Statistics of the real roots of real random polynomials

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Optimal and random point configurations

IHP, June 27- July 1st

G. S., S. N. Majumdar, Phys. Rev. Lett. **99**, 060603 (2007), arXiv:0705.2648,

J. Stat. Phys. **132**, 235-273 (2008), arXiv:0803.4396,

J. Stat. Phys. **135**, 587-598 (2009), arXiv:0902.1027

Real random polynomials

$$P_n(x) = \sum_{i=0}^{n} a_i x^i$$

$a_i \equiv$ ind. random variables,
$\mathbb{E}(a_i) = 0$, $\mathbb{E}(a_i a_j) = \sigma_i^2 \delta_{ij}$

$$P_n(\lambda_i) = 0 \; , \;\; \lambda_1, \cdots, \lambda_d \in \mathbb{R}$$

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General question: statistics of λ_i's ?

Introduction

- Bloch, Pólya, *On the roots of certain algebraic equations* (1932),
- Littlewood, Offord, *On the number of real roots of a random algebraic equation* (1939),
- Kac, *On the average number of real roots of a random algebraic equation*, (1943),
- Edelman, Kostlan, *How many zeros of random polynomials are real ? (1999)*,

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- Kac, *On the average number of real roots of a random algebraic equation*, (1943),
- Bogomolny, Bohigas, Leboeuf, *Quantum chaotic dynamics and random polynomials*, (1992),
- Edelman, Kostlan, *How many zeros of random polynomials are real ? (1999)*,
- Y. Castin *et al.*, *Seeing zeros of random polynomials: quantized vortices in the ideal Bose gas*, (2005)

Introduction: topics of this talk

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Condensation of the roots of P_n on the real axis

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2. Random polynomials having few or no real roots

   Probability of no real root and first-passage problems of stochastic processes

Outline

1. Condensation of the roots of random polynomials on the real axis
   - Motivations: Kac's polynomials and beyond
   - Condensation transition
   - Derivation of the results
2. Polynomials having few real roots
   - Motivation: roots of Kac's random polynomials
   - First passage problems and persistence
   - Derivation: mapping to a Gaussian Stationary Process
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Motivations : Kac's polynomials

Real Kac's polynomials

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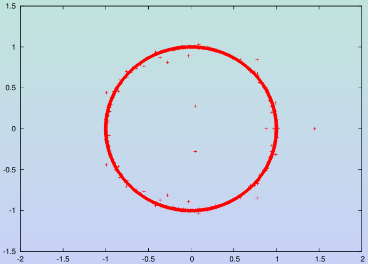
Complex roots



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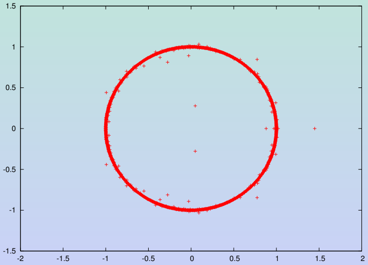

Real roots

$\mathbb{E}(N_n) \equiv$ average number of roots on the real axis M. Kac '43

$$\mathbb{E}(N_n) = \frac{2}{\pi} \log n + \mathcal{O}(1)$$

Motivations : Kac's polynomials

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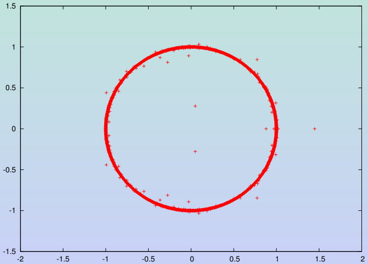

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$$\mathbb{E}(N_n) \sim \frac{2}{\pi} \log n \ll n$$

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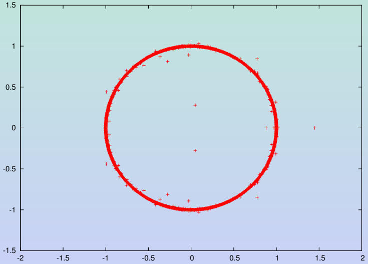

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Q : how can one increase $\mathbb{E}(N_n)$ by modifying $\mathbb{E}(a_i^2)$?

