From random interpolation nodes to complex geometry and optimal transport

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Optimal and random point configurations: from statistical physics to approximation theory (IHP Paris, 2016)

Outline

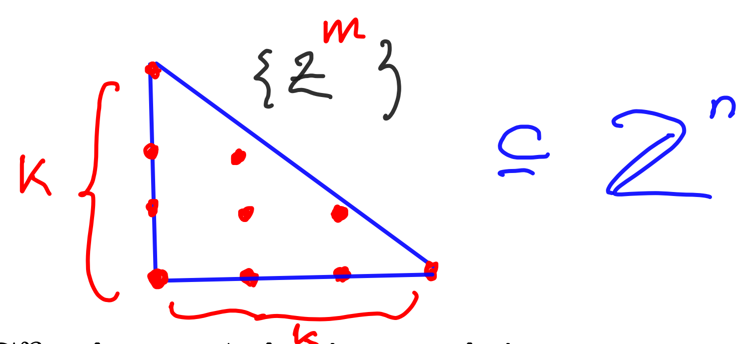
- Optimal interpolation nodes in $\mathbb{C}^n$
- Statistical mechanics in and out of equilbrium
- Einstein's equations for Kähler metrics on $\mathbb{C}^n$
- Tropicalization and sticky particles in $\mathbb{R}^n$

A. Optimal interpolation nodes

Denote by

- $\mathcal{P}_k(\mathbb{C}^n)$ the space of all *polynomials* $p_k(z)$ on $\mathbb{C}^n$ of total degree $\leq k$.

- $N_k := \dim H_k(\mathbb{C}^n) = \frac{1}{n!}k^n + o(k^n)$



- $\phi(z)$, a given *weight function* on $\mathbb{C}^n$, i.e. ϕ is lsc with super log growth:

$$\phi(z) \geq (1+\epsilon)\log|z|^2 + O(1), \quad |z| \to \infty$$

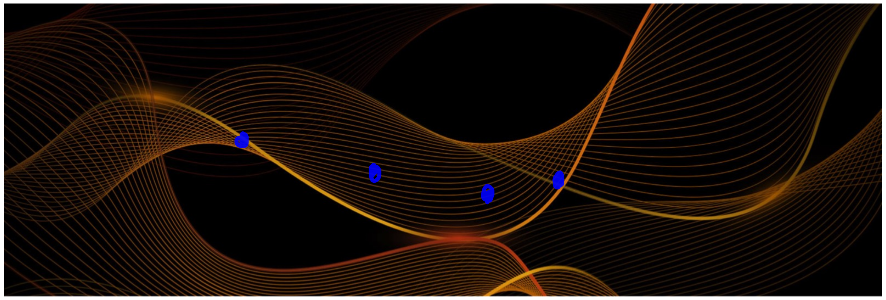
The *weighted sup norm* of a polynomial $p_k \in \mathcal{P}_k(\mathbb{C}^n)$ is then defined by

$$\|p_k\|_{k\phi} := \sup_{z\in\mathbb{C}^n} |p_k(z)|e^{-k\phi(z)} < \infty$$

The interpolation problem

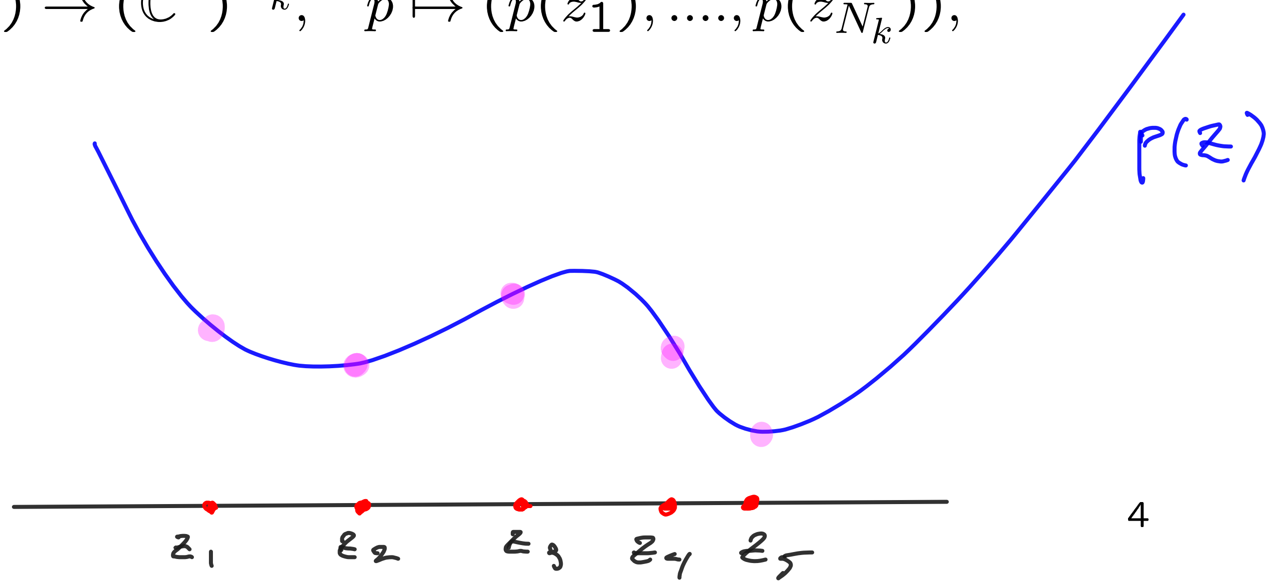
A polynomial $p_k(z)$ is uniquely determined by its values at any given generic configuration $(z_1, ...z_{N_k})$ of N_k points.

