Universality for the hard edge

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The hard edge

First consider $n \times m$ matrices M of independent (real, complex, quaternion) Gaussians and form the appropriately scaled $MM^\dagger$.

For the counting measure of eigenvalues: if say $\frac{m}{n} \to \gamma \geq 1$,

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}(\lambda) \to \frac{1}{2\pi\lambda} \sqrt{(\lambda - L)(R - \lambda)}\, d\lambda$$

where $L = (1 - \sqrt{\gamma})^2$ and $R = (1 + \sqrt{\gamma})^2$

When $\gamma > 1$ both edges are "soft", and we have (classical) Tracy-Widom fluctuations (in terms of Painlevé II).

When $\gamma = 1$, then $L = 0$ and have a different phenomenon as the eigenvalues now feel the "hard edge" of the origin.

In fact, if $m = n + a$ as $n \uparrow \infty$ there is a one-parameter family of limit laws for λ_{min} indexed by a (first discovered by Tracy-Widom in terms of Painlevé IV).

And if take $a \to \infty$ after the fact one recovers the (soft-edge) Tracy-Widom laws.

Beta ensembles

For the soft edge it is more immediate to consider "beta Hermite ensembles" generalizing GOE or GUE. These are the measures on $\mathbb{R}^n$ with density proportional to

$$\prod_{i \neq j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^n w(\lambda_i), \quad w(\lambda) = e^{-\frac{\beta}{4} n \lambda^2}.$$

with any $\beta > 0$.

Similarly, the appropriate generalization of the Wishart type matrices are the so called "beta Laguerre ensembles" where the above is replaced by

$$\prod_{i \neq j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^n w(\lambda_i), \quad w(\lambda) = \lambda^{\frac{\beta}{2}(m-n)-1} e^{-\frac{\beta}{2} n \lambda}$$

restricted to $\mathbb{R}_+^n$.

Tuned for the hard edge we set $m - n = a$ in the latter density which is then sensible for any $a > -1$.

Random operators

Ramírez-R-Virág established that the limiting general beta soft-edge point process is described by the "stochastic Airy operator"

$$\mathsf{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x)$$

acting on $\mathbb{R}_+$ with a Dirichlet boundary condition at the origin. Here $b'(x)$ is a white noise.

The corresponding hard edge operator was worked out by Ramírez-R and reads:

$$\mathsf{L}_{\beta,a} = -e^x \left(\frac{d^2}{dx^2} - (a + \frac{2}{\sqrt{\beta}} b'(x)) \frac{d}{dx} \right),$$

again on $\mathbb{R}_+$ with a Dirichlet boundary condition at the origin.

While H_β requires some care to make sense of one can write down $\mathsf{L}_{\beta,a}^{-1}$ explicitly and check that it is (almost surely) trace class.

Hermite/Laguerre Tridiagonals

These (due to Dmitriu-Edelman) in a sense generalize the standard Householder tri/bi-diagonalization algorithm.

For the $e^{-n\beta\lambda^2/2}$ weight on the line build the $n \times n$ random Jacobi matrix $T(A, B)$ where

$$A_k \sim N(0, 2) \text{ for } k = 1, \ldots, n, \quad B_k \sim \chi_{\beta(n-k)} \text{ for } k = 1, \ldots, n-1$$

and all independent. Then (for whatever $\beta > 0$) the matrix $T(A, B)$ has the desired eigenvalue law (after a scaling by $\frac{1}{\sqrt{n\beta}}$).

For the Laguerre-like case, you take instead a bidiagonal matrix $M(X, Y)$ with the X's on diagonal and the Y's above where:

$$X_k \sim \chi_{\beta(k+a)} \text{ for } k = 1, \ldots, n, \quad Y_k \sim \chi_{\beta k} \text{ for } k = 1, \ldots, n-1,$$

again all independent. Then MM^T does the job.

Soft edge easier to "see"

The scaling for Tracy-Widom is of course that $n^{2/3}(\lambda_{max} - 2) \Rightarrow TW$, so what you want to see is that

$$n^{2/3}\Big(2 - T(A,B)\Big) \to \mathsf{H}_\beta$$

in an appropriate sense. And formally $\chi_{\beta(n-k)} \sim \sqrt{\beta n} - \frac{\sqrt{\beta}}{2}\frac{k}{n} + \frac{1}{\sqrt{2}}G$ gives that the left hand side ($n^{2/3}$ times) resembles the discrete Laplacian with potential

$$\frac{1}{2n^{1/3}}T(0,(1,2,3,\dots)) + \beta^{-1/2}n^{1/6}\,T(\mathbf{G},\mathbf{G}').$$

In a real sense TW_β is identified through the integrated potential:

$$n^{2/3}\sum_{k=1}^{xn^{1/3}}(\mathcal{E} - (A_k + 2B_k)) \Rightarrow \frac{x^2}{2} + \frac{2}{\sqrt{\beta}}b(x)$$

where $\mathcal{E}$ is the edge (here happens to be $\mathcal{E} = 2$).

