Random normal matrices with hard edge spectrum

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1. Set up

Plasma β-ensembles

$$\zeta = \{\zeta_j\}_{j=1}^n \subset \mathbb{C}, \quad Z_n = \int_{\mathbb{C}^n} e^{-\beta H_n},$$

where

$$H_n(\zeta) = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum Q(\zeta_j),$$

$Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ is given.

$\beta = 1$ random normal matrix model

Equilibrium and droplet

- $$\frac{1}{n}\sum_{1}^{n} \delta_{\zeta_j} \to \sigma$$

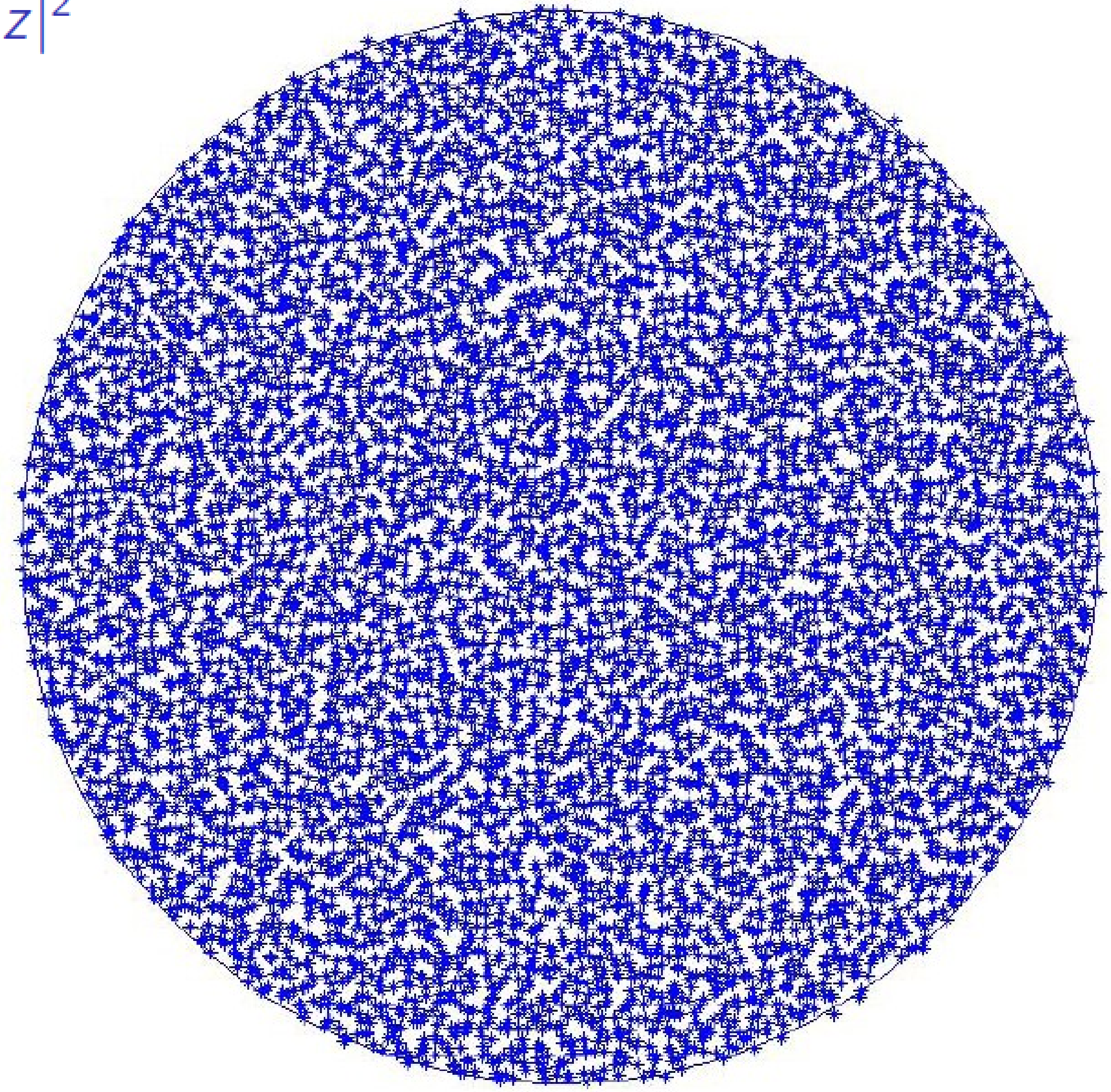
  σ is probability measure of minimal Q-energy