Problem: how to distribute the "interpolation nodes" $z_1, z_2, .., z_N$ in order to be able to reconstruct any $p_k \in \mathcal{P}_k(\mathbb{C}^n)$ from its values at the nodes in an optimal way (wrt the weighted norm)?



Definition: A configuration of points $(z_1, .., z_{N_k})$ on $\mathbb{C}^n$ is said to be *optimal* (wrt the weight ϕ) if it maximizes the weighted determinant of the corresponding evaluation *map*

$$\mathcal{P}_k(\mathbb{C}^n) \xrightarrow{A} (\mathbb{C}^n)^{N_k}, \quad p \mapsto (p(z_1), ..., p(z_{N_k})),$$



i.e. $(z_1, .., z_{N_k})$ on $\mathbb{C}^n$ is said to be *optimal* if its maximizes the weighted determinant

$$|A(z_1, ..., z_N)|e^{-k\phi(z_1)} \cdots e^{-k\phi(z_{N_k})}$$

of the corresponding $N_k \times N_k$ matrix A :

$$A(z_1, z_2, ...z_{N_k}) := (e_i(z_j))_{1 \leq i,j \leq N_k}$$

where e_i is some fixed basis in $\mathcal{P}_k(\mathbb{C}^n)$.

- For example, can take a multinomial basis:

$$e_m(z) = z^m := m \in \mathbb{Z}^n \cap k\Delta$$

Why this optimality condition?

- The "worst" configurations $(z_1, z_2, ...z_{N_k})$ are those satisfying

$$\det A(z_1, z_2, ...z_{N_k}) = 0$$

  Then the corresponding evaluation map is not even invertible, i.e. we cannot interpolate all values.

- Hence, the *optimal* configurations should be those *maximizing* the norm of $\det A(z_1, z_2, ...z_{N_k})$

This ensures that interpolation is possible and *stable* (by Cramer's rule)

The classical setting in $\mathbb{C}$

When $n = 1$ we have

$$N_k = k + 1.$$

The corresponding polynomial $\det A(z_1, z_2, ...z_k)$ factorizes in products of $(z_i - z_j)$

$$\det A(z_1, z_2, ...z_{k+1}) = \prod_{i<j} (z_i - z_j)$$

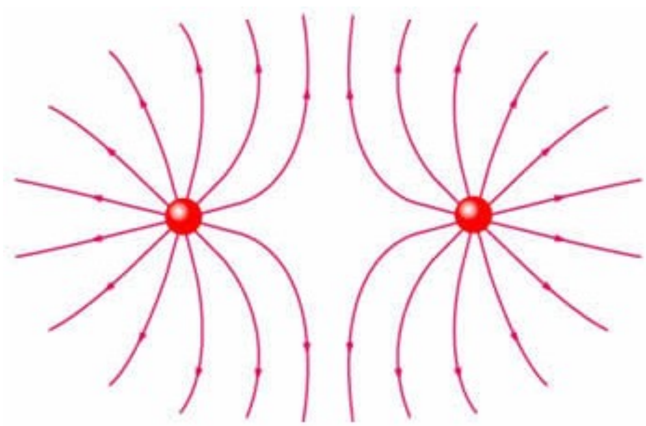
(since it vanishes for $z_i = z_j$). Hence,

$$-\frac{1}{k} \log |\det A(z_1, z_2, ...z_{N_k})|_{k\phi} = -\frac{1}{(N-1)}\frac{1}{2} \sum_{i \neq j \leq N} \log |z_i - z_j| + \sum_{i=1}^{N} \phi(z_i)$$

is the mean field energy of N interacting Coulomb charges $z_1, ..., z_N$ in $\mathbb{C}$ confined by the potential ϕ.

