Local single ring theorem

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Eigenvalues vs Singular values

Horn's question:

Given a generic non-Hermitian square matrix $X \in M_N(\mathbb{C})$, what is the relationship between its eigenvalues and singular values?

EVs: $\lambda_1(X), \lambda_2(X), \dots, \lambda_N(X)$, descending in magnitude

SVs (EVs of $\sqrt{XX^*}$): $s_1(X), s_2(X), \dots, s_N(X)$, descending

Answer: [Weyl's inequalities]

For any $X \in M_N(\mathbb{C})$ and any $1 \leqslant k \leqslant N$

$$\prod_{\ell=1}^{k} |\lambda_\ell(X)| \leqslant \prod_{\ell=1}^{k} s_\ell(X),$$

and equality holds when $k = N$ (both sides are $|\det(X)|$).

A randomized question

Random X with given SVs

$$X = USV^*$$

$S = \text{diag}(s_1, \ldots, s_N)$ is given, U, V independent Haar unitary.

Question Is there any typical behavior of the set of EVs given the set of the SVs, if one selects U and V uniformly?

Distribution of N numbers: the empirical measure

Empirical spectral distribution: For any $X \in M_N(\mathbb{C})$, Borel set $\mathcal{D} \subset \mathbb{C}$,

$$\mu_X = \frac{1}{N} \sum_i \delta_{\lambda_i(X)}, \qquad \text{i.e.} \qquad \mu_X(\mathcal{D}) = \frac{|\{i : \lambda_i(X) \in \mathcal{D}\}|}{N}.$$

Specifically, we are interested in the weak limit of the measure μ_X, whose definition involves free additive convolution.

Stieltjes transform

Definition: For any probab. measure μ on $\mathbb{R}$, its Stieltjes transform $m_\mu(z)$ is

$$m_\mu(z) = \int \frac{1}{\lambda - z} \mathrm{d}\mu(\lambda), \qquad z \in \mathbb{C}^+.$$

Inverse formula: one to one correspondence between measure and its Stieltjes transform: density of μ given by

$$\rho(E) = \frac{1}{\pi} \lim_{\eta \downarrow 0} \mathrm{Im}\, m_\mu(E + \mathrm{i}\eta).$$

Notation: For μ_A and μ_B, we use $\mu_A \boxplus \mu_B$ to denote their free additive convolution, and use $m_{\mu_A}(z)$, $m_{\mu_B}(z)$ and $m_{\mu_A \boxplus \mu_B}(z)$ to denote their Stieltjes transforms.

Free convolution via subordination

Definition via subordination functions [Voiculescu '93, Biane '98, Belinschi-Bercovici '07, Chistyakov-Götze '11]

There exist unique analytic $\omega_A, \omega_B : \mathbb{C}^+ \to \mathbb{C}^+$, s.t. $\mathrm{Im}\,\omega_k(z) \geqslant \mathrm{Im}\,z$ and $\lim_{\eta \uparrow \infty} \frac{\omega_k(\mathrm{i}\eta)}{\mathrm{i}\eta} = 1$ for $k = A, B$, such that

$$m_{\mu_A \boxplus \mu_B}(z) \quad := \quad m_{\mu_A}(\omega_B(z)) = m_{\mu_B}(\omega_A(z)), \qquad (*)$$
$$-[m_{\mu_A}(\omega_B(z))]^{-1} = \omega_A(z) + \omega_B(z) - z.$$

- $\omega_A(z), \omega_B(z)$: subordination functions
- $(*)$: self-consistent equation (SCE)

Another definition R-transform [Voiculescu '86]

Additive model

Let $A = A_N$ and $B = B_N$ be two deterministic Hermitian matrices with ESD μ_A and μ_B. Let U be Haar unitary matrix.

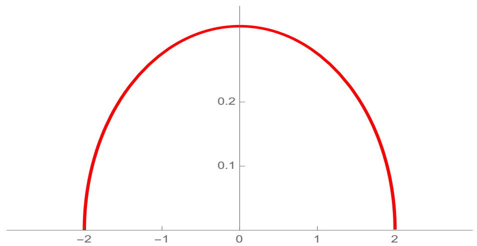
> **Theorem** [Voiculescu '91] Let $H = A + UBU^*$ and $\mu_H := \frac{1}{N}\sum \delta_{\lambda_i(H)}$. Under certain mild conditions, $\mu_H \Rightarrow \mu_A \boxplus \mu_B$ almost surely, as $N \to \infty$.

