Discrete Beta ensembles

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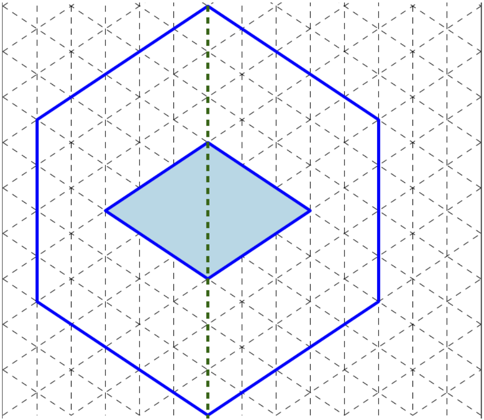
Joint works with A. Borodin, G. Borot, V. Gorin

Lozenges tiling of a 0



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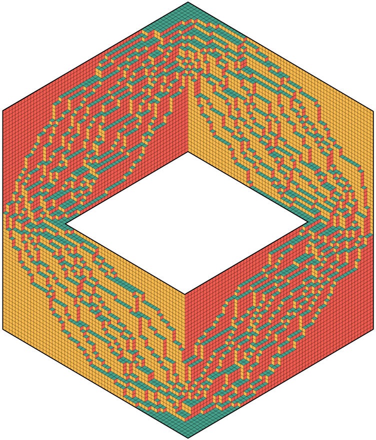
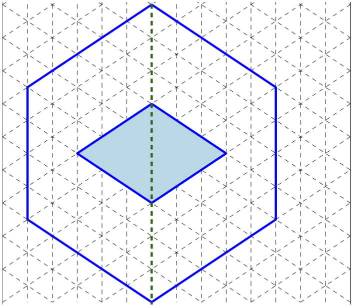


Leonid Petrov's simulations

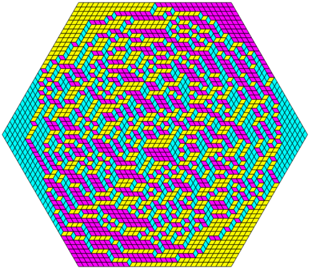
Outlook of the results

Take the tiling uniformly at random in a 0. Consider a vertical line in the middle of the tiling: it meets either horizontal tiles or the border of a tile in one of the two other directions. Let $\ell_1, \ldots, \ell_N$ be the positions of the horizontal tiles.



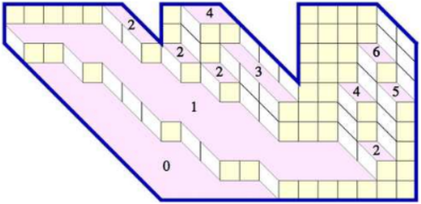
- $\frac{1}{N}\sum_{i=1}^{N} f(\ell_i/N) \mapsto \int f(x)d\mu(x)$ *a.s* for all f continuous.
- Assume the number of horizontal tiles in each connected component is given. Take f analytic. $\sum_{i=1}^{N}(f(\ell_i/N) - \mathbb{E}[f(\ell_i/N)])$ converges towards a centered Gaussian, with covariance as given by random matrix analogue.

Previous results



Petrov (2012) generalizes this result to a family of polygons (Gelfand-Tsetlin patterns), which includes the hexagon. Facets are allowed.

Kenyon (2004). Assume that there is no facets ($0 < \frac{d\mu}{dx} < 1$) [for the hexagon, the boundary is taken to be the limiting arctic circle (to avoid facets)]. Then, the fluctuations of the surface is given up to isomorphism by the Gaussian Free Field.



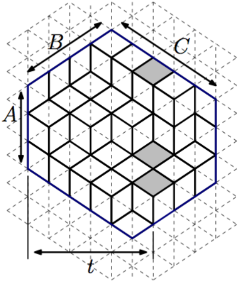
Tiling distribution

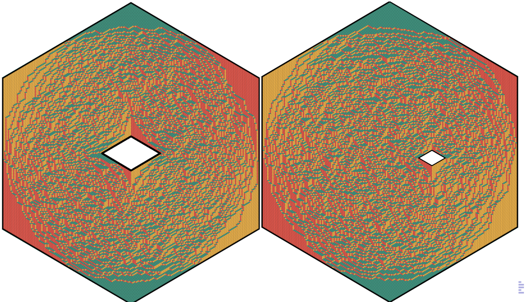
Let $\ell_i^h, 1 \leq i \leq N_h$, be the positions of the horizontal tiles in the $h-th$ connected component.