- $S := \mathrm{supp}(\sigma)$

- If $Q \in C^2$ in $\mathrm{nbh}(S)$, then

  $$\sigma = \frac{1}{\pi}\partial\bar{\partial}Q \cdot 1_S$$

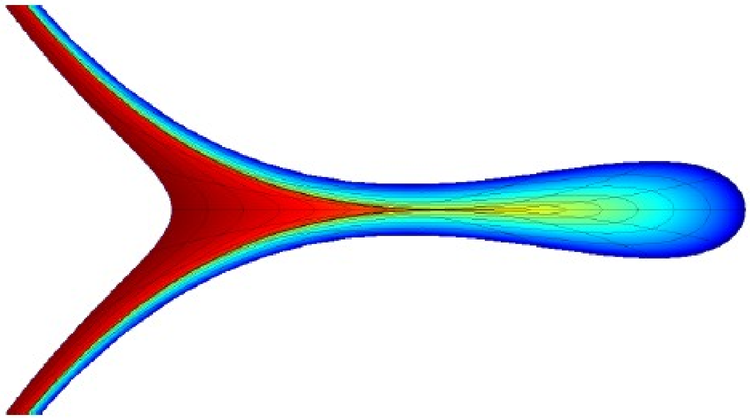
$Q(z) = |z|^2$



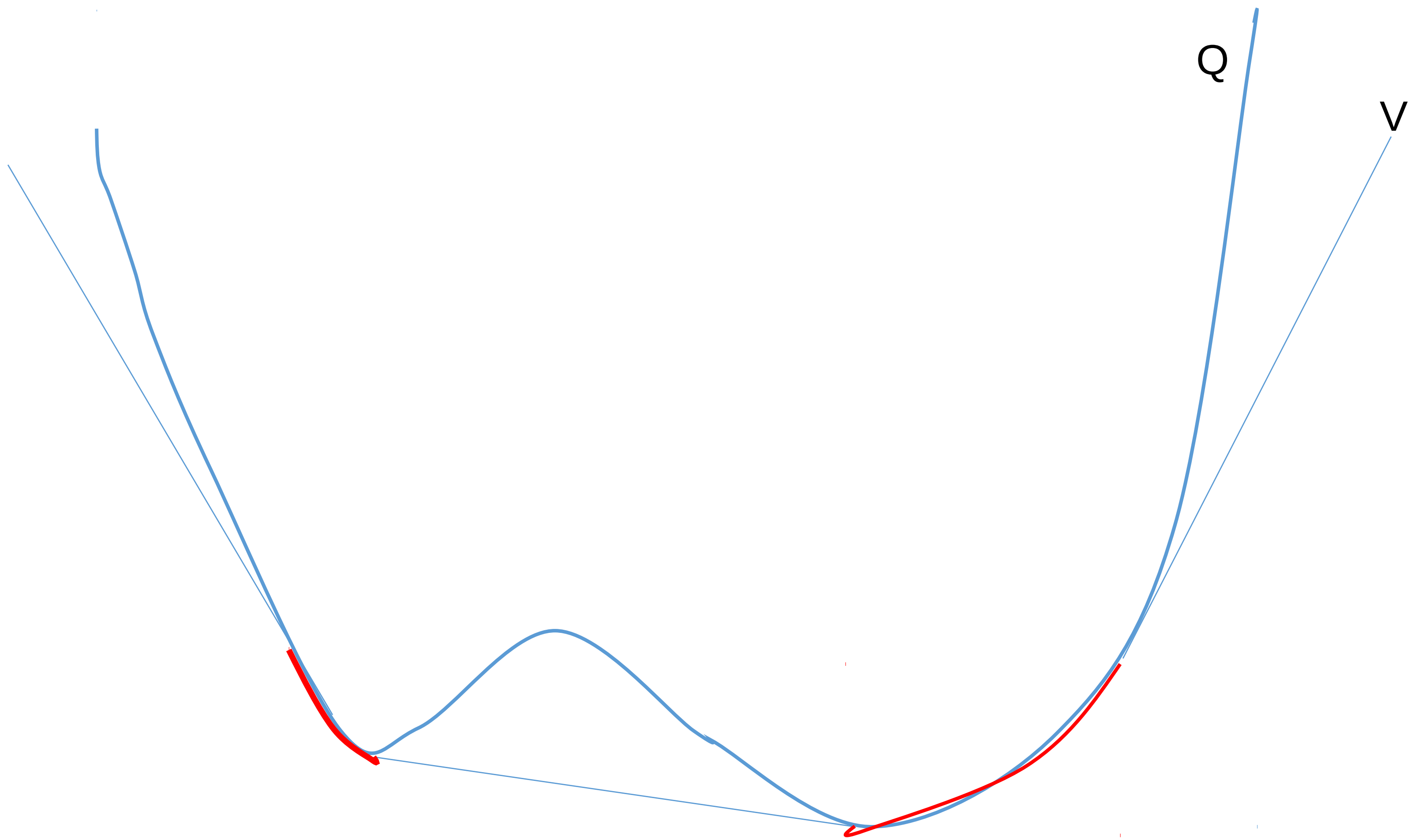
Hele-Shaw flow

$$S_t = \operatorname{supp}(\sigma_t)$$

σ_t equilibrium measure of mass t



Obstacle problem



$$S_t = \{Q = \check{Q}_t\}$$

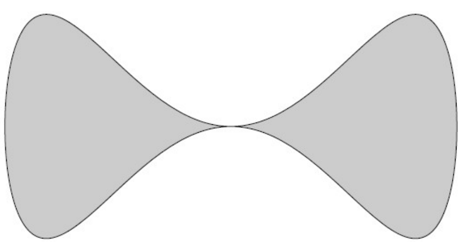
$\check{Q}_t$ is maximal subharmonic $V \leq Q$, $V \sim t \log |z|^2$ at ∞

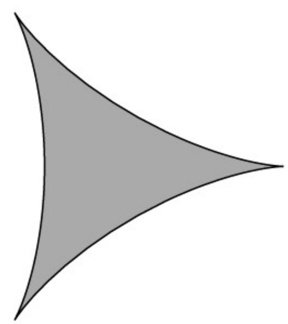
"Free boundary"

Assumptions: $S = S(Q)$

$$Q \in C^{\omega}(\mathrm{nbh}\ S), \quad \Delta Q > 0 \text{ on } S$$

Regular and singular boundary points:





"Hard edge"

$$Q^S := \begin{cases} Q & \text{on } S \\ +\infty & \text{on } \mathbb{C} \setminus S \end{cases}$$