Other proofs [Speicher'93, Biane'98, Collins'03, Pastur-Vasilchuk'00]

Remark Voiculescu's result identifies the law of the sum of two large Hermitian matrices in a randomly chosen relative basis.

Examples

semicircle ⊞ semicircle



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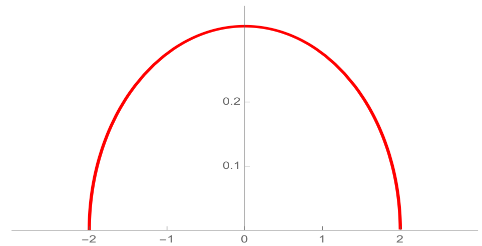

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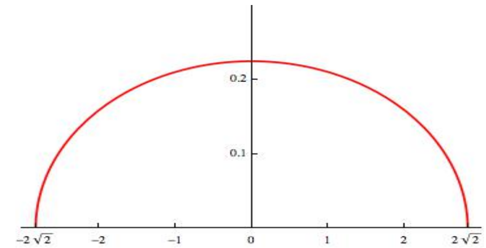
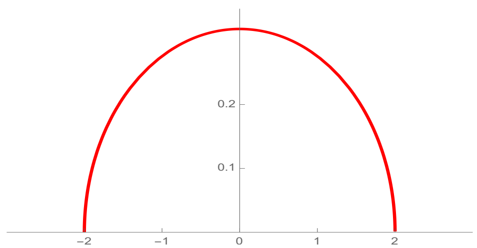


semicircle ⊞ Bernoulli



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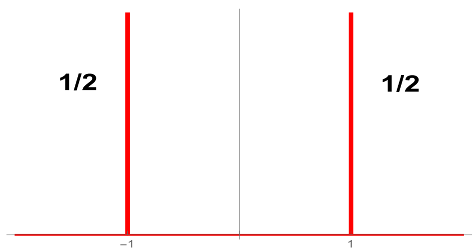

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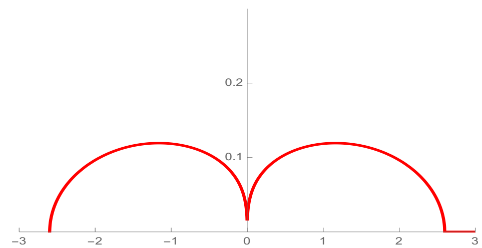
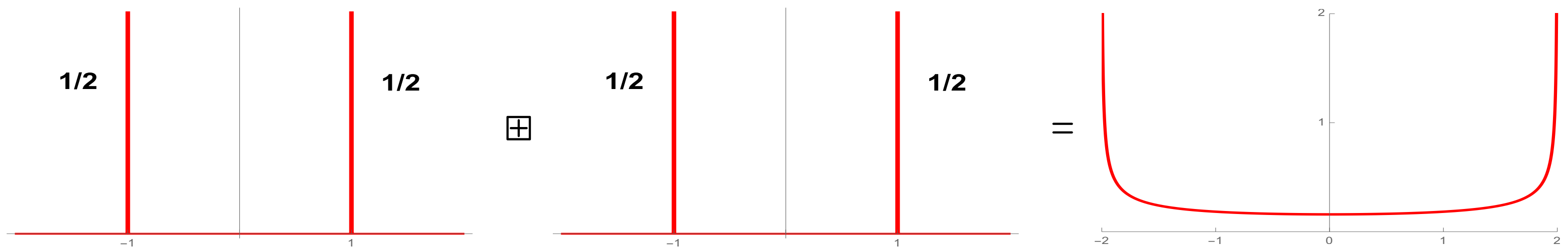
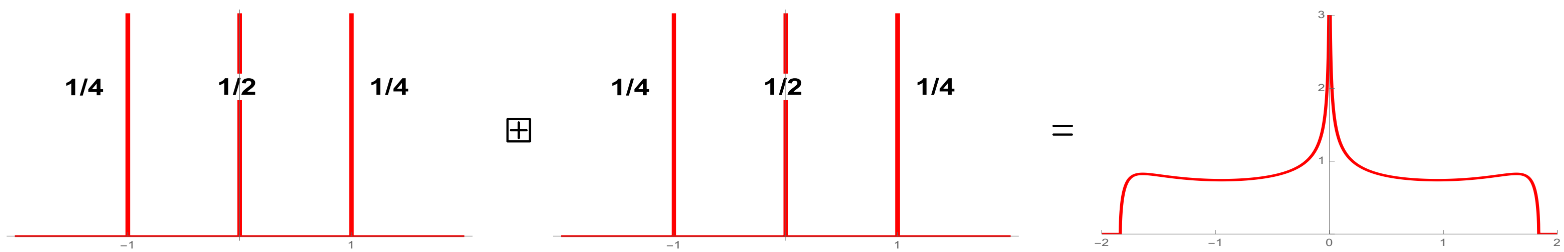