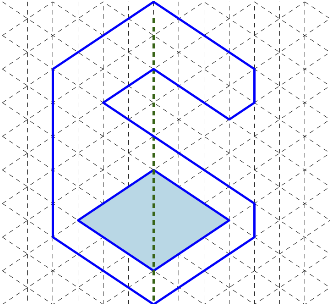
$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{h \leq h'} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N_{h'}, i<j}} |\ell_i^h - \ell_j^{h'}|^2 \prod_{i=1}^{N} w_h(\ell_i^h)$$

where $\ell_i^h \geq \ell_{i-1}^h + 1$.

For some w, this is the distribution of the N horizontal lozenges in a vertical line of a uniformly distributed tiling:





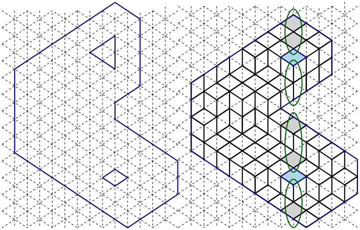


Let N_i be the number of horizontal tiles in the i-th connected component starting from bottom. The distribution of the $N = N_1 + N_2 + N_3$ horizontal lozenges in the central vertical line of a uniformly distributed tiling of the 6 is

$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{h \le h'} \prod_{\substack{1 \le i \le N_h \\ 1 \le j \le N_{h'}, i<j}} |\ell_j^{h'} - \ell_i^h|^{\theta_{h,h'}} \prod_{\substack{1 \le i \le N_h \\ h}} w_h(\ell_i^h)$$

where $\theta_{h,h'} = 2$ if $h = h'$ or $h = 1, h' = 2$, $\theta_{h,h'} = 1$ if h or $h' = 3$.

More examples



Let $N_i$be the number of horizontal tiles in the i-th connected component starting from bottom. The geometry dictates: $N_1 + N_2 = 3$ and $N_3 + N_4 = 2$. With $\theta_{h,h'} = (1_{h-h' \leq 1} + 1)/2$:

$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{h,h'} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N_{h'}, i \neq j}} (\ell_j^{h'} - \ell_i^h)^{\theta_{h,h'}}$$

$$\times \prod_{h=1,2} \prod_{1 \leq i \leq N_h} (\ell_i^h - 9)(\ell_i^h - 8)(\ell_i^h - 2)(\ell_i^h + 1)$$

$$\times \prod_{h=3,4} \prod_{1 \leq i \leq N_h} (\ell_i^h - 9)(\ell_i^h - 8)(\ell_i^h - 2)$$

Results

Consider for $\ell^h_{i+1} \geq \ell^h_i + 1$, $\ell^h \in [a_h, b_h]$, $a_{h+1} \geq b_h + cN$, $c > 0$, the probability measure:

$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{h \leq h'} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N_{h'}, i<j}} |\ell^h_i - \ell^{h'}_j|^2 \prod_{h=1}^{k} \prod_{i=1}^{N_h} w_h(\ell^h_i)$$

Theorem (Saff-Totic 97, ...)

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Theorem (Saff-Totic 97, ...)

Assume $N_h/N \mapsto \varepsilon_h$ $w_h(x) \simeq e^{-NV_h(x/N)+O(\log N)}$, V_h *smooth enough, and*

$$\lim_{|x|\to\infty} V_h(x)/\log|x|\varepsilon_h > 1\,.$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N_h} \delta_{\ell^h_i/N} \to \mu^h_\varepsilon \qquad a.s$$

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-$\frac{w_h(x)}{w_h(x-1)} = \frac{\phi^+_{N,h}(x)}{\phi^-_{N,h}(x)}$, $\phi^\pm_{N,h}$ *analytic*, $\phi^\pm_{N,h} = \phi^\pm_h + \frac{1}{N}\phi^\pm_{1,h} + o(\frac{1}{N})$.