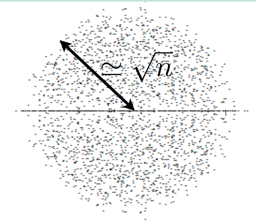
Beyond Kac's polynomials

- Weyl polynomials

$$W_n(x) = \sum_{i=0}^{n} a_i x^i$$

$a_i \equiv$ independent random variables,
$\mathbb{E}(a_i) = 0$, $\mathbb{E}(a_i a_j) = \sigma_i{}^2 \delta_{ij}$, $\sigma_i = \frac{1}{\sqrt{i!}}$

Complex roots



Real roots

$$\mathbb{E}(N_n) = \frac{2}{\pi}\sqrt{n} + o(\sqrt{n})$$

Beyond Kac's polynomials

- Littlewood & Offord's random polynomials

$$L_n(x) = \sum_{i=0}^{n} a_i x^i$$

$$a_i = \frac{\epsilon_i}{\sqrt{(i!)^i}} \, , \epsilon_i = \pm 1$$

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* All roots are real with probability one: $N_n = \mathbb{E}(N_n) = n$

* (Quasi)-periodic structure:

$$x_0 = 0 \,, x_m = m^m m! \,, m \leq n$$

one root in $[x_{m-1}, x_m]$ or in $[-x_{m-1}, -x_m]$ with probability one

Kac's polynomials and beyond

1. Kac polynomials

$$K_n(x) = \sum_{i=0}^{n} a_i x^i \,, \mathbb{E}(a_i^2) = \sigma^2 \implies \mathbb{E}(N_n) \propto \log n$$

2. Weyl polynomials

$$W_n(x) = \sum_{i=0}^{n} a_i x^i \,, \mathbb{E}(a_i^2) = (i!)^{-1} \implies \mathbb{E}(N_n) \propto \sqrt{n}$$

3. Littlewood-Offord polynomials

$$L_n(x) = \sum_{i=0}^{n} a_i x^i \,, \mathbb{E}(a_i^2) = (i!)^{-i} \implies \mathbb{E}(N_n) \propto n$$

Kac's polynomials and beyond

1. Kac polynomials: $K_n(x) = \sum_{i=0}^{n} a_i x^i$, $\mathbb{E}(a_i^2) \sim e^{-i^0}$

   $\mathbb{E}(N_n) \propto \log n$

2. Weyl polynomials: $W_n(x) = \sum_{i=0}^{n} a_i x^i$, $\mathbb{E}(a_i^2) \sim e^{-i \ln i}$

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A family of random polynomials indexed by α

$$P_n(x) = \sum_{i=0}^{n} a_i x^i \; , \; \mathbb{E}(a_i^2) = e^{-i^\alpha}$$

Outline

Condensation transition: average number of real roots

A family of random polynomials indexed by α

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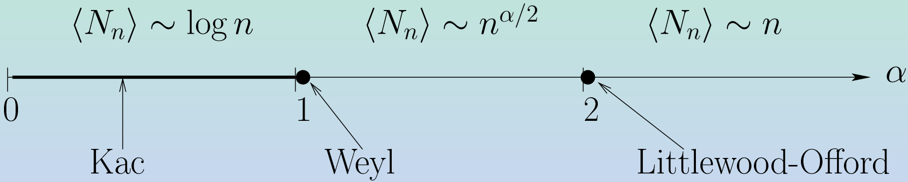

Condensation transition: the density

- A change of variable:

$$Y = \left(\frac{2}{\alpha} \ln x \right)^{\frac{1}{\alpha - 1}}$$

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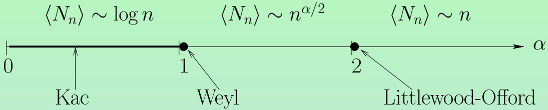