Hence, when $n = 1$ optimal interpolation nodes in $\mathbb{C}$ (i.e. *Fekete points*) correspond to minimizers of the electrostatic energy

$$E_\phi(z_1, ..., z_N) = \frac{1}{N-1} \sum_{i,j} \log |z_i - z_j| + \sum_{i=1}^{N} \phi(z_i)$$



Classical results (Gauss, Fekete, Polya, Szegö, ...) then show that when $k(= N-1) \to \infty$ optimal nodes $(z_1^*, z_2^*, ...z_N^*)$ converge weakly:

$$\frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{z_i^*} \to \mu_\phi,$$

where μ_ϕ is the *unique* minimizer of the "continuous" logarithmic weighted energy:

$$E(\mu) := -\frac{1}{2} \int \log |z-w| \, d\mu(z) d\mu(w) + \int \phi(z) d\mu(z)$$

as μ ranges over all probability measures on $\mathbb{C}$

- The measure μ_ϕ is called the weighted *equilibrium measure* in potential theory

The higher dimensional setting: $n \geq 1$

Conjecture:

- (Leja, '50s) Any sequence $(z_1^*, z_2^*, ...z_{N_k}^*)$ of optimal interpolation nodes has a unique limit as $k \to \infty$:

$$\frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{z_i^*} \to \mu_\phi,$$

weakly, for a probability measure μ_ϕ.

- (Siciak '80s) The measure μ_ϕ should be the weighted *pluripotential equilibrium measure*

Siciak's equilibrium measure is defined in terms of pluripotential theory which is a non-linear generalization to $\mathbb{C}^n$ of classical potential theory in $\mathbb{C}$:

- Replace subharmonic functions on $\mathbb{C}$ with *plurisubharmonic* functions $\psi(z)$ on $\mathbb{C}^n$:

$$\partial\bar{\partial}\psi := \left(\frac{\partial^2\psi}{\partial z_i \partial \bar{z}_j}\right) \geq 0$$

- Replace the Laplacian on $\mathbb{C}$ with the complex *Monge-Ampère operator* on $\mathbb{C}^n$:

$$MA(\psi) := \frac{i}{\pi}(\partial\bar{\partial}\psi)^{\wedge n} \sim \det\left(\frac{\partial^2\psi}{\partial z_i \partial \bar{z}_j}\right)_{i,j\leq n}$$

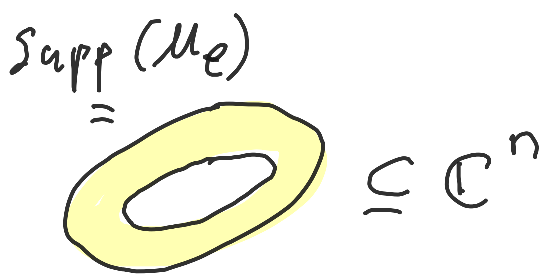
Then Siciak's pluripotential equilibrium measure μ_ϕ wrt the weight ϕ is defined by

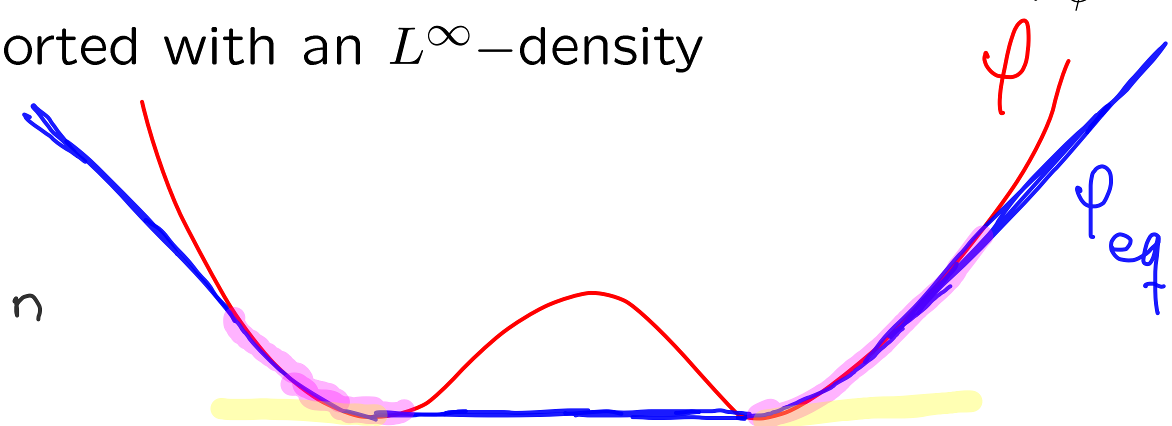
$$\mu_\phi := MA(\phi_{eq}),$$

where ϕ_{eq} is defined as an envelope with obstacle ϕ :