Generalization ("local droplets"): S is a compact set such that

$$\sigma(Q^S) = \frac{1}{\pi} \Delta Q \cdot 1_S$$

Scaling

$$\{\zeta_j\}_1^n \mapsto \{z_j\}_1^n$$

where

$$z_j = c\sqrt{n}(\zeta_j - p), \quad c := \sqrt{\Delta Q(p)}$$

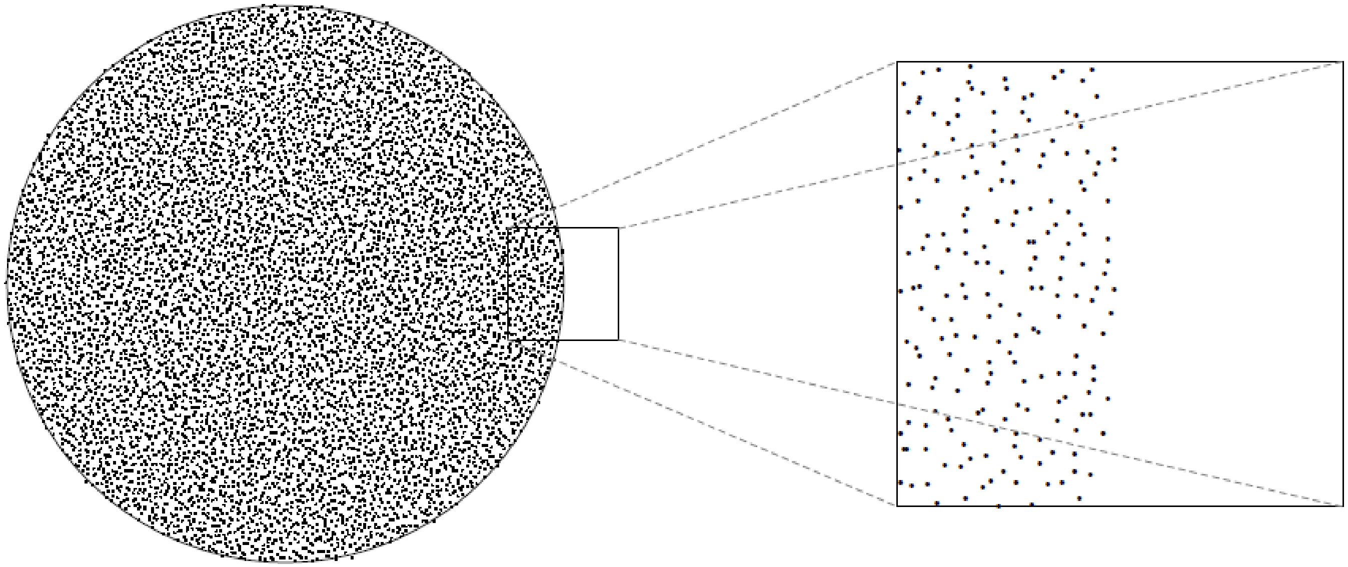
Generalization: $p_n \to p$,

$$z_j = \sqrt{n}(\zeta_j - p_n).$$

Notation: $\mathbb{S}$ limiting shape of rescaled droplets (e.g. halfplane or strip)

Q: limiting point processes?

[Cf. Airy, Bessel in determinantal 1D case]

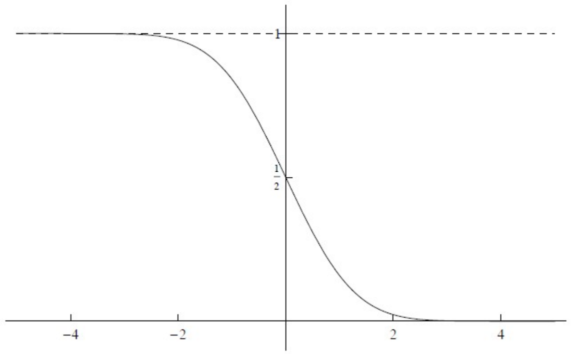


$$\mathbb{S} = \{x \leq 0\}$$

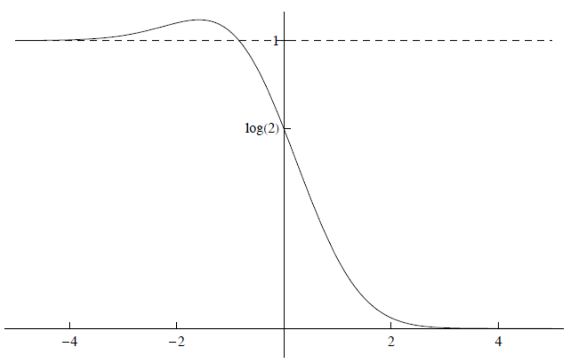
1-point function at a regular boundary point