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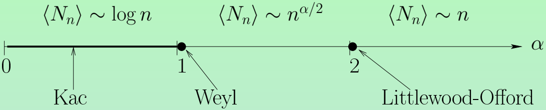
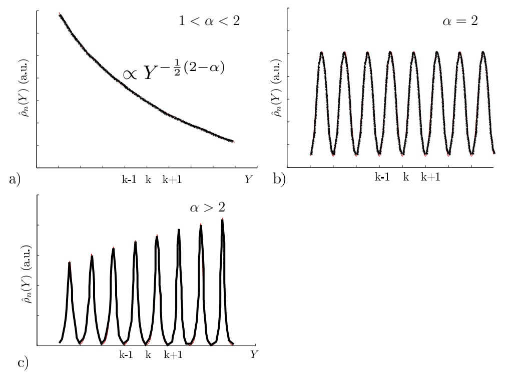


- The density $\hat{\rho}_n(Y)$ across the transition



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Average density of real roots

$$P_n(\lambda_i) = 0 \, , \; \lambda_1, \cdots, \lambda_d \in \mathbb{R}$$

$$\rho_n(x) = \sum_{i=1}^{d} \mathbb{E}[\delta(x - \lambda_i)]$$

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Average density of real roots

$$
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After some algebra...

$$\rho_n(x) = \frac{\sqrt{c_n(x)(c_n'(x)/x + c_n''(x)) - [c_n'(x)]^2}}{2\pi c_n(x)}\,,$$

$$c_n(x) = \mathbb{E}[P_n(x)P_n(x)] = \sum_{k=0}^{n} e^{-k^\alpha} x^{2k}$$

Average density of real roots

- Saddle point calculation

$$\rho_n(x) = \frac{\sqrt{c_n(x)(c_n'(x)/x + c_n''(x)) - [c_n'(x)]^2}}{2\pi c_n(x)} \,,$$

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$$\phi(u,x) = u^\alpha - 2u \ln x$$

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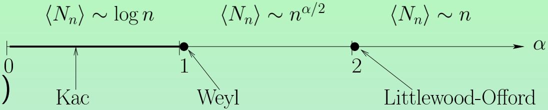
- 3 different cases depending on α

1. $\alpha < 1$: $u^*(x) = n$
2. $1 < \alpha < 2$: $u^*(x) < n$ & $\partial_u^2 \phi(u^*(x),x) \to 0$, $x \to \infty$
3. $\alpha > 2$: $u^*(x) < n$ & $\partial_u^2 \phi(u^*(x),x) \to \infty$, $x \to \infty$

Condensation transition: to summarize

- A change of variable:

$$Y = \left(\frac{2}{\alpha}\ln x\right)^{\frac{1}{\alpha-1}} = u^*(x)$$



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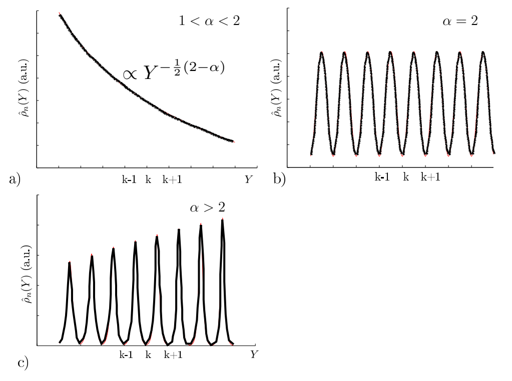

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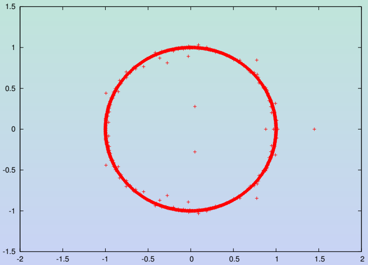

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Motivations : Real roots of Kac's polynomials

JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 15, Number 4, Pages 857–892
S 0894-0347(02)00386-7
Article electronically published on May 16, 2002

RANDOM POLYNOMIALS HAVING FEW OR NO REAL ZEROS

AMIR DEMBO, BJORN POONEN, QI-MAN SHAO, AND OFER ZEITOUNI

$q_0(n) \equiv$ Probability that $K_n(x)$ has no real root in $[0, 1]$

$$q_0(n) \propto n^{-\gamma}$$

$$\text{with} \quad \gamma = 0.19(1) \quad \text{(Numerics)}$$

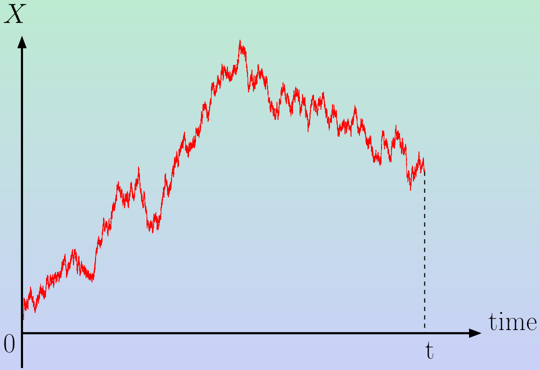
Outline

Introduction

Persistence probability $p_0(t)$

- $X(t) \equiv$ stochastic random variable evolving in time t, $\mathbb{E}[X(t)] = 0$
- Persistence probability

$$p_0(t) \equiv \text{Proba. that } X \text{ has not changed sign up to time } t$$



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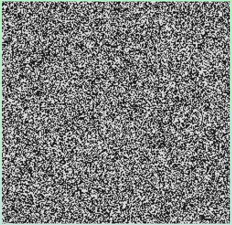
Persistence in spatially extended systems

- phase ordering kinetics

Introduction: Phase ordering kinetics

- Glauber dynamics of 2*d* Ising model at $T = 0$, $H_{\rm Ising} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$
  $\sigma_i = \pm 1$



$t_1 = 0$

Introduction: Phase ordering kinetics

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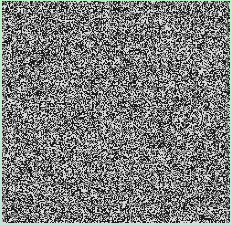

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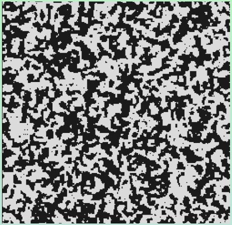

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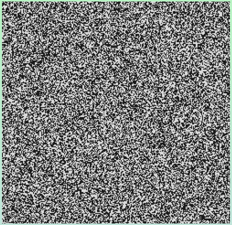

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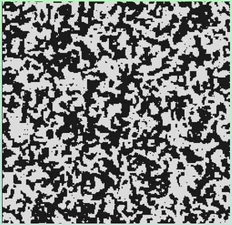

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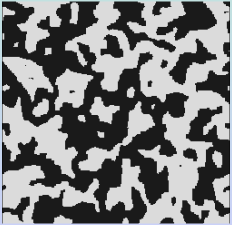

$t_3 = 10^4$

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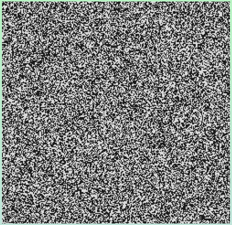
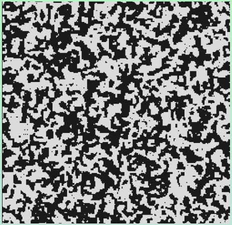
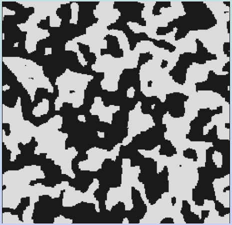