Then

$$\left(\sum_{i=1}^{N_h} (f_h(\ell_i^h/N) - \mathbb{E}[f_h(\ell_i^h/N)]) \right)_h \Rightarrow N(0, \Sigma(f)) \quad \forall f \ \textit{real analytic}.$$

Discrete and Continuous covariance

Take V smooth going to infinity fast enough and

$$dP_V^N := \frac{1}{Z_V^N} \prod_{i<j} |\lambda_i - \lambda_j|^{2\theta} e^{-N \sum V(\lambda_i)} \prod d\lambda_i \,.$$

Then,

$$\frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) \mapsto \mu_V(f) \quad a.s.$$

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The covariance is the same than in the discrete case if the support of the equilibrium measures are the same:

$\Sigma((z-.)^{-1},(w-.)^{-1}) :=$

$$\frac{1}{(z-w)^2}\left(1 - \frac{zw - \frac{1}{2}(\alpha_1+\beta_1)(z+w)+\alpha_1\beta_1}{\sqrt{(z-\alpha_1)(z-\beta_1)}\sqrt{(w-\alpha_1)(w-\beta_1)}}\right).$$

Results: Discrete β-ensembles ($\beta = 2\theta$)

$\ell^h_{i+1} - \ell^h_i - \theta_{h,h} \in \mathbb{N}$.

$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{h \leq h'} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N'_h, i<j}} \frac{\Gamma(\ell^{h'}_j - \ell^h_i + 1)\Gamma(\ell^{h'}_j - \ell^h_i + \theta_{h,h'})}{\Gamma(\ell^{h'}_j - \ell^h_i)\Gamma(\ell^{h'}_j - \ell^h_i + 1 - \theta_{h,h'})} \prod w_h(\ell^h_i)$$

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Theorem (Borodin-Gorin-G 15' Borot-Gorin-G 16')

Assume $N_h/N \mapsto \varepsilon_h$, $w_h(x) \simeq e^{-NV_h(x/N)}$, $(\theta_{h,h'})_{h,h'} \ge 0$, $\theta_{h,h} > 0$.
Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N_k} \delta_{\ell^h_i/N} \to \mu^h_\varepsilon \quad a.s\,,$$

Assume $\{0 < \frac{d\mu^h_\varepsilon}{dx} < \theta^{-1}_{hh}\}$ *connected. w is off critical.*
$\frac{w_h(x)}{w_h(x-1)} = \frac{\phi^+_{N,h}(x)}{\phi^-_{N,h}(x)}$, $\phi^\pm_{N,h}$ *analytic*, $\phi^\pm_{N,h} = \phi^\pm_h + \frac{1}{N}\phi^\pm_{1,h} + o(\frac{1}{N})$.

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Work in progress[Borot-Gorin-G 16']

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being the distribution of the vertical line D_N in the tiling of a region Ω_N, $\frac{1}{N}\Omega_N$ (resp. $\frac{1}{N}D_N$) converging to Ω (resp. D). Ω planar.

Then, $\Sigma(f)$ is the covariance of the linear statistics of f under the Gaussian Free field in Ω (with metric dictated by local densities), with Dirichlet conditions at the boundary of the liquid region.

Work in progress[Borot-Gorin-G 16'] : Discrete β-ensembles

$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{h \le h'} \prod_{\substack{1 \le i \le N_h \\ 1 \le j \le N'_h, i<j}} \frac{\Gamma(\ell_j^{h'} - \ell_i^h + 1)\Gamma(\ell_j^{h'} - \ell_i^h + \theta_{h,h'})}{\Gamma(\ell_j^{h'} - \ell_i^h)\Gamma(\ell_j^{h'} - \ell_i^h + 1 - \theta_{h,h'})} w_h(\ell_i^h)$$

Same hypotheses as before and $\phi_N^{\pm} = \sum_{k=0}^{K} \frac{1}{N^k} \phi_k^{\pm} + o(N^{-K})$.
Then

$$\frac{1}{N^2} \ln Z_N = \sum_{k=0}^{K} N^{-k} c_k + o(N^{-K})$$

c_k are defined recursively. The expansion is different from its continuous analogue.

Work in progress[Borot-Gorin-G 16'] : Discrete β-ensembles

$$\mathbb{P}_w(\ell) \propto \prod_{h\le h'} \prod_{\substack{1\le i\le N_h \\ 1\le j\le N'_h, i<j}} \frac{\Gamma(\ell_j^{h'}-\ell_i^h+1)\Gamma(\ell_j^{h'}-\ell_i^h+\theta_{h,h'})}{\Gamma(\ell_j^{h'}-\ell_i^h)\Gamma(\ell_j^{h'}-\ell_i^h+1-\theta_{h,h'})} \prod w_h(\ell_i^h)$$

