f(\lambda))\mathbb{B}^d$

$f(\lambda) \sim c\lambda^{-\frac{2}{d+1}}$



Comparison between K_λ and the floating body

- *Expectation*

Bárány & Larman (1988):

$$c\mathrm{Vol}(K(v \leq 1/\lambda)) \leq \mathrm{Vol}(K) - \mathbb{E}[\mathrm{Vol}(K_\lambda)] \leq C\mathrm{Vol}(K(v \leq 1/\lambda))$$

- *Variance*

Bárány & Reitzner (2010):

$$c\lambda^{-1}\mathrm{Vol}(K(v \leq 1/\lambda)) \leq \mathrm{Var}[\mathrm{Vol}(K_\lambda)]$$

- *Sandwiching (polytope case)*

Bárány & Reitzner (2010b):

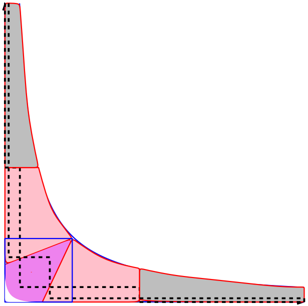
$$\mathbb{P}[\partial K_\lambda \not\subset [K(v \geq s) \setminus K(v \geq T)]] = O\left((\log(\lambda))^{-4d^2}\right)$$

$$s := \frac{c}{\lambda(\log(\lambda))^{4d^2+d-1}},\ T := c' \frac{\log(\log(\lambda))}{\lambda}$$

Additivity of the variance over the vertices

- $\mathcal{V}(K) :=$ set of vertices of K
- $p_\delta(v) :=$ *parallelepiped* with volume δ^d at v where $\delta = \exp(-(\log^{\frac{1}{d}}(\lambda)))$
- $Z_v := (k+1)^{-1} \sum_{x \in \mathcal{P}_\lambda \cap p_\delta(v)} \#\{k\text{-faces containing } x\}$

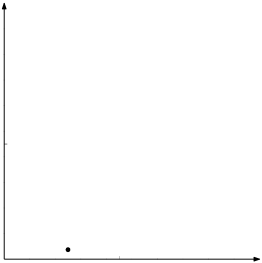
$$\mathrm{Var}[f_k(K_\lambda)] = \sum_{v \in \mathcal{V}(K)} \mathrm{Var}[Z_v] + o(\mathrm{Var}[f_k(K_\lambda)]).$$



Scaling transform in the vicinity of a vertex

- K identified with $(0,\infty)^d$ after scaling transformation
  Floating body $K(v = \frac{t}{\lambda}) = \{(z_1, \cdots, z_d) \in (0,\infty)^d : \prod_{i=1}^d z_i = c\frac{t}{\lambda}\}$
  $V := \{(y_1, \cdots, y_d) \in \mathbb{R}^d : \sum_{i=1}^d y_i = 0\} \cong \mathbb{R}^{d-1}$
- *Scaling transform*:

$$T^{(\lambda)} : \begin{cases} (0,\infty)^d & \longrightarrow \quad V \times \mathbb{R} \\ (z_1, \cdots, z_d) & \longmapsto \quad \left(\mathrm{proj}_V(\log(z)), \frac{1}{d}\log(\lambda \prod_{i=1}^d z_i)\right) \end{cases}$$



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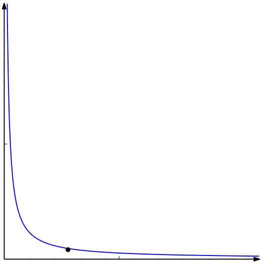

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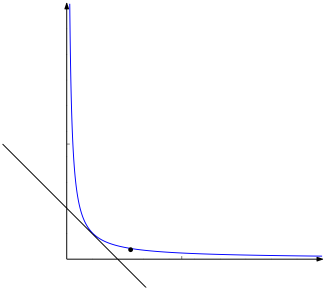

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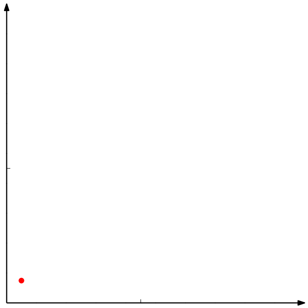
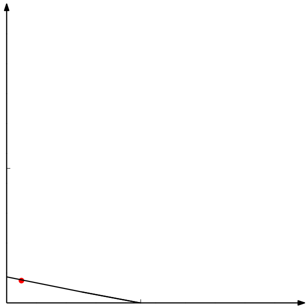
- *Convergence of $\mathcal{P}_\lambda$*: $T^\lambda(\mathcal{P}_\lambda) \xrightarrow{D} \mathcal{P}$ where

  $\mathcal{P} :=$ Poisson point process in $\mathbb{R}^{d-1}\times\mathbb{R}$ of intensity measure $\sqrt{d}e^{dh}\mathrm{d}v\mathrm{d}h$

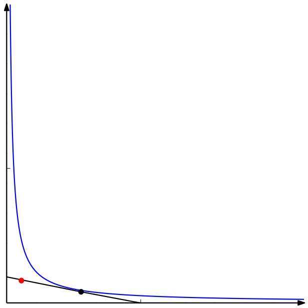
Dual characterization of extreme points



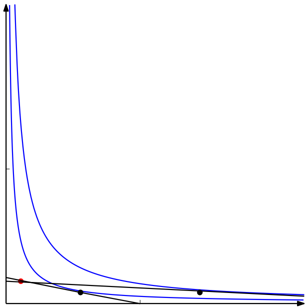
Dual characterization of extreme points



Dual characterization of extreme points



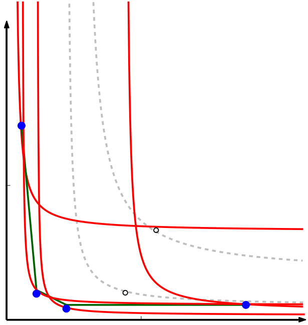
Dual characterization of extreme points



Dual characterization of extreme points

Each point $z \in (0, \infty)^d$ generates a petal $S(z)$, i.e. the set of all tangency points of surfaces $K(v = \frac{t}{\lambda})$, $t > 0$, with the hyperplanes containing z.

z is *cone-extreme* iff $S(z)$ is not fully covered by the other petals.

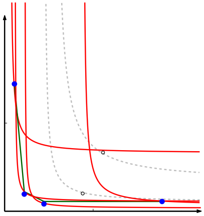


Action of the scaling transform

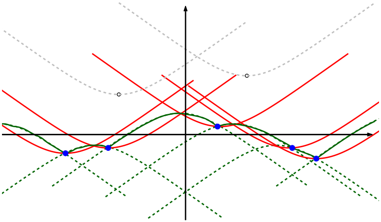
$$G(v) := \log\left(\frac{1}{d}\sum_{k=1}^{d} e^{\ell_k(v)}\right), \quad v = (\ell_1(v), \cdots, \ell_d(v)) \in V$$

$$\Pi^{\uparrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \geq G(-v)\},\ \Pi^{\downarrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \leq -G(v)\}$$

Floating bodies	horizontal half-spaces
Boundary of the convex hull	Union of portions down cone-like grains
Petal	Translate of $\partial\Pi^{\uparrow}$
Extreme point	$(x + \Pi^{\uparrow})$ not fully covered



$\longrightarrow$



Thank you for your attention!