$$\phi_{eq}(z) = \sup\{\psi(z) : \ \psi \leq \phi\}$$

- When ϕ is smooth with superlogarithmic growth μ_ϕ is compactly supported with an $L^\infty-$density

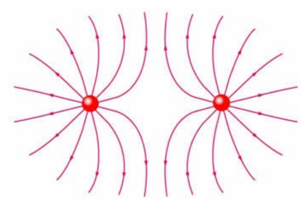




The Leja-Siciak conjecture was settled ('08) in a joint work with Sebastien Boucksom (Paris) and David Witt Nyström (Chalmers)

But the solution prompts some new questions:

- What is the "physical" interpretation of optimal interpolation nodes when $n > 1$?



- How to locate (nearly) optimal nodes dynamically

The idea is to introduce the $N-$particle "interaction energy"

$$E(z_1, ..., z_{N_k}) := -\frac{1}{k} \log |\det A(z_1, z_2, ... z_{N_k})|,$$

where

$$A(z_1, z_2, ... z_{N_k}) = (e_i(z_j))_{i,j \leq N_k}$$

in terms of a fixed multinomal base $e_1, .., e_{N_k}$ in $\mathcal{P}_k(\mathbb{C}^n)$

- The interaction energy is *symmetric*
- It is *repulsive*
- It is independent of the base, up to an overall additive constant

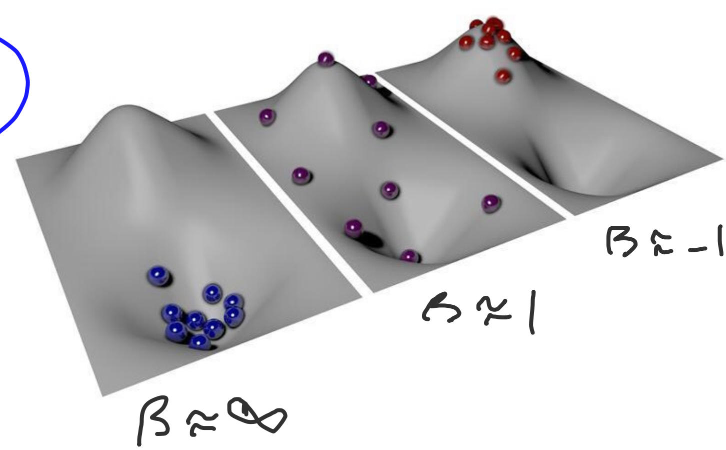
- It is highly *non-linear* in $(z_1, ..., z_{N_k})$ when $n > 1$

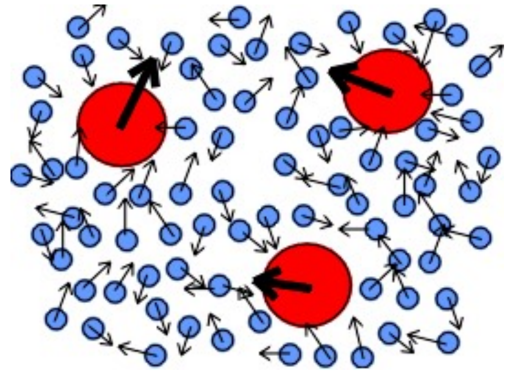
B. Statistical mechanics in and out of equilibrium

Consider an ensemble of N identical particles $x_1, ..., x_N$ on a Riemannian manifold X interacting by a symmetric energy function $E^{(N)}(x_1, x_2, ..., x_N)$.