Same hypotheses as before but $\{0 < \frac{d\mu^h}{dx} < \theta_{hh}^{-1}\}$ is not connected or N_h random. Assume $\phi_N^{\pm} = \sum_{k=0}^{2} \frac{1}{N^k}\phi_k^{\pm} + o(N^{-2})$. Then

$$\sum_{i=1}^{N_h} (f_h(\ell_i^h/N) - \mathbb{E}[f_h(\ell_i^h/N)])$$

converges only under subsequences in general. For instance, if the liquid region has two connected subsets S_1, S_2, the number of horizontal tiles in S_1 is equivalent to a discrete Gaussian centered at $N\mu_h^{\varepsilon}(S_1) - \lfloor N\mu_h^{\varepsilon}(S_1)\rfloor$ (c.f Kriecherbauer-Shcherbina 10', Borot-G 13')

Classical proof of CLT : Stein's method

Let P_n be a family of distribution and L_n some operator on a set $\mathcal{F}$ of functions so that

$$P_n(L_n f) = 0 \quad \forall f \in \mathcal{F}.$$

Assume P_n is tight and L_n goes to L. Assume moreover there exists a unique P so that

$$P(Lf) = 0 \quad \forall f \in \mathcal{F}.$$

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Then P_n converges towards P. If $L_n = L + \frac{1}{n}L_1 + \frac{1}{n^2}L_2 \cdots$, and L invertible, one can hope that

$$P_n(f) = P(f) + \frac{1}{n}P(L_1 L^{-1} f) + \cdots\,.$$

General strategy to study the fluctuations of $M_N(f) = \sum_{i=1}^{N} f(\lambda_i)$

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- Obtain the asymptotics of

$$\ln \mathbb{E}[e^{\lambda M_N(f)}] = \int_0^\lambda \mathbb{E}_{tf}[M_N(f)]dt\,.$$

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  - getting rid of small terms (e.g. by concentration of measure theory) to get approximately closed equations,
  - linearizing the equations around the limit and solve the linear equation.

Example: Convergence for β-ensembles

$$dP^N_{\beta,V}(\lambda) = \frac{1}{Z^N_{\beta,V}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp\{-N\beta \sum_{i=1}^N V(\lambda_i)\} \prod d\lambda_i$$

Theorem

$L_N = \frac{1}{N}\sum \delta_{\lambda_i}$ converges $P^N_{\beta,V}$-almost surely towards the unique minimizer μ_V of

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Proof.

$$\frac{dP^N_{\beta,V}(\lambda)}{d\lambda_N} \simeq e^{-\beta N^2 (I(L_N) - \inf I(\mu))}\,.$$

Fluctuations for β-ensembles : The Schwinger-Dyson equations [Johansson 97']

$$dP^N_{\beta,V}(\lambda) = \frac{1}{Z^N_{\beta,V}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp\{-N\beta \sum_{i=1}^N V(\lambda_i)\} \prod d\lambda_i$$

The empirical measure $L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ satisfies the Schwinger-Dyson equations

$$\int \left(\frac{1}{2} \iint \frac{f(x)-f(y)}{x-y} dL_N(x) dL_N(y) - \int V'(x) f(x) dL_N(x) \right) dP^N_{\beta,V}$$

$$= \frac{1}{N}\left(\frac{1}{2} - \frac{1}{\beta}\right) \int \int f'(x) dL_N(x) dP^N_{\beta,V} .$$

(this is integration by parts)

Fluctuations for β-ensembles : Linearization

If V is analytic, taking $f(x) = (z-x)^{-1}$, we find that $W_N(z) = \int (z-x)^{-1} dL_N(x), W_V(z) = \int (z-x)^{-1} d\mu_V(x), \Delta W_N = W_N - W_V$ satisfy

$$K\mathbb{E}[\Delta W_N](z) = -\frac{1}{N}(\frac{1}{2} - \frac{1}{\beta})\partial_z \mathbb{E}[W_N(z)] - \mathbb{E}[\Delta W_N(z)^2]$$

with

$$Kf(z) = 2W_V(z)f(z) - \oint \frac{d\xi}{2i\pi} \frac{1}{z-\xi} V'(\xi) f(\xi) .$$

Fluctuations for β-ensembles : Linearization

If V is analytic, taking $f(x)=(z-x)^{-1}$, we find that $W_N(z)=\int(z-x)^{-1}dL_N(x), W_V(z)=\int(z-x)^{-1}d\mu_V(x), \Delta W_N = W_N - W_V$ satisfy

$$K\mathbb{E}[\Delta W_N](z) = -\frac{1}{N}(\frac{1}{2}-\frac{1}{\beta})\partial_z\mathbb{E}[W_N(z)] - \mathbb{E}[\Delta W_N(z)^2]$$

with

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By concentration, $N\mathbb{E}[\Delta W_N(z)^2]$ is small, hence if K is invertible

$$\lim_{N\to\infty} N\mathbb{E}[\Delta W_N](z) = -(\frac{1}{2}-\frac{1}{\beta})K^{-1}\partial_z W_V(z) =: W_V^1(z)$$