Bernoulli ⊞ Bernoulli



three point masses ⊞ three point masses



Bulk regime where the density is bounded below and above.

Single ring theorem

Random X with given SVs $X = USV^*$, U, V indept Haar

Empirical Singular value distribution $\mu_S := \frac{1}{N}\sum \delta_{s_i(X)} \Rightarrow \mu_\infty$

Brown measure (associated with μ_∞) : denoted by ν_∞ , given by

$$\boxed{\mathrm{d}\nu_\infty(w) = \frac{1}{2\pi}\Delta_w\Big(\int_{\mathbb{R}} \log|u|\mu_{\infty,|w|}(\mathrm{d}u)\Big)\mathrm{d}w \wedge \mathrm{d}\bar{w}, \qquad w \in \mathbb{C}.}$$

Here Δ_w is the Laplacian w.r.t. $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$, and

$$\mu_{\infty,|w|} := \mu_\infty^{\mathrm{sym}} \boxplus \delta_{|w|}^{\mathrm{sym}},$$

where $\mu^{\mathrm{sym}}(I) = (\mu(I) + \mu(-I))/2$.

> **Single ring theorem** [Guionnet-Krishnapur-Zeitouni '11] Under several technical assumptions, $\mu_X = \frac{1}{N}\sum_i \delta_{\lambda_i(X)}$ converges weakly (in probab.) to the Brown measure ν_∞. In addition, the support of ν_∞ is a single ring on $\mathbb{C}$, with the inner radius $r_- := \left[\int x^{-2}\mathrm{d}\mu_\infty(x)\right]^{-\frac{1}{2}}$ and outer radius $r_+ := \left[\int x^2\mathrm{d}\mu_\infty(x)\right]^{\frac{1}{2}}$

Remarks

Remark 1 Single ring theorem was discovered in [Feinberg-Zee '97], for a special class of non-Hermitian matrices, without full rigor.

Remark 2 In [Guionnet-Krishnapur-Zeitouni '11], there are several hard-to-check assumptions. One of them on the smallest singular value of $X - z, z \in \mathbb{C}$ was removed in [Rudelson-Vershynin '14].

Remark 3 The Brown measure ν_∞ was previously analyzed in [Haagerup-Larsen '00]

Non-asymptotic counterpart of ν_∞

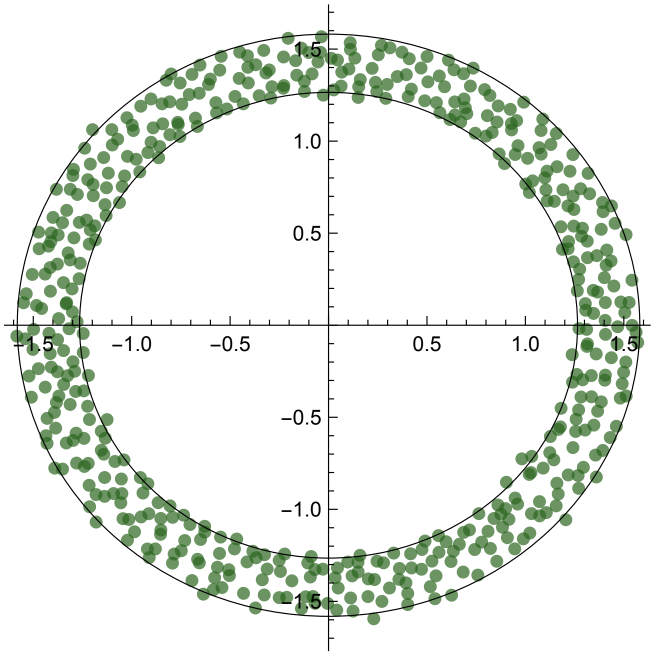
Replacing μ_∞ by μ_S, we define the non-asymptotic counterpart of ν_∞