- *(Statics)* At inverse temperature $\beta(= \beta_N)$ the corresponding equilibrium state is represented by the Boltzmann-Gibbs measure

$$\mu_\beta^{(N)} := \frac{e^{-\beta E^{(N)}}}{Z_\beta} dV^{\otimes N}$$





- *(Dynamics)* The relaxation to equlibrium, at inverse temperature β, is described by

$$dx_i(t) = -\nabla_{x_i} E^{(N)}(x_1, ..., x_N)dt + \frac{\sqrt{2}}{\sqrt{\beta}} dB_i(t),$$

(the overdamped Langevin equation)

The Boltzmann-Gibbs measure is stationary for the dynamics and arises when $t \to \infty$

Random interpolation nodes

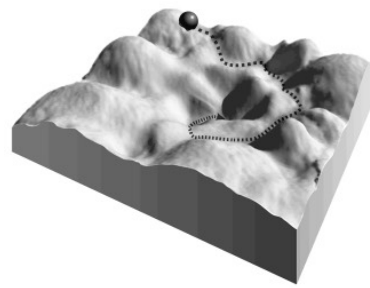
In our setting we take

$$E^{(N)}(z_1, ..., z_{N_k}) := -\frac{1}{k} \log |\det A(z_1, z_2, ...z_{N_k})|_{k\phi}$$

and try to study the corresponding large $N-$limit

- In other words we think of random interpolation nodes as forming a statistical mechanical system.

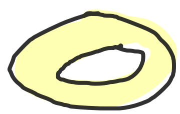
- The problem is to show that a coherent large-scale structure emerges in the "thermodynamic" limit



- For Monte-Carlo approaches to optimal interpolation nodes the relevant case is the "zero-temperature limit":

$$\beta := \lim_{N\to\infty} \beta_N = \infty$$

(the "liquid phase")



- But some suprising connection to complex geometry/math. physics appears at a finite inverse temperature β (the "gas phase")

In the special case when $\beta_{N_k} = 2k$ the Boltzmann-Gibbs model becomes

$$\mu_\beta^{(N)} := \frac{e^{-\beta E^{(N)}}}{Z_\beta} dV^{\otimes N} = \frac{1}{Z_\beta} |(\det A)(z_1, z_2, ... z_{N_k})|_{k\phi}^2 dV^{\otimes N},$$

where $(\det A)(z_1, z_2, ... z_{N_k})$ is a Vandermonde determinant raised to the power 2

- Hence $\mu_\beta^{(N)}$ then defines a *determinantal point process* and asymptotically $\beta_{N_k} = 2k \to \beta := \infty$
- But in the positive temperature case $\beta < \infty$ the power is $\sim 1/k$

The static case

Theorem (B. 08' , '13): Given a $\beta \in]0, \infty]$ the corresponding random measure on $\mathbb{C}^n$

$$\frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{z_i}$$

converges in law to a deterministic measure μ_β on $\mathbb{C}^n$

- In fact, the convergence is exponential in the sense of large deviation theory

- When $\beta = \infty$ the limiting measure μ_β is the equilibrium measure corresponding to the weight ϕ

- For a finite β the measure μ_β can be written as

$$\mu_\beta = MA(\psi_\beta),$$

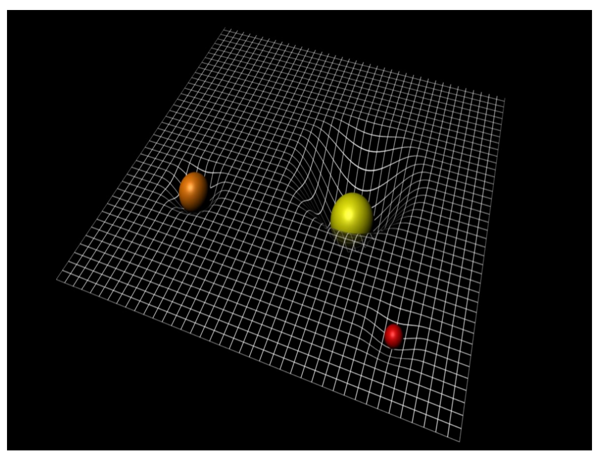
where ψ_β is smooth with logarithmic growth and solves the PDE

$$MA(\psi_\beta) = e^{\beta(\psi_\beta - \phi)} dV$$

(Aubin, Yau,...)

Einstein's equations for Kähler-metrics on $\mathbb{C}^n$

The measure μ_β is the volume form of a certain Riemannian metric g_β solving Einstein's equations on the "universe" $X := \mathbb{C}^n$ (with Euclidean signature).