$t_1 = 0$



$t_2 = 10^2$



$t_3 = 10^4$

$t_4 = 10^6$

Introduction

Persistence probability $p_0(t)$

- $X(t) \equiv$ stochastic random variable evolving in time t, $\mathbb{E}[X(t)] = 0$
- Persistence probability

$$p_0(t) \equiv \text{Proba. that } X \text{ has not changed sign up to time } t$$

Persistence in spatially extended systems

- phase ordering kinetics ('94-)

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$$p_0(t) \propto t^{-\theta_p}$$

A. J. Bray, S. N. Majumdar, G. S., Adv. Phys. **62**, pp 225-361 (2013), arXiv:1304.1195

"Persistence and First-Passage Properties in Non-equilibrium Systems"

Motivations : persistence for the diffusion equation

Diffusion equation with random initial conditions

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\mathbb{E}(\phi(x,0)\phi(x',0)) = \delta^d(x-x')$$

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- Ordering dynamics for $O(N)$-symmetric spin models in the limit $N \to \infty$
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Single length scale
$\ell(t) \propto t^{1/2}$

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Persistence $p_0(t,L)$ for a d-dim. system of linear size L

$p_0(t,L) \equiv$ Proba. that $\phi(x,t)$ has not changed sign up to t

S. N. Majumdar, C. Sire, A. J. Bray and S. J. Cornell, PRL 96

B. Derrida, V. Hakim and R. Zeitak, PRL 96

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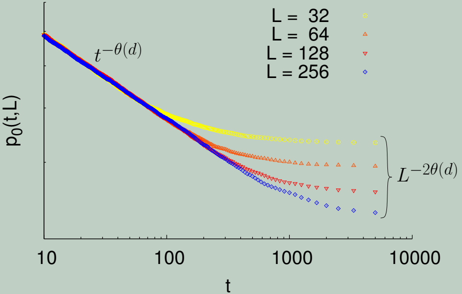

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$p_0(t,L) \equiv$ Proba. that $\phi(x,t)$ has not changed sign up to t

$$p_0(t,L) \propto L^{-2\theta(d)} h(t/L^2)$$

$$\theta(1) = 0.1207$$
$$\theta(2) = 0.1875 \quad , \quad \text{Numerics}$$

Purpose : a link between random polynomials & diffusion equation

Generalized Kac's polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i$$

$a_i \equiv$ Gaussian random variables, $\mathbb{E}(a_i) = 0$, $\mathbb{E}(a_i a_j) = \delta_{ij}$

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$$q_0(n) \propto n^{-b(d)}$$

Persistence of diffusion

$$p_0(t, L) \propto L^{-2\theta(d)}$$

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G. S., S. N. Majumdar 07

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G. S., S. N. Majumdar 07

A. Dembo, S. Mukherjee 15

Outline

Persistence of diffusion equation

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\mathbb{E}(\phi(x,0)\phi(x',0)) = \delta^d(x-x')$$

$$\phi(x,t) = \int_{|y|<L} d^d y G(x-y,t)\phi(y,0)$$

$$G(x,t) = (4\pi t)^{-\frac{d}{2}} \exp\left(-x^2/4t\right)$$

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Mapping of $\phi(x,t)$ to a Gaussian stationary process

1. Normalized process $X(t) = \frac{\phi(x,t)}{[\mathbb{E}[\phi(x,t)^2]]^{1/2}}$

$$\mathbb{E}(X(t)X(t')) \sim \begin{cases} \left(4\frac{tt'}{(t+t')^2}\right)^{\frac{d}{4}}, & t,t' \ll L^2 \\ 1, & t,t' \gg L^2 \end{cases}$$

2. New time variable $T = \log t$, for $t \ll L^2$

$$\mathbb{E}\left(X(T)X(T')\right) = \left[\cosh((T-T')/2)\right]^{-d/2}$$

Persistence for a Gaussian stationary process (GSP)