$$\mathbb{S} = \{x \leq 0\}$$

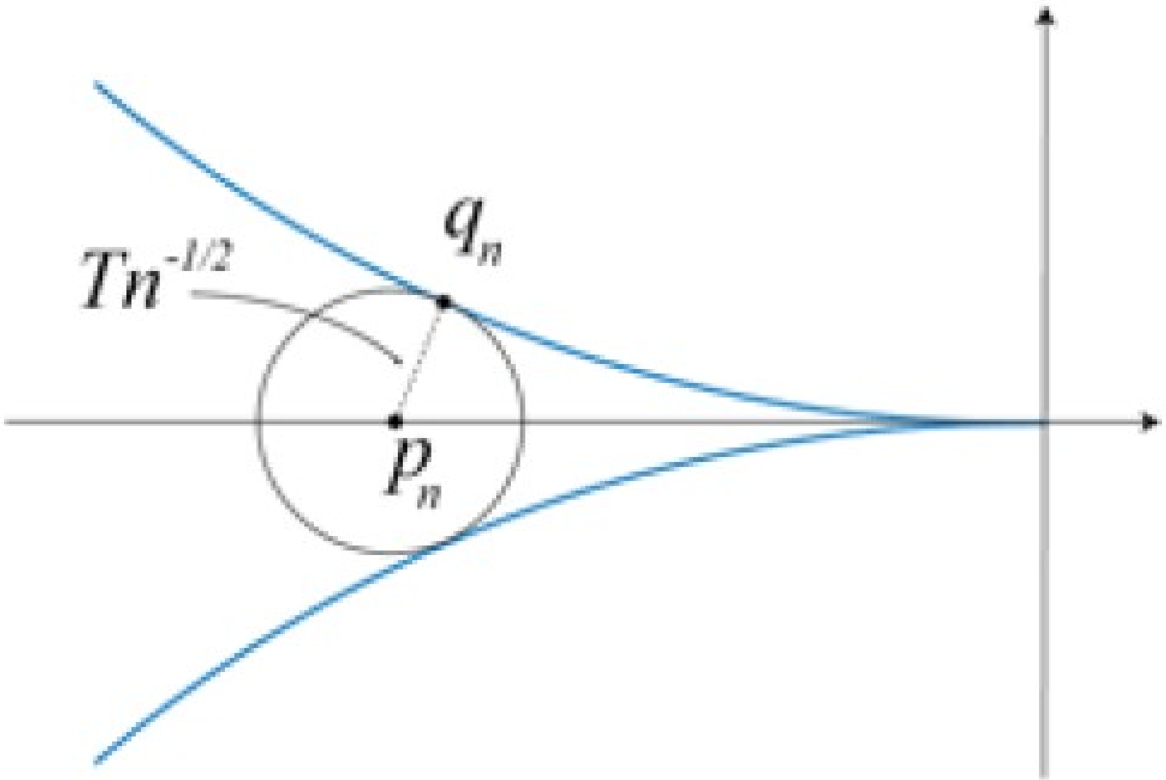
free boundary



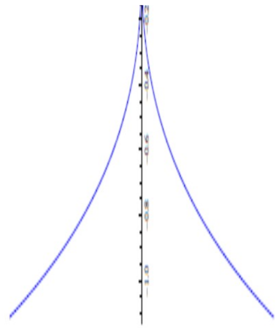
hard edge



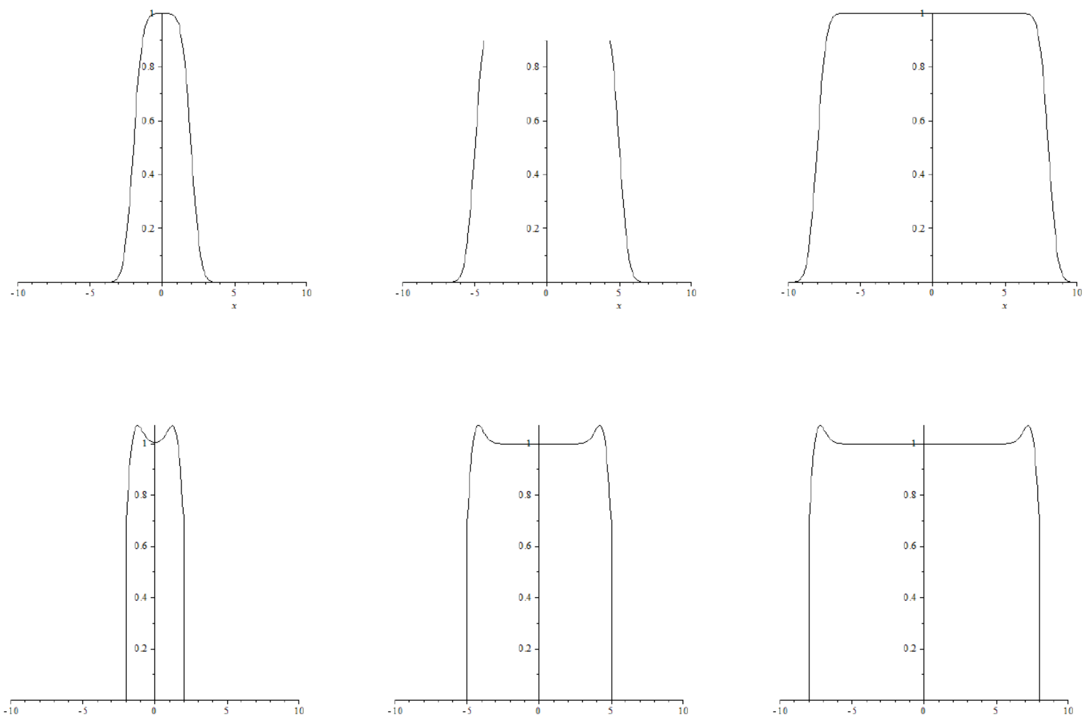
Scaling at a singular boundary point



$$\mathbb{S} = \{-T \leq y \leq T\}$$



$$\mathbb{S} = \{-T \leq x \leq T\}$$



free boundary

hard edge

2. Compactness and analyticity

Det-processes

$$R^{(k)}(\zeta_1, \dots, \zeta_k) = \det \mathbf{K}(\zeta_i, \zeta_j)$$

$R^{(k)} \sim \epsilon^{-k}\mathbb{P}(\text{at least one particle in each } B(\zeta_j, \epsilon)$.