The point is that any smooth function spsh ψ on $\mathbb{C}^n$ defines a Riemannian metric g_ψ on $\mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n$

$$g = \text{Re } (\partial\bar{\partial}\psi) \oplus \text{Re } (\partial\bar{\partial}\psi)$$

(= a *Kähler metric*). By definition,

$$dV_g = MA(\psi)$$

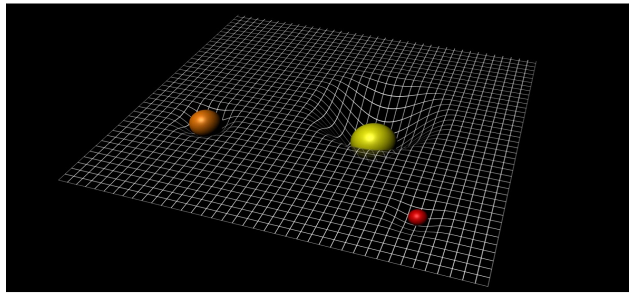
The limiting complex MA-equation

$$MA(\psi) = e^{\beta(\psi-\phi)} dV$$

is equivalent to the (twisted) Kähler-Einstein equation

$$\text{Ric } g_\psi = -\beta g_\psi + \beta T_\phi,$$

where T_ϕ is a symmetric two-tensor



Recall that Einstein's equations for a metric g on a "universe" X can be written as

$$\text{Ric } g - \Lambda g = T,$$

where Λ is the *cosmological constant* and T is the (trace-reduced) *energy-momentum tensor*

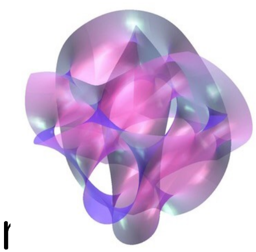
- Here

$$\text{Ric } g_\psi = -\beta g_\psi + \beta T_\phi$$

and hence $\Lambda = -\beta$ and the weight ϕ determines $T (= \partial\bar{\partial}\phi)$

- In the standard physical setup g is a Lorentzian metric ("space-time")

- But here we are concerned with the case when g is a Riemannian metric (such solutions appear, for example, as gravitational instantons in Hawking's space-time foam)



- When X has a complex structure J one looks for Riemann metrics compatible with J, i.e. *Kähler metrics*

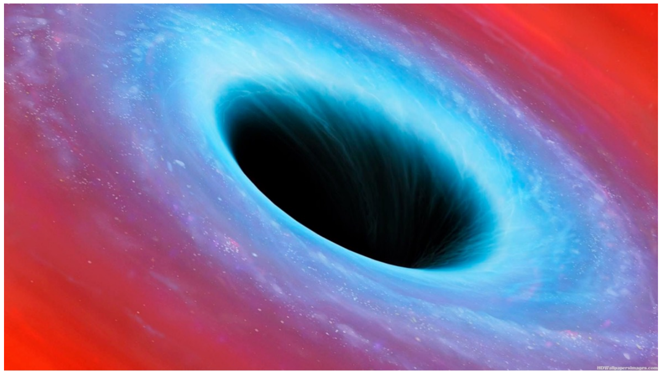
Physical interpretation?

- Hence, at positive temperature, random interpolation nodes in equilibrium yield a microscopic/statistical mechanical description of Einstein's equations (in Euclidean singature)

- Quantum gravity? Emergent gravity?...

- There are some intruiging relations to the thermodynamics of black holes

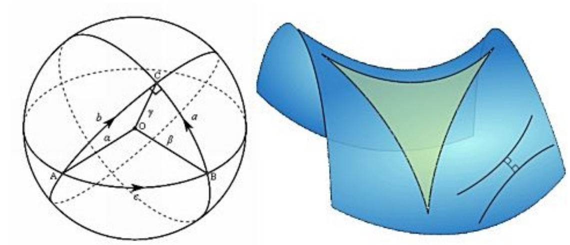
But that's a different story...



Positive cosmological constant/Negative temperature states

Since

$$\Lambda = -\beta,$$

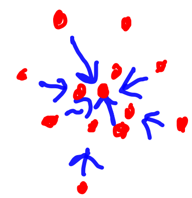


to get a *positive* cosmological constan Λ we would need $\beta < 0$.

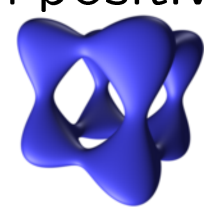
- Since the Boltzmann-Gibbs density $\sim e^{-\beta E^{(N)}}$ this corresponds to keeping $\beta > 0$, but switching the sign of the interaction energy $E^{(N)}$ to make it *attractive.*

- But then it turns out that there is a critical β_{cr} such that there are no (stable) solutions for $\beta > \beta_{cr}$

- (phenomen of collaps, concentration...)



- This is related to the Yau-Tian-Donaldson conjecture concerning the existence of Kähler-Einstein metrics with positive Ricci curvature on a complex algebraic variety X



- Convergence of the Boltzmann-Gibbs measures in this attractive setting is open, in general.