- $X(T)$ is a GSP with correlations

$$\begin{aligned} \mathbb{E}(X(T)X(T')) &= a(T - T') \\ a(T) &= (\cosh(T/2))^{-d/2} \end{aligned}$$

- Persistence probability $\mathcal{P}_0(T)$ (by Slepian's lemma)

For $T \gg 1$ $a(T) \propto \exp\left(-\frac{d}{2} T\right) \Rightarrow \mathcal{P}_0(T) \propto \exp\left(-\theta(d) T\right)$

- Reverting back to $t = \exp(T)$

$$p_0(t, L) \sim t^{-\theta(d)} \quad 1 \ll t \ll L^2$$

Persistence of diffusion equation

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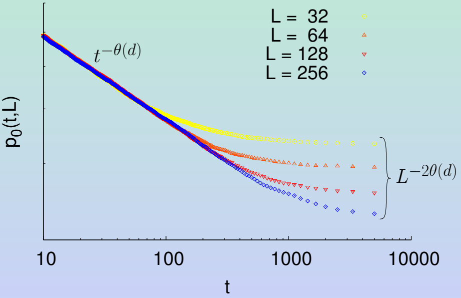

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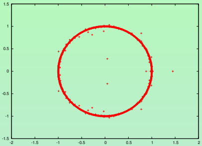
$$p_0(t,L) \propto L^{-2\theta(d)} h(t/L^2) \qquad\qquad h(u) \sim \begin{cases} u \sim u^{-\theta(d)} \,, & u \ll 1 \\ u \sim c^{\rm st} \,, & u \gg 1 \end{cases}$$

Real roots of generalized Kac's polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i$$



Averaged density of real roots

$$\rho_n(x) \;=\; \mathbb{E}[|K_n'(x)|\,\delta(K_n(x))]$$

Real roots concentrate around $x = \pm 1$

$$\rho_n(\pm 1) \sim A_d n \qquad A_d = \frac{2\sqrt{d/(d+4)}}{\pi(d+2)}$$

Real roots of generalized Kac's polynomials

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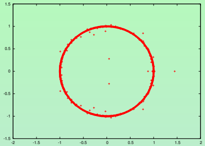
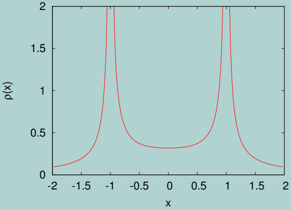


Averaged density of real roots for $n \to \infty$

$$\rho_\infty(x) = \frac{\left(\mathrm{Li}_{-1-d/2}(x^2)(1+\mathrm{Li}_{1-d/2}(x^2)) - \mathrm{Li}^2_{-d/2}(x^2)\right)^{\frac{1}{2}}}{\pi|x|(1+\mathrm{Li}_{1-d/2}(x^2))}$$



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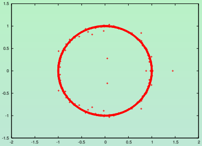
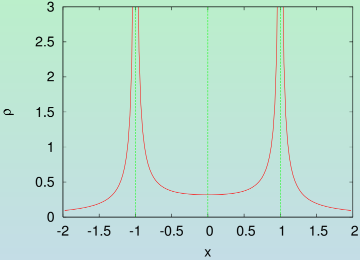


Mean number of real roots in $[0, 1]$: Kac-Rice formula

$$\mathbb{E}(N_n[0,1]) = \int_0^1 \rho_n(x)\,dx \sim \frac{1}{2\pi}\sqrt{\frac{d}{2}}\log n$$

Probability of no real root for $K_n(x)$



Dembo *et al.* '02

Statistical independence of $K_n(x)$ in the 4 sub-intervals

$\Longrightarrow$ Focus on the interval [0, 1]