RNM model:

$$\mathbf{K}_n(\zeta, \eta) = \sum_{j=1}^{n-1} p_j(\zeta)\overline{p_j(\eta)}e^{-nQ(\zeta)/2}e^{-nQ(\eta)/2},$$

p_j's are ON polynomials in $L^2\left(e^{-nQ}\right)$.

Rescaling: if $z = \sqrt{n}\zeta$ and $w = \sqrt{n}\eta$, then

$$K_n(z, w) = \frac{1}{n}\mathbf{K}_n(\zeta, \eta).$$

If $K_n(z, w) \to K(z, w)$, then limiting process exists, is unique.

Ginibre(∞)

Example: $Q(\zeta) = |\zeta|^2$ and $p = 0$, then

$$K_n(z, w) = \sum_0^{n-1} \frac{(z\bar{w})^j}{j!} e^{-|z|^2/2} e^{-|w|^2/2}$$

converges to

$$G(z, w) = e^{z\bar{w}} e^{-|z|^2/2} e^{-|w|^2/2}$$

Universality in bulk: if $p_n \in \mathrm{int}(S)$ and $\sqrt{n} \cdot \mathrm{dist}(p_n, \partial S) \to \infty$, then G is the limiting process

Compactness and analyticity

Theorem

Every subsequence of the point processes $\{K_n\}$ has a further subsequence which converges to a det-process (maybe zero)

Theorem

Every limiting process has the form

$K = G\Psi$ *(free boundary), or*
$K = G\Psi 1_{\mathbb{S}\times\mathbb{S}}$ *(hard edge),*

where $\Psi = \Psi(z, w)$ *is a Hermitian entire function.*

Proof

- The case of a regular point on hard edge boundary. For simplicity, $\Delta Q \equiv 1$, and $p = 0$. Rescaling: $z = \sqrt{n}\zeta$, $w = \sqrt{n}\eta$
- Rescaled correlation kernels

$$K_n = K_n^{\#} \Psi_n \, 1_{\sqrt{n}S \times \sqrt{n}S}$$

where

$$K_n^{\#}(z, w) = ne^{n[Q(\zeta,\eta) - Q(\zeta)/2 - Q(\eta)/2]}$$

and

$$\Psi_n(z, w) = \frac{1}{n}\mathbf{k_n}(\zeta, \eta)e^{-nQ(\zeta,\eta)};$$

$Q(\zeta, \eta)$ is Hermitian extension, and $\mathbf{k}_n$ is the reprokernel in $\mathcal{P}_n^2\left(e^{-nQ^S}\right)$

- Lemma 1. There are cocycles c_n such that $c_n K_n^{\#} \to G$
- Lemma 2. $\forall \eta \in \mathbb{C}$, $\mathbf{k}_n(\eta, \eta) \lesssim ne^{n\check{Q}(\eta)}$

Proof of last lemma

- Consider S_t with $(1-t) \asymp \frac{1}{\sqrt{n}}$. The points on the exterior boundary of S_t are at distance $\asymp \frac{1}{\sqrt{n}}$ from the boundary of S
- Fix η and let

$$q_n(\zeta) = \frac{\mathbf{k}_n(\zeta, \eta)}{\sqrt{\mathbf{k}_n(\eta, \eta)}}.$$

- We have $\|q_n\|^2_{nQ^S} = 1$ so

$$|q_n|^2 e^{-nQ} \le Cn \quad \text{on } S_t.$$

- Denote $u_n = \frac{1}{n} \log |q_n|^2$. Then

$$u_n - \check{Q}_t - \frac{C + \log n}{n} \le 0 \quad \text{on } S_t.$$

- LHS is subharmonic in S_t^c and $\sim (1-t) \log |\zeta|^2$ at ∞. Therefore,

$$u_n \le \check{Q}_t + \frac{C + \log n}{n} + (1-t) G_t \quad \text{on } S_t^c$$

and we conclude

$$u_n \le \check{Q} + \frac{C' + \log n}{n} \quad \text{on } S \text{ and therefore everywhere.}$$

Zero-one law

Theorem

Either $K \equiv 0$, *or* $\Psi(z, z) > 0$ *for all* $z \in \mathbb{C}$

Theorem

Suppose the limiting droplet $\mathbb{S}$ *exists. Then (every)* K *is trivial iff* $\mathbb{S}$ *has zero area.*

Holomorphic kernel

$$L(z,w) := e^{z\bar{w}}\Psi(z,w).$$

Theorem

L is the reprokernel of some Hilbert space $\mathcal{H}$ of entire functions. Moreover,

$$\mathcal{H} \overset{\text{contr}}{\hookrightarrow} A^2\left(e^{-|z|^2}\right) \quad \text{(free boundary)}$$

and

$$\mathcal{H} \overset{\text{contr}}{\hookrightarrow} A^2\left(e^{-|z|^2} \cdot 1_{\mathbb{S}}\right) \quad \text{(hard edge)}$$

Cf. de Branges spaces $\mathcal{B}(E)$ with

$$E = e^{-i\pi z}, \quad E = \mathrm{Ai}' - i\mathrm{Ai}, \quad E = \sqrt{z}J_0'(\sqrt{z}) - iJ_0(\sqrt{z}).$$