$$\boxed{\mathrm{d}\nu_S(w) := \frac{1}{2\pi}\Delta_w\Big(\int_{\mathbb{R}} \log|u|\mu_{S,|w|}(\mathrm{d}u)\Big)\mathrm{d}w \wedge \mathrm{d}\bar{w}, \qquad w \in \mathbb{C}}$$

where $\mu_{S,|w|} = \mu_S^{\mathrm{sym}} \boxplus \delta_{|w|}^{\mathrm{sym}}$.

Remark Write $\mu_X - \nu_\infty = (\mu_X - \nu_S) + (\nu_S - \nu_\infty)$. For convergence speed of μ_X, it would be more appropriate to work with $\mu_X - \nu_S$ since $\nu_S \Rightarrow \nu_\infty$ can be arbitrarily slow.

Example



$\mu_S := \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$

inner radius:
$r_- := \left[\int x^{-2} \mathrm{d}\mu_S(x)\right]^{-\frac{1}{2}}$

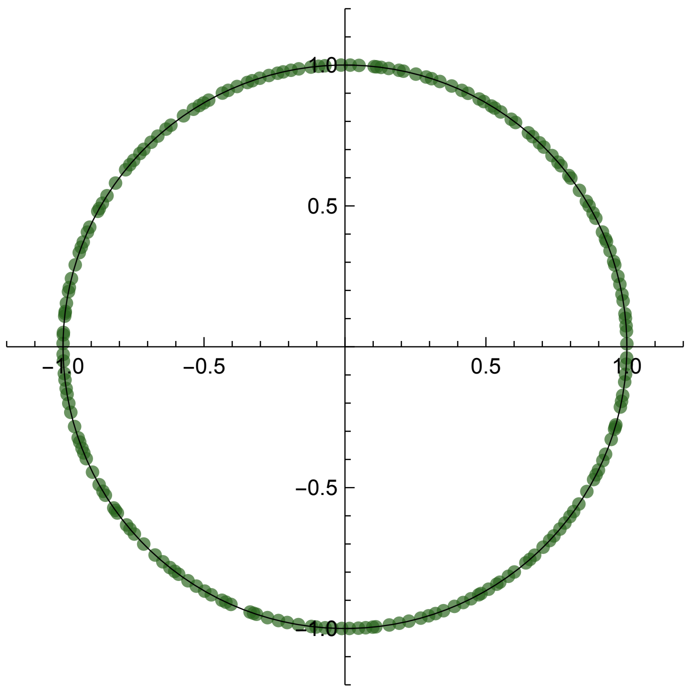
outer radius:
$r_+ := \left[\int x^{2} \mathrm{d}\mu_S(x)\right]^{\frac{1}{2}}$

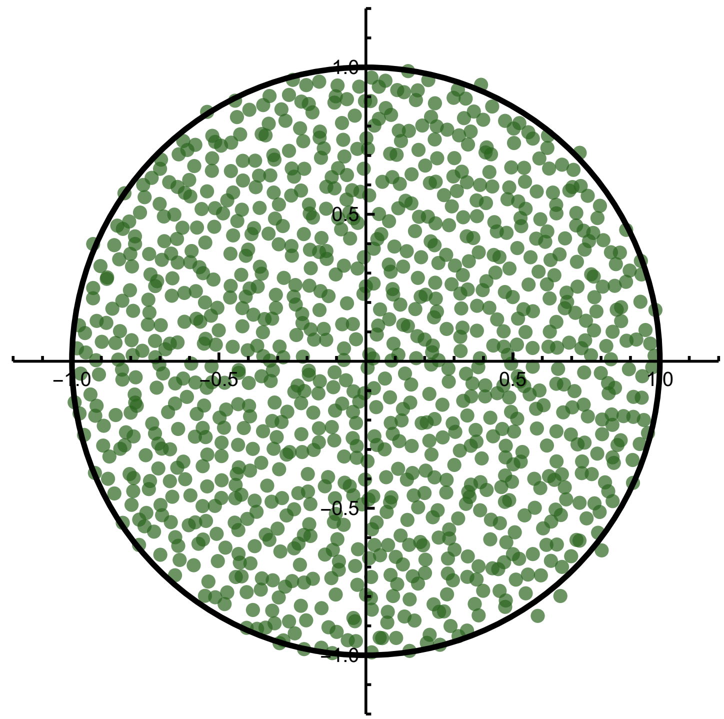
Properties

1: ν_S possesses a radially-symmetric density.