Hard edge goes through inverses

The formula advertised before is that

$$(-\mathsf{L})^{-1} = KK^T$$

where K is the kernel operator

$$K(s,t) = \frac{1}{\sqrt{t}} \left(\frac{s}{t}\right)^{a/2} \exp\left[\int_s^t \frac{db_u}{\sqrt{\beta u}}\right] 1_{s<t}$$

(after a change of variables). This arises from the matrix model having the form MM^T and the explicit expression

$$[M^{-1}]_{k,\ell} = \frac{(-1)^{k+\ell}}{M_{\ell,\ell}} \prod_{j=k}^{\ell} \frac{M_{j,j+1}}{M_{j,j}}$$

for bi-diagonals. Then the simple facts

$$\frac{X_{nt}}{\sqrt{\beta n}} \Rightarrow \sqrt{t}, \qquad \sum_{k=ns}^{nt} \log \frac{Y_k}{X_k} \Rightarrow \frac{a}{2} \log\left(\frac{s}{t}\right) + \int_s^t \frac{db_u}{\sqrt{\beta u}}$$

identify the hard edge operator.

Universality (for general potentials)

Now one wants to consider weights $e^{-n\beta V(x)}\mathbf{1}_{\mathbf{R}}$ or $x^{\gamma}e^{-n\beta V(x)}\mathbf{1}_{\mathbf{R}_+}$ generalizing β-Hermite/Laguerre.

Again there are tridiagonal models – give a conceptually simply explanation for soft/hard edge universality.

Krishnapur-R-Virág showed that that for $\beta > 0$ and V a strictly convex polynomial see the Stochastic Airy Operator at the soft edge.

There are better universality results out there (Bourgade-Erdös-Yau, Bekerman-Figalli-Guionnet)

What is new is a similar statement for the hard edge. Though now we need $\beta \geq 1$.

Quick look at soft edge

Draw $(\mathbf{A},\mathbf{B})$ according to the law with density

$$\propto \text{exp}\left(-n\beta \text{tr} V(T(a,b))\right)\prod_{k=1}^{n-1} b_k^{\beta(n-k)-1} = \text{exp}\left(-n\beta H(\mathbf{a},\mathbf{b})\right)$$

Then $T((\mathbf{A},\mathbf{B})$ has eigenvalue density $\prod|\lambda_i-\lambda_j|^\beta e^{-n\sum \beta V(\lambda_i)}$.

And you want to get your hands on: for Φ a nice test function of the first $O(n^{1/3})$ coordinates,

$$\int \Phi(\mathbf{a},\mathbf{b}) e^{-n\beta H(\mathbf{a},\mathbf{b})} d\mathbf{a} d\mathbf{b} = \int \left[\int \Phi(a,b) e^{-n\beta H_q(\mathbf{a},\mathbf{b})} d\mathbf{a} d\mathbf{b}\right] dQ(q)$$

where the cryptic H_q indicates a conditional Hamiltonian/measure.

Now the finite range nature of the potential V produces an exponential decay of dependence of the minimizers of H_q on the particulars of q.

Course Hamiltonians/minimizers

The full Hamiltonian is

$$H(\mathbf{a}, \mathbf{b}) = \mathrm{tr} V\Big(T(\mathbf{a}, \mathbf{b})\Big) - \sum_{k=1}^{n-1} \Big(1 - \frac{k}{n} - \frac{1}{n\beta}\Big) \log b_k.$$

Fix an index $k \sim nt$, and take the point of view that the minimizing a_k, b_k should be locally constant. By counting the paths that comprise the trace arrive at "course" Hamiltonian in two variables:

$$H_0(a, b) = [1] V\Big(a + b(z + \frac{1}{z})\Big) - (1 - t) \log b$$

whose minimizer $a(t), b(t)$ you can compute enough information about in order to expand around.

Has further meaning via the time-dependent equilibrium problem: the minimizing measure for

$$\int \frac{1}{1-t} V(x) d\mu(x) - \int \int \log |x - y| d\mu(x) d\mu(y)$$

has support $[a(t) - 2b(t), a(t) + 2b(t)]$.

General Wishart tridiagonals

Let $M = M(\mathbf{x}, \mathbf{y})$ by the bidiagonal matrix with $\mathbf{x} = (x_1, x_2, \dots)$ on the main diagonal and $\mathbf{y} = (y_1, y_2, ...)$ just above.