Mass one equation

Conjecture: we should have $\mathcal{H} \overset{\text{iso}}{\hookrightarrow} A^2(\cdots)$, i.e.

$$\forall z \in \mathbb{C}, \int_{\mathbb{C}} B(z,w)\, dA(w) = 1$$

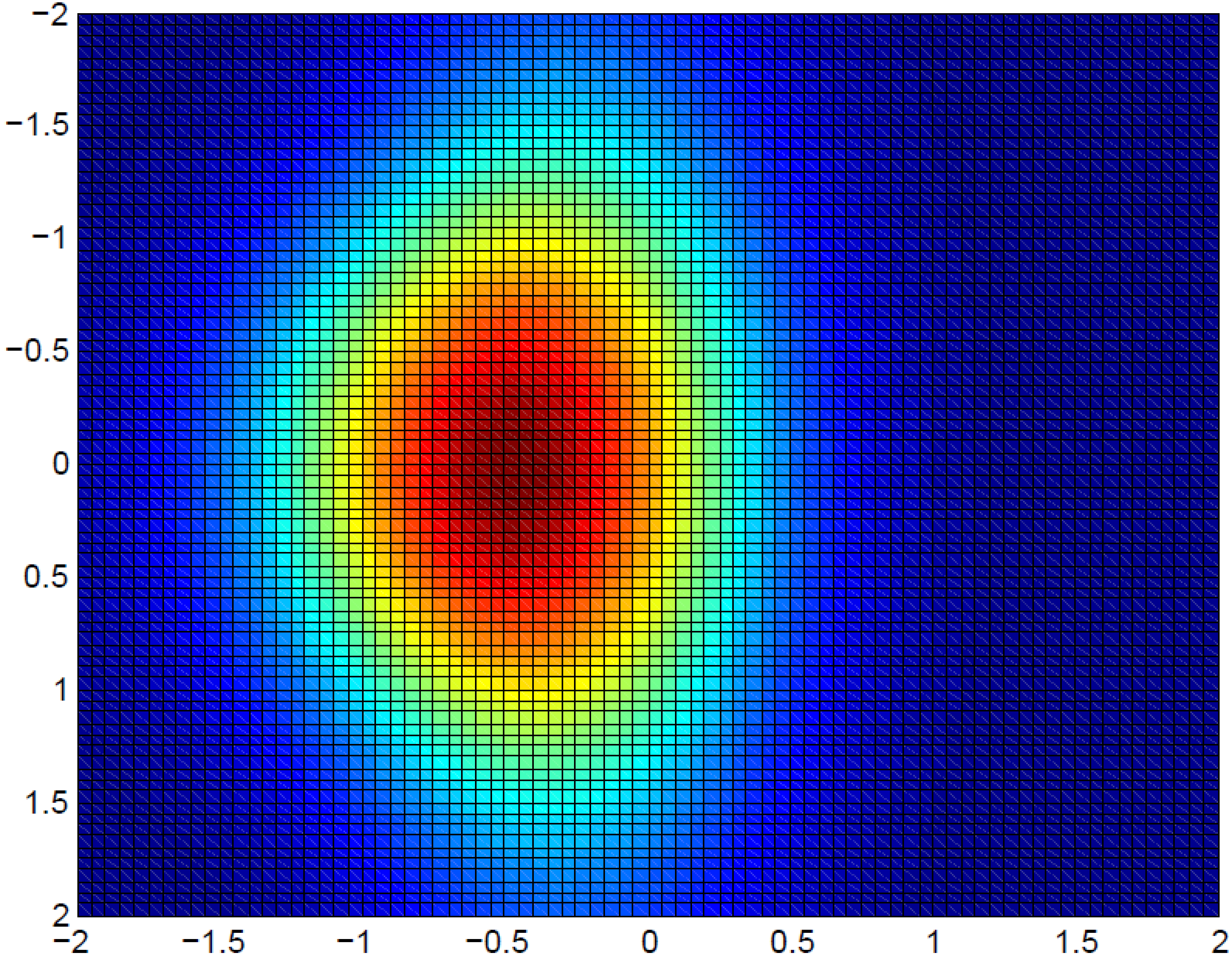
or

$$\forall z \in \mathbb{S}, \int_{\mathbb{S}} B(z,w)\, dA(w) = 1,$$

where

$$B(z,w) = \frac{|K(z,w)|^2}{K(z,z)}.$$

[We assume $K \not\equiv 0$]