2: The support of ν_S is a single ring.

Special cases





Circular Unitary Ensemble

X: Haar Unitary Matrix

$\mu_S = \delta_1$

Ginibre Ensemble

X: i.i.d. Gaussian matrix

$\mu_S(\mathrm{d}x) \approx \frac{1}{\pi}\sqrt{4 - x^2}\mathbf{1}_{(0,2)}(x)\mathrm{d}x$

Our question: Local law

Global law For any fixed continuity set $\mathcal{D} \subset \mathbb{C}$ of ν_S,

$$\boxed{\frac{\mu_X(\mathcal{D}) - \nu_S(\mathcal{D})}{|\mathcal{D}|} \overset{\mathrm{P}}{\Rightarrow} 0.} \qquad (*)$$

Our question (local law) Does the convergence still hold if $|\mathcal{D}| = o(1)$, and how small can $|\mathcal{D}|$ be? (**Answer** $\frac{1}{N}$)

Remark Global law cannot exclude the existence of big hole or eigenvalue clustering on a scale of $o(1)$, but local law can.

Remark Actually, the LHS of $(*)$ is bounded by $\frac{1}{N|\mathcal{D}|}$ for all $|\mathcal{D}| \gg \frac{1}{N}$, which implies a convergence rate $\frac{1}{N}$.

Local single ring theorem (bulk)

Local single ring theorem (bulk) [B.- Erdős-Schnelli '16] Suppose $\|S\| \sim 1$ and $\mu_S \Rightarrow \mu_\infty$ is not one point mass. Let $|w_0| \in [r_- + \tau, r_+ - \tau]$ for some small $\tau > 0$. Let $f : \mathbb{C} \to \mathbb{R}$ be compactly supported, smooth, s.t $\|f\|_\infty \leqslant C$, $\|f'\|_\infty \leqslant N^C$. Then for $\alpha \in (0, 1/2]$, we have

$$N^{2\alpha}\left|\int_{\mathbb{C}} f(N^\alpha(w - w_0))\big[\mu_X(\mathrm{d}w) - \nu_S(\mathrm{d}w)\big]\right| \prec N^{-1+2\alpha}\|\Delta f\|_{L^1(\mathbb{C})}.$$

Remark If $f(x) \approx \mathbf{1}(x \in \widetilde{\mathcal{D}})$, then $f(N^\alpha(w - w_0)) \approx \mathbf{1}(w \in w_0 + N^{-\alpha}\widetilde{\mathcal{D}}) =: \mathbf{1}(w \in \mathcal{D})$.

Previous work [Benaych-Georges '15] ($|\mathcal{D}| \geqslant (\log N)^{-1/2}$).

Notation $A \prec B$: $|A| \leqslant N^\varepsilon |B|$ with high probability for any given $\varepsilon > 0$.

Related work: Local circular law

Ginibre ensemble can be extended by considering i.i.d. entries (no unitary invariance). Global/local circular laws have been widely studied.

Global [Ginibre '65] (complex Gaussian), [Girko '84] (independent entries, without full rigor), [Bai '97] (i.i.d., bounded density), [Tao-Vu, '10] (i.i.d., second moment).....

Local [Bourgade-Yau-Yin '14], [Yin '14], [Tao-Vu '15](bulk/edge local laws, optimal scale)

Method (of Bourgade-Yau-Yin):