Central limit theorem

For f analytic, we deduce that

$$\begin{aligned}
\ln \mathbb{E}_V[e^{\lambda M_N(f)}] &= \int_0^\lambda \mathbb{E}_{V-\frac{t}{N}f}[M_N(f)]dt \\
&= N\int_0^\lambda \mu_{V-\frac{t}{N}f}(f)dt \\
&\quad + \int_0^\lambda \int_C \frac{d\xi}{2i\pi} f(\xi)\mathbb{E}_{V-\frac{t}{N}f}[\Delta W_N(\xi)]dt \\
&= \lambda N \mu_V(f) + \frac{\lambda^2}{2} Q_V(f,f) + \lambda L_V(f) + o(1)
\end{aligned}$$

with

$$\begin{aligned}
Q_V(f,g) &= -D_V[\mu_V(g)][f] \\
L_V(g) &= \frac{1}{2i\pi}\int_C g(\xi) W_V^1(\xi) d\xi
\end{aligned}$$

Generalization to the discrete setting

$$\mathbb{P}(\ell) = \frac{1}{Z_N} \prod_{1 \le i < j \le N} \frac{\Gamma(\ell_j - \ell_i + 1)\Gamma(\ell_j - \ell_i + \theta)}{\Gamma(\ell_j - \ell_i)\Gamma(\ell_j - \ell_i + 1 - \theta)} \prod_{i=1}^{N} w(\ell_i)$$

- The convergence is similar, but the discrete setting imposes $\frac{d\mu}{dx} \le \frac{1}{\theta}$ (c.f. Feral 08') .

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• Analogues of Schwinger-Dyson equation was given by Nekrasov:

If $\frac{w(x)}{w(x-1)} = \frac{\phi_N^+(x)}{\phi_N^-(x)}$ with $\phi_N^{\pm}(\xi)$ holomorphic in a domain $\mathcal{M}_N \subset \mathbb{C}$,

$$\phi_N^-(\xi) \cdot \mathbb{E}_{\mathbb{P}} \left[\prod_{i=1}^{N} \left(1 - \frac{\theta}{\xi - \ell_i}\right) \right] + \phi_N^+(\xi) \cdot \mathbb{E}_{\mathbb{P}} \left[\prod_{i=1}^{N} \left(1 + \frac{\theta}{\xi - \ell_i - 1}\right) \right]$$

is holomorphic in $\mathcal{M}_N$ (check residues vanish).

Nekrasov equation analysis

Assume $\phi_N^{\pm}(N\xi) \simeq \phi^{\pm}(\xi) + \frac{1}{N}\phi_1^{\pm}(\xi) + o(\frac{1}{N})$ and $\{0 < \frac{d\mu}{dx} < \frac{1}{\theta}\} = [a, b]$.

Nekrasov equation asymptotically becomes with $\Delta W_N(z) = \frac{1}{N}\sum(z - \ell_i/N)^{-1} - W(z)$, $W(z) = \int(z-x)^{-1}d\mu(x)$,

$$Q(\xi)\mathbb{E}[\Delta W_N(\xi)] = \frac{1}{N}f(\xi) + R_N(\xi) + o(\frac{1}{N})$$

with $Q(\xi) = \phi^-(\xi)e^{-\theta W(\xi)} - \phi^+(z)e^{\theta W(\xi)}$, f known and R_N analytic.

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Assume off criticality : $Q(\xi) = H(\xi)\sqrt{(\xi - a)(b - \xi)}$ with H analytic, non vanishing. Integrating over a contour around the support of the measure gives

$$\sqrt{(\xi - a)(b - \xi)}\mathbb{E}[\Delta W_N(\xi)] = \frac{1}{N}\frac{1}{2\pi i}\int_C \frac{H(z)^{-1}f(z)}{\xi - z}dz + o(\frac{1}{N}).$$

Conclusion

- The case where the interaction is given by $\prod_{i \neq j} |\ell_i - \ell_j|^\theta$, $\theta \neq 1, 2$ is open.
- In the continuous setting, similar analysis allows to build approximate transport and prove universality for local fluctuations. In the discrete setting, Local fluctuations are unknown for $\theta \neq 2$.
- Nekrasov equations include more general models, cf quivers.