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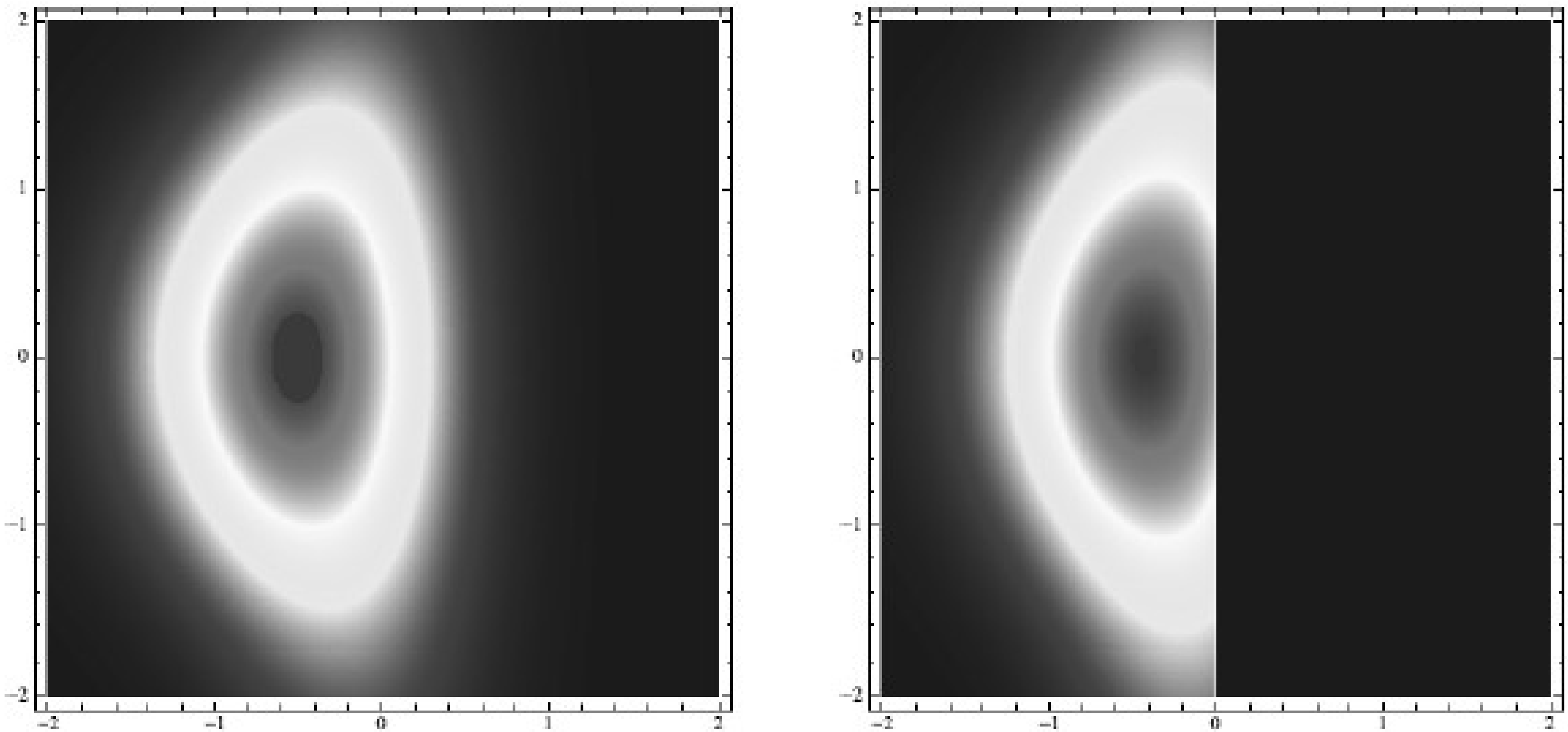

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3. Rescaling Ward's identities

S.E.T.

$$v \mapsto W[v] \quad \text{(vector field} \ \mapsto \ \text{random variable)}$$

For β-ensembles:

$$W_n^+[v] = \beta B_n(v) - \beta n \mathrm{Tr}_n(v \partial Q) + \mathrm{Tr}_n(\partial v)$$

$$B_n(v) = \frac{1}{2} \sum \sum_{j \neq k} \frac{v(z_j) - v(z_k)}{z_j - z_k}$$

- Notation: $\mathrm{Tr}_n(f) = \sum f(z_j)$
- W^+ is $\mathbb{C}$-linear part of $v \mapsto W_v$
- $W_n[v] = W_n^+[v] + \overline{W_n^+[v]}$

Ward's identities

$$\mathbb{E} W_n^+[v] = 0$$

If $\rho = \rho_n$ is the density field, a (1,1)-differential,

$$\int f \rho_n = \frac{1}{n} \sum f(z_j),$$

then

$$\mathbb{E} \mathcal{L}_v \left[\rho(z_1)\rho(z_2)\dots\right] = \mathbb{E} W_v \left[\rho(z_1)\rho(z_2)\dots\right],$$

where $\mathcal{L}_v$ is Lie derivative.

Ward's equation

Define

$$C(z) = \int_{\mathbb{C}} \frac{B(z,w)}{z-w}\, dA(w),$$

or

$$C(z) = \int_{\mathbb{S}} \frac{B(z,w)}{z-w}\, dA(w).$$

Theorem

The equation

$$\bar{\partial} C = R - 1 - \Delta(\log R)$$

holds pointwise in $\mathbb{C}$ or in $\mathbb{S}$.

4. Translation invariant solutions

We will consider free boundary and hard edge cases with

$$\mathbb{S} = \{x \le 0\} \quad \text{or} \quad \mathbb{S} = \{-\tau \le x \le \tau\}$$

Mass one and Ward's equations are (vertically) translation invariant. We'll find all their t.i. solutions

$$\Psi(z + it, w + it) \equiv \Psi(z, w).$$

Clearly,

$$\Psi(z, w) = \Phi(z + \bar{w}), \qquad \Phi \in \mathcal{A}(\mathbb{C}).$$

Key idea: use inverse Fourier transform for the purpose of analytic continuation

$$\Phi(z) = \int_{\mathbb{R}} e^{izt} \, (\Phi_{\mathbb{R}})\hat{}(t) \; dt, \quad (z \in \mathbb{C})$$

Gaussian reprersentation

$$\gamma(t) := \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

Theorem (free boundary case)

Let $K(z, w) = G(z, w)\Phi(z + \bar{w})$ be a t.i. limiting kernel. Then

$$\Phi_{\mathbb{R}} = \gamma * f, \quad 0 \leq f \leq 1.$$

Proof

- $L(z,w) = e^{z\bar{w}}\Phi(z+\bar{w})$ is a PDK on $i\mathbb{R}$

$$\sum \alpha_j \overline{\alpha_k} e^{x_j x_k} \Phi(ix_j - ix_k) \geq 0.$$

- Define $V(z)$ by $\Phi(iz) = V(z)e^{z^2/2}$. Then

$$\sum \alpha_j \overline{\alpha_k} e^{x_j^2/2} e^{x_k^2/2} V(x_j - x_k) \geq 0,$$

  so $V(x) = \hat{\mu}(x)$ for some positive finite μ.