Girko's Hermitization + Local law for Hermitian matrix

Girko's Hermitization

Logarithmic potential

$$\mathcal{P}_\mu(w) := -\int_{\mathbb{C}} \log|\lambda - w|\mu(d\lambda)$$

Example 1

$$\mathcal{P}_{\mu_X}(w) = -\frac{1}{N}\sum_i \log|\lambda_i(X) - w| = -\frac{1}{N}\log\det|X - w|$$

$$= -\frac{1}{2N}\log\det|(X-w)(X-w)^*| =: \boxed{-\frac{1}{2N}\log\det|H_w|}$$

where

$$H_w = \begin{pmatrix} & X - w \\ X^* - w^* & \end{pmatrix}$$

Example 2

$$\mathcal{P}_{\nu_S}(w) = -\frac{1}{2\pi}\int_{\mathbb{C}} \log|\lambda - w|\Delta_\lambda\Big(\int_{\mathbb{R}} \log|u|\mu_{S,|\lambda|}(du)\Big)d\lambda \wedge d\bar{\lambda}$$

$$= \boxed{-\int_{\mathbb{R}} \log|u|\mu_{S,|w|}(du)}$$

Reduction to log determinant

A fact For any smooth and compactly supported $F : \mathbb{C} \to \mathbb{R}$

$$2\pi \int_{\mathbb{C}} F(\lambda)\mu(\mathrm{d}\lambda) = -\int_{\mathbb{C}} \Delta_w F(w) \cdot \mathcal{P}_\mu(w)\mathrm{d}w \wedge \mathrm{d}\bar{w}$$

since $2\pi F(\lambda) = \int_{\mathbb{C}} \Delta_w F(w) \log|w-\lambda| \mathrm{d}w \wedge \mathrm{d}\bar{w}$.

Consequence For (local) single ring theorem, it suffices to estimate

$$|\mathcal{P}_{\mu_X}(w) - \mathcal{P}_{\nu_S}(w)| = \left| \frac{1}{2N} \log\det|H_w| - \int_{\mathbb{R}} \log|u| \mu_{S,|w|}(\mathrm{d}u) \right|.$$

Task For optimal local single ring theorem, one needs

$$\left| \frac{1}{2N} \log\det|H_w| - \int_{\mathbb{R}} \log|u| \mu_{S,|w|}(\mathrm{d}u) \right| \prec \frac{1}{N}$$

Remark For global law, an $o(1)$ bound will be sufficient.

Reduction to Stieltjes transform

ESD of H_w: $\mu_{H_w} = \frac{1}{2N}\sum_{i=1}^{2N}\delta_{\lambda_i(H_w)}$

We can rewrite

$$\left|\frac{1}{2N}\log\det|H_w| - \int_{\mathbb{R}}\log|u|\mu_{S,|w|}(\mathrm{d}u)\right| = \left|\int_{\mathbb{R}}\log|u|\mathrm{d}\big(\mu_{H_w}-\mu_{S,|w|}\big)\right|$$

A basic equation (used in [Tao-Vu '15])

$$\int_{\mathbb{R}}\log|u|\mu(\mathrm{d}u) = \int_{\mathbb{R}}\log|u-\mathrm{i}K|\mu(\mathrm{d}u) - \int_0^K \mathrm{Im}\, m_\mu(\mathrm{i}\eta)\mathrm{d}\eta.$$

Choosing $K = N^L$,

$$\int_{\mathbb{R}}\log|u-\mathrm{i}K|\mathrm{d}\big(\mu_{H_w}-\mu_{S,|w|}\big) \ll \frac{1}{N}.$$

Further task

$$\boxed{\left|\int_0^{N^L}\mathrm{Im}\Big(m_{\mu_{H_w}}(\mathrm{i}\eta) - m_{\mu_{S,|w|}}(\mathrm{i}\eta)\Big)\mathrm{d}\eta\right| \prec \frac{1}{N}}.$$

Local law for Stieltjes transform

Theorem [B.-Erdős-Schnelli '16] : Suppose that $|w| \in [r_- + \tau, r_+ - \tau]$ for some small $\tau > 0$, we have the following uniformly in $\eta > 0$

$$\left|\mathrm{Im}\big(m_{\mu_{H_w}}(\mathrm{i}\eta) - m_{\mu_{S,|w|}}(\mathrm{i}\eta)\big)\right| \prec \frac{1}{N\eta}$$

The above is not sufficient to control the integral over $[0, N^L]$. For the tiny η regime, $\eta \in [0, N^{-L}]$, we need

Theorem [Rudelson-Vershynin'14] There exists positive constants $c > 0$ and $C < \infty$, s.t.

$$\mathbb{P}\Big(\min_i |\lambda_i(H_w)| \leqslant \frac{t}{|w|}\Big) \leqslant \Big(\frac{t}{|w|}\Big)^c N^C.$$

The above provides an upper bound for $\mathrm{Im} m_{\mu_{H_w}}(\mathrm{i}\eta) \leqslant \frac{\eta}{\min_i |\lambda_i(H_w)|^2 + \eta^2}$ when $\eta \to 0$.