Draw $(\mathbf{X}, \mathbf{Y})$ from the density

$$P_V(x_1, \dots x_n, y_1, \dots, y_{n-1}) = c \,\mathsf{exp}\left[-n\beta \mathsf{tr} V(MM^T)\right] \prod_{k=1}^{n} x_k^{k+a-1} \prod_{k=1}^{n-1} y_k^{k-1},$$

then the tridiagonal matrix $M(X,Y)M(X,Y)^T$ has joint eigenvalue law

$$c' \prod_{k=1}^{n} \lambda_k^{\frac{\beta}{2}a-1} e^{-n\beta V(\lambda_k)} \times \prod_{j \neq k} |\lambda_j - \lambda_k|^\beta.$$

Hard edge universality

Assume now that $x \mapsto V(x^2)$ is a strictly convex polynomial and $\beta \geq 1$.

With $V(x) = \sum_{m=1}^{d} g_m x^m$ define ϕ to be the (unique) solution to

$$t = \sum_{m=1}^{d} m \binom{2m}{m} g_m \phi(t)^{2m} \qquad \text{for } t \in [0,1]$$

as well as

$$\theta(t) = c \left(\int_0^t \frac{du}{\phi(u)} \right)^2 \qquad \text{normalized so that } \theta(1) = 1.$$

Then, with the bidiagonal M drawn from the law P_V it holds that

$$(M)^{-1}(ns, nt) = \frac{1}{X_{nt}} \prod_{ns}^{nt} \frac{X_k}{Y_k} \to \frac{1}{\sqrt{\phi(s)\phi(t)}} \left(\frac{\theta(s)}{\theta(t)} \right)^{\frac{a}{2}+\frac{1}{4}} \exp \left[\frac{1}{\sqrt{\beta}} \int_{\theta(s)}^{\theta(t)} \frac{db_z}{\sqrt{z}} \right]$$

as operators. Note when $V(x) = x$ have that $\phi(t) = \sqrt{t}$ and $\theta(t) = t$ and this is the limit kernel K that we saw before.

Some ingredients

First, what is this function ϕ?

The full Hamiltonian reads

$$H(\mathbf{x},\mathbf{y}) = \mathrm{tr} V\left(M(\mathbf{x},\mathbf{y})M(\mathbf{x},\mathbf{y})^T\right) - \sum_{k=1}^{n}(\frac{k}{n} + \frac{a}{n} - \frac{1}{n\beta})\log x_k - \sum_{k=1}^{n-1}(\frac{k}{n} - \frac{1}{n\beta})\log y_k$$

Following the same philosophy as in the soft edge we fix about a continuum index $k \sim nt$ and use the guess that the minimizer $k \mapsto x_k, y_k$ is locally constant. Dropping nuisance factors this leads to

$$H_0(x,y) = \sum_{m=1}^{d}\left(g_m \sum_{\ell=0}^{m}\binom{m}{\ell}^2 x^{2\ell}y^{2m-2\ell}\right) - t\log x - t\log y$$

and ϕ is the common minimizing x and y.

Though keep in mind that since the limiting potential arises from $\sum_{k=ns}^{nt}\log\left(\frac{X_k}{Y_k}\right)$, this can't give the full story, even for the mean.

Issue 1: Non-locality

The fact that the functional CLT is required over $O(n)$ variables simply means that you have to employ a classical block strategy (the soft edge calculation is a "one block" problem).

The field is split up into "good" blocks of length $O(n^{\gamma})$ for γ small, and "bad" blocks of length $O(\log n)$.

The latter are small enough not to contribute, but large enough that the variables in consecutive good blocks decorrelate in the limit.

The decorrelation again comes the decay of dependence of the minimizers on any (reasonable) boundary condition. This yields that the Gaussian approximation in any good block doesn't feel the variables outside.

Basically had this tool developed for the soft edge (though now need more care as errors "pile up").

Issue 2: Require finer estimates of the minimizer(s)

The proposed limit theorem requires say: with $k \mapsto x_0(k), y_0(k)$ the true (global) or any appropriate conditional minimizer,

$$\sum_{ns}^{nt} \log \frac{x_0(k)}{y_0(k)} \to \left(\frac{a}{2} + \frac{1}{4}\right) \log \frac{\theta(t)}{\theta(s)} - \frac{1}{2} \log \frac{\phi(t)}{\phi(s)},$$

and certainly our first order approximation $x_0 \sim y_0 \sim \phi$ doesn't cut it.

Here we really rely on convexity, as we have that: with c the convexity constant for H,

$$||(\mathbf{x}, \mathbf{y}) - (\mathbf{x}', \mathbf{y}')||_2 \leq \frac{1}{c} ||\nabla H(x, y) - \nabla H(x', y')||_2.$$

And then putting in the ansatz

$$x_0(k) = \phi(k/n) + \frac{1}{n}\bar{x}(k/n) + \cdots, \quad y_0(k) = \phi(k/n) + \frac{1}{n}\bar{y}(k/n) + \cdots$$

an exact calculation of ∇H will provide as sharp an approximation to the actual minimizer as you may want/need.