The dynamic case

We now consider the *time-dependent* random measures

$$\frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{z_i(t)}, \quad \text{on } \mathbb{C}^n$$

where $(z_1(t), ...z_N(t))$ evolves according to the stochastic gradient flow of $E_\phi^{(N)}$ on $(\mathbb{C}^n)^N$ assuming that $z_1(0), ...z_N(0)$ are iid with joint law μ_0.

Conjecture (B.): As $N \to \infty$ the random measures above converge in law to a deterministic curve $\mu(t)$ of probability measures emanating from μ_0.

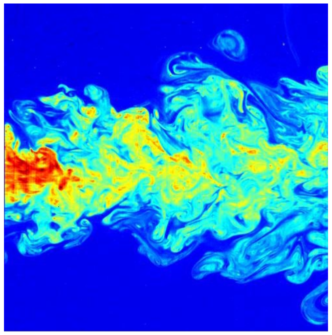
In general, if a system of SDEs on X^N admits such a large N limit $\mu(t)$, then *propagation of chaos* is said to hold

(Boltzmann, Kac, Snitzmann...)

More precisely, in the present setting the deterministic limit $\mu(t)$ conjecturally evolves according to the following drift-diffusion equation on $\mathbb{C}^n$:

$$\frac{\partial \mu(t)}{\partial t} = \frac{1}{\beta}\Delta\mu(t) - \pm\nabla\cdot(\mu(t)\nabla(\psi(t)-\phi))$$

$$MA(\psi(t)) = \mu(t)$$



(assuming propagation of chaos, in a strong sense, one can show that the equation above has to hold)

- This conjecture is consistent with the static case: $\mu(t) \equiv MA(\psi) = e^{\pm\beta(\psi-\phi)}$ is a stationary solution

This conjecture seems very challenging and there are many hurdles:

- Even defining the evolution equations is non-trivial due to the singularities of $E^{(N)}$

- Even the simplest case $n = 1$ is open!

The case $n = 1$

- The repulsive case with $\beta = \infty$ appears in supra-conductivity where the particles are vortices (Ambrosio-Serfaty,...)

- The attractive case with $\beta \leq \beta_{cr}$ coincides with the Keller-Segel system in chemotaxis (recent progress by Fournier-Jourdain,..)

Tropicalization and a sticky particle system in $\mathbb{R}^n$

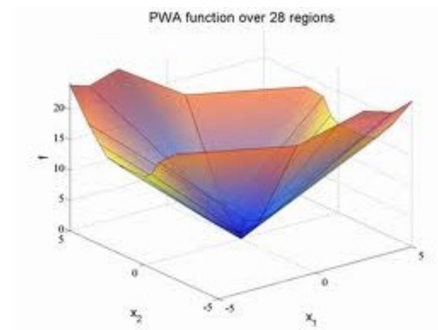
In the "attractive case" the stochastic gradient flows on $\mathbb{C}^d := (\mathbb{C}^n)^N$ above have the following form

$$d\boldsymbol{z}(t) = -\nabla \log |P(\boldsymbol{z})|^2 + \frac{\sqrt{2}}{\sqrt{\beta}} d\boldsymbol{B}(t),$$

for a a polynomial $P(\boldsymbol{z})$ on $\mathbb{C}^d$ (the Vandermonde determinant).

The philosphy of Tropicalization: replace an elusive problem for polynomials in $\mathbb{C}^d$ with a simpler one for piece-wise affine convex function in $\mathbb{R}^d$:

$$\sum_{\boldsymbol{m}\in I} c_{\boldsymbol{m}} \boldsymbol{z}^{\boldsymbol{m}} \rightsquigarrow \max_{\boldsymbol{m}\in I} \boldsymbol{x}\cdot\boldsymbol{m}$$



Equivalently, the psh function on $\mathbb{C}^d$

$$\Psi(\boldsymbol{z}) := \log|P(\boldsymbol{z})|$$

is replaced by a convex function on $\mathbb{R}^d$:

$$\varphi(\boldsymbol{x}) := \lim_{k\to\infty} k^{-1}\Psi(e^{k\boldsymbol{x}})|$$