$P_0(x, n) \equiv$ Proba. that $K_n(x)$ has no real root in $[0, x]$

Probability of no real root for $K_n(x)$

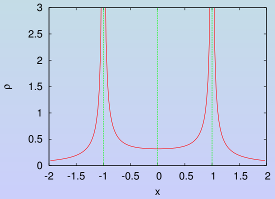
- Two-point correlator

$$C_n(x,y) = \mathbb{E}(K_n(x)K_n(y)) = \sum_{i=0}^{n-1} i^{(d-2)/2}(xy)^i$$

- Normalization

$$\mathcal{C}_n(x,y) = \frac{C_n(x,y)}{(C_n(x,x))^{\frac{1}{2}}(C_n(y,y))^{\frac{1}{2}}}$$

- Change of variable



$$x = 1 - \frac{1}{t} \quad , \quad t \gg 1$$

Probability of no real root for $K_n(x)$

Normalized correlator in the scaling limit

- Scaling limit

$$t \gg 1 \quad , \quad n \gg 1 \quad \text{keeping} \quad \tilde{t} = \frac{t}{n} \quad \text{fixed}$$

- $\mathcal{C}_n(t,t') \to \mathcal{C}(\tilde{t},\tilde{t}')$ with the asymptotic behaviors

$$\mathcal{C}(\tilde{t},\tilde{t}') \sim \begin{cases} \left(4\frac{\tilde{t}\tilde{t}'}{(\tilde{t}+\tilde{t}')^2}\right)^{\frac{d}{4}} , & \tilde{t},\tilde{t}' \ll 1 \\ 1 , & \tilde{t},\tilde{t}' \gg 1 \end{cases}$$

Persistence of diffusion equation (reminder)

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t) \qquad \phi(x,t) = \int d^d y G(x-y)\phi(y,0)$$

$$\mathbb{E}(\phi(x,0)\phi(x',0)) = \delta^d(x-x') \qquad G(x,t) = (4\pi t)^{-\frac{d}{2}} \exp\left(-x^2/4t\right)$$

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2. Persistence probability $p_0(t, L)$

$$p_0(t,L) \propto L^{-2\theta(d)} h(t/L^2)$$

Probability of no real root for $K_n(x)$

$$\mathcal{C}(\tilde{t},\tilde{t}') \sim \begin{cases} \left(4\frac{\tilde{t}\tilde{t}'}{(\tilde{t}+\tilde{t}')^2}\right)^{\frac{d}{4}} , & \tilde{t},\tilde{t}' \ll 1 \\ 1 , & \tilde{t},\tilde{t}' \gg 1 \end{cases}$$

$P_0(x,n) \equiv$ Proba. that $K_n(x)$ has no real root in $[0,x]$

Scaling form for $P_0(x,n)$

$$P_0(x,n) \propto n^{-\theta(d)} \tilde{h}(n(1-x))$$

$$\tilde{h}(u) \sim \begin{cases} c^{st} , & u \ll 1 \\ u^{\theta(d)} , & u \gg 1 \end{cases}$$

Probability of no real root for $K_n(x)$

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$$P_0(x,n) \propto n^{-\theta(d)} \tilde{h}(n(1-x))$$

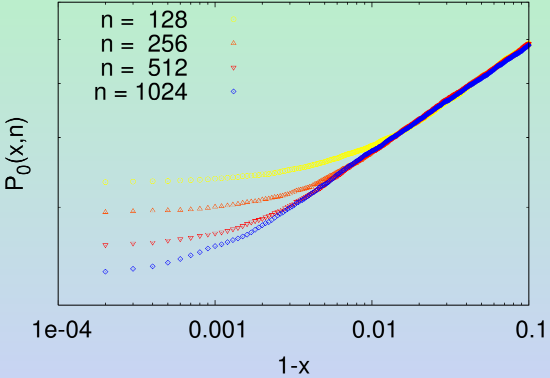
$$\tilde{h}(u) \sim \begin{cases} c^{st} \quad , & u \ll 1 \\ u^{\theta(d)} \quad , & u \gg 1 \end{cases}$$