- We have $\Phi_{\mathbb{R}} = \gamma * \nu$ for $d\nu = e^{t^2/2} d\mu$

- Since $\mathcal{H}(L) \overset{\text{contr}}{\hookrightarrow} A^2\left(e^{-|z|^2}\right)$, we have $L_1(z,w) := e^{z\bar{w}} - L(z,w)$ is a PDK, so $1 - \Phi_{\mathbb{R}} = \gamma * \nu_1$ and $\nu + \nu_1 \equiv 1$

Free boundary solutions

$$K(z,w) = G(z,w)\Phi(z+\bar{w}); \quad \Phi_{\mathbb{R}} = \gamma * f$$

Theorem

- $K \in (\text{mass one})$ *iff* $f = 1_E$, $E \subset \mathbb{R}$
- $K \in (\text{Ward})$ *iff* $f = 1_E$ *and* E *is connected*

Proof

- Lemma:
$$\int_{\mathbb{C}} e^{|w|^2} e^{iwt} e^{i\bar{w}s}\ dA(w) = e^{-st}$$
- Mass one:
$$\forall x \in \mathbb{R},\ \int_{\mathbb{C}} e^{-|w|^2} |\Phi(z+w)|^2 = \Phi(x)$$
or $(F := \Phi_{\mathbb{R}})$
$$F(x) = \iint_{(s,t)} e^{ix(t+s)} e^{-st} \hat{F}(t)\hat{F}(s)$$
- Subtract
$$F(x) = \iint_{(s,t)} e^{ix(t+s)} e^{-st} \hat{F}(t)\hat{1}(s)$$
to get $(F_1 := 1 - F)$
$$0 = \iint_{(s,t)} e^{ix(t+s)} e^{-st} \hat{F}(t)\hat{F_1}(s)$$
- Gaussian representation $F = \gamma * f$, $F_1 = \gamma * f_1$, $f_1 = 1 - f$ gives $\hat{f} * \hat{f_1} = 0$ or $f = f^2$.

Hard edge solutions with $\mathbb{S} = \{x \leq 0\}$

$$K(z,w) = G(z,w)\Phi(z+\bar{w})\cdot 1_{\mathbb{S}}(z)1_{\mathbb{S}}(w); \quad \Phi_{\mathbb{R}} = \gamma * h$$

Theorem

- $K \in (\text{mass one})$ *iff* $h = 1_E/F$, *where*

$$F = \gamma * 1_{(-\infty,0)},$$

$$\forall A > 0, \quad \int_E e^{At}\, dt < \infty$$

- $K \in (\text{Ward})$ *iff* $h = 1_E/F$, *and* $E = (a,b)$ *with* $b < \infty$

Hard edge solutions with $\mathbb{S} = \{-\tau \leq x \leq \tau\}$

$$\Phi_{\mathbb{R}} = \gamma * h_\tau$$

Theorem

- $K \in (\text{mass one})$ *iff* $h_\tau = 1_E / F_\tau$, *where*

$$F_\tau = \gamma * 1_{(-2\tau, 2\tau)},$$

  and

$$\forall A > 0, \quad \int_E e^{A|t|} dt < \infty$$

- $K \in (\text{Ward})$ *iff* $h_\tau = 1_E / F_\tau$, *and* E *is a finite interval*

5. Concluding remarks

Radially symmetric potentials

$$Q(\zeta) = Q(|\zeta|)$$

Theorem

If p is on a hard edge boundary, then the limiting point process exists and

$$K(z,w) = G(z,w)\Phi(z+\bar{w}), \quad (\Re z \leq 0, \; \Re w \leq 0)$$

with

$$\Phi(z) = \int_{-\infty}^{0} \frac{e^{-(t-z)^2/2}}{\int_t^\infty e^{-s^2/2} ds} \, dt.$$

Non-TI solutions

- Twisted convolutions:

$$(f \star g)(z) = \int_{\mathbb{C}} f(z-w)g(w)e^{i\Im(\bar{z}w)}dA(w)$$

- Hermitian extension of the 1-point function:

$$\Psi(z,w) = \int_{\mathbb{C}} e^{i(z\bar{t}+\bar{w}t)}\hat{R}(t)\, dA(t).$$

- Gauss representation:

$$\hat{R} = \Gamma r, \quad \widehat{1-R} = \Gamma r_1, \quad \Gamma(z) := e^{-|z|^2/2}$$

- Mass one equation:

$$\hat{r} \star \hat{r}_1 = 0$$

- Ward's equation has a similar form

Rescaling by n^α at singular points

$$\Delta C = (I - \Delta)\partial R$$

Mercy!

Equations for β-ensembles

$$\partial C = R - 1 - \frac{1}{\beta}\Delta \log R$$

$\{\sqrt{\beta} z_j\}$ satisfy Ward($\beta = 1$) but mass β

Guess the form of $R^{(2)}$ such that Ward(β) implied mass one