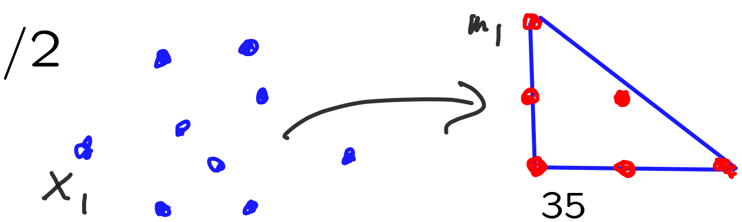
- Here this means that the *Vandermonde determinant on* $\mathbb{C}^d = \mathbb{C}^{nN}$,

$$\sum_{\sigma \in S_N} (-1)^{\text{sign}(\sigma)} z_1^{m_{\sigma(1)}} \cdots z_{N_k}^{m_{\sigma(N)}}$$

is replaced by the *tropical permanent*

$$E_{trop}^{(N)}(x_1, ..., x_N) := \max_{\sigma \in S_N} \left(x_1 \cdot m_{\sigma(1)} + ... + x_N \cdot m_{\sigma(N)} \right)$$

In terms of *discrete optimal transport* this is minus the *minimal cost* to transport the N points $\{x_1, x_2, ..., x_N\}$ in $\mathbb{R}^n$ to the N lattice points $\{m_1, ..., m_N\} \in k\Delta$ with respect to the quadratic cost function $c(x, y) := -x \cdot y \sim |x - y|^2/2$



Guided by the philospophy of tropicalization we replace the SDEs on $\mathbb{C}^{nN}$ by the following SDEs on $\mathbb{R}^{nN}$:

$$d\boldsymbol{x}(t) = -\nabla E^{(N)}_{trop}(x_1, x_2, ..., x_N) + \frac{\sqrt{2}}{\sqrt{\beta_N}} d\boldsymbol{B}(t),$$

where

$$E^{(N)}_{trop}(x_1, ..., x_N) := \max_{\sigma \in S_N} \left(x_1 \cdot m_{\sigma(1)} + ... + x_N \cdot m_{\sigma(N)} \right)$$

Thm : (B.-Önnheim '15). As $N \to \infty$ propagation of chaos holds for the SDEs above

More precisely, the deterministic limit μ_t on $\mathbb{R}^n$ evolves according to

$$\frac{\partial \mu(t)}{\partial t} = \frac{1}{\beta}\Delta\mu(t) + \nabla \cdot (\mu(t)\nabla(\varphi(t))$$

$$MA(\varphi(t)) = \mu(t)$$

where $\varphi(t) := \varphi(t,x)$ is convex on $\mathbb{R}^n$ with given asymptotics as $|x| \to \infty$:

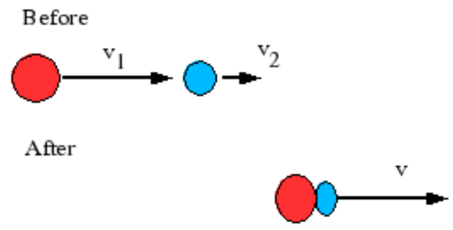
$$\varphi(t,x) = \max\{|x_1|, ..., |x_n|\} + o(|x|)$$

The proof is based on a new propagation of chaos result for stochastic gradient flows of quasi-convex interaction energies.

- The key technical tool is the theory of Wasserstein gradient flows of Ambrosio et al.

A sticky particle system in $\mathbb{R}^n$

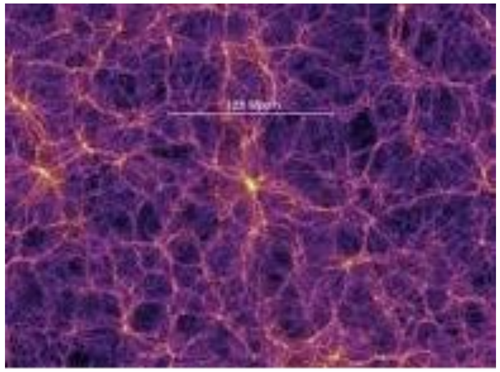
- In the deterministic setting ($\beta_N = \infty$) the particles move at *constant speed* generically
- Indeed, the interaction energy $E_{trop}^{(N)}(x_1, ..., x_N)$ is piece-wise affine
- Convexity of $E_{trop}^{(N)}(x_1, ..., x_N)$ means that the system is *attractive,* which leads to "sticky" behaviour
- As a consequence, when $\beta = \infty$ the particles aggregate into a single particle $x_*(t)$ in a finite time, moving at constant speed



- Wen $n = 1$ the deterministic system coincides with the *sticky particle system* on $\mathbb{R}$ (originating in cosmology; the Zeldovich model).

- Adding a small noise ($\beta_N \to \infty$) correspons to the *adhesion model* in cosmology when $n = 1$

- The case $n > 1$ is closely related to Brenier's generalization of the Zeldovich model



Statistical mechanics of interpolation nodes ($T = 0$)

$T \neq 0$ ⟿ Kähler-Einstein metrics

$T < 0$ ⟿ Sticky particle systems

Thank you!