$q_0(n) \equiv$ Probability that $K_n(x)$ has no real root in $[0,1]$

$$q_0(n) = P(1,n) \sim n^{-\theta(d)}$$

Numerical check of the scaling form

Numerical computation of $P_0(x, n)$ for $d = 2$



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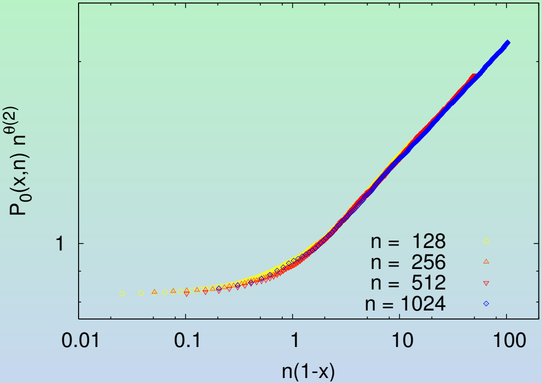

$$P_0(x, n) \propto n^{-\theta(d)} \tilde{h}(n(1 - x))$$

Conclusion

- A condensation phenomenon of the roots on the real axis
- A link between diffusion equation and random polynomials

Proba. of no real root

$$q_0(n) \propto n^{-b(d)}$$

Persistence of diffusion

$$p_0(t, L) \propto L^{-2\theta(d)}$$

$$b(d) = \theta(d)$$

1. Universality see A. Dembo, S. Mukherjee 15
2. Towards exact results for $\theta(d)$, $1/(4\sqrt{3}) \leq \theta(2) \leq 1/4$
   G. Molchan 12 W. Li, Q. M. Shao 02

   see also D. Zaporozhets 06

A heuristic argument

- Diffusion equation

$$\begin{aligned}
\phi(x=0,t) &= (4\pi t)^{-d/2} \int_{0<|\mathbf{x}|<L} d^d\mathbf{x}\ \exp\left(-\frac{\mathbf{x}^2}{4t}\right) \phi(\mathbf{x},0) \\
&= \frac{S_d^{1/2}}{(4\pi t)^{d/2}} \int_0^L dr\ r^{\frac{1}{2}(d-1)} e^{-\frac{r^2}{4t}}\ \Psi(r) \\
\Psi(r) &= S_d^{-1/2} r^{-\frac{1}{2}(d-1)} \lim_{\Delta r \to 0} \frac{1}{\Delta r} \int_{r<|\mathbf{x}|<r+\Delta r} d^d\mathbf{x}\ \phi(\mathbf{x},0) \\
\mathbb{E}(\Psi(r)\Psi(r')) &= \delta(r-r')
\end{aligned}$$

A heuristic argument

- Diffusion equation

$$\phi(x=0,t) \propto \int_0^{L^2} du\, u^{\frac{d-2}{4}} e^{-\frac{u}{t}}\, \tilde{\Psi}(u)$$

$$\mathbb{E}\left(\tilde{\Psi}(u)\tilde{\Psi}(u')\right) = \delta(u-u')$$

A heuristic argument

- Diffusion equation

$$\begin{aligned}\phi(x=0,t) &\propto \int_0^{L^2} du\, u^{\frac{d-2}{4}} e^{-\frac{u}{t}}\, \tilde{\Psi}(u) \\ \mathbb{E}\,(\tilde{\Psi}(u)\tilde{\Psi}(u')) &= \delta(u-u')\end{aligned}$$

- Random polynomials : $K_n(x) = a_0 + \sum_{i=1}^{n} a_i i^{\frac{d-2}{4}} x^i$

$$\begin{aligned}K_n(1-1/t) &\sim a_0 + \sum_{i=1}^{n} i^{\frac{d-2}{4}} e^{-\frac{i}{t}} a_i \\ &\sim \int_0^{n} du\, u^{\frac{d-2}{4}} e^{-\frac{u}{t}}\, a(u) \\ \mathbb{E}\,(a(u)a(u')) &= \delta(u-u')\end{aligned}$$