We still need an upper bound of $\mathrm{Im} m_{\mu_{S,|w|}}(\mathrm{i}\eta)$ for small η.

0 is in the bulk of $\mu_{S,|w|}$

Theorem [B.-Erdős-Schnelli '16] : Let $J = \big[r_- + \tau, r_+ - \tau\big]$ for any given (small) $\tau > 0$. For $\mu_{S,|w|} := \mu_S^{\text{sym}} \boxplus \delta_{|w|}^{\text{sym}}$, we have

$$\inf_{|w|\in J} \inf_{\eta\in[0,K]} \big|m_{\mu_{S,|w|}}(\mathrm{i}\eta)\big| \geqslant c, \qquad \sup_{|w|\in J} \sup_{\eta\in[0,K]} \big|m_{\mu_{S,|w|}}(\mathrm{i}\eta)\big| \leqslant C.$$

Consequently, 0 is in the bulk of $\mu_{S,|w|}$.

Consequence To derive the bulk local law of the non-Hermitian matrix, one only needs the bulk local law for the Hermitian matrix, since

$$\Big|\int_{N^{-L}}^{N^L} \mathrm{Im}\Big(m_{\mu_{H_w}}(\mathrm{i}\eta) - m_{\mu_{S,|w|}}(\mathrm{i}\eta)\Big)\mathrm{d}\eta\Big| \prec \int_{N^{-L}}^{N^L} \frac{1}{N\eta}\mathrm{d}\eta \prec \frac{1}{N}.$$

Block additive model

Write

$$A = -\begin{pmatrix} & w \\ w^* & \end{pmatrix}, \qquad B := \begin{pmatrix} & S \\ S & \end{pmatrix}, \qquad \mathcal{U} := \begin{pmatrix} U & \\ & V \end{pmatrix}$$

With the above notation

$$H_w = \begin{pmatrix} & X - w \\ X^* - w^* & \end{pmatrix} = \begin{pmatrix} & USV^* - w \\ VSU^* - w^* & \end{pmatrix} = A + \mathcal{U}B\mathcal{U}^*.$$

Observe that $\mu_A = \delta_{|w|}^{\mathsf{sym}}$, $\mu_B = \mu_S^{\mathsf{sym}}$.

Our aim

Prove the local law for block additive model with parameter $z = 0 + \mathrm{i}\eta$.

Two ingredients

- Stability of SCE at $z = 0 + \mathrm{i}\eta$
- Approximate SCE for block additive model

Local stability in the bulk

Recall

$$m_{A\boxplus B}(z) \quad := \quad m_A(\omega_B(z)) = m_B(\omega_A(z)),$$
$$-[m_A(\omega_B(z))]^{-1} = \omega_A(z) + \omega_B(z) - z.$$

Write it as $\Phi_{\mu_A,\mu_B}(\omega_A(z), \omega_B(z), z) = 0$ with

$$\Phi_{\mu_A,\mu_B}(\omega_1, \omega_2, z) := \begin{pmatrix} -[m_A(\omega_2)]^{-1} - \omega_1 - \omega_2 + z \\ -[m_B(\omega_1)]^{-1} - \omega_1 - \omega_2 + z \end{pmatrix}$$

and prove a stability result [B.-Erdős-Schnelli '15], i.e., if

$$\Phi_{\mu_A,\mu_B}(\omega_A^c(z), \omega_B^c(z), z) = \mathbf{r}(z),$$

and $|\omega_{A/B}^c(z) - \omega_{A/B}(z)| \leqslant \delta$, then

$$|\omega_{A/B}^c(z) - \omega_{A/B}(z)| \leqslant C\|\mathbf{r}(z)\|.$$

if $\mathrm{Re} z \in$ bulk of $\mu_A \boxplus \mu_B$ and $\mathrm{Im} z \geqslant 0$.

Approximate SCE for block additive model

Green function: $G(z) := (H_w - z)^{-1}$, note

$$m_{\mu_{H_w}}(z) = \frac{1}{N}\sum \frac{1}{\lambda_i(H_w) - z} = \mathrm{tr}\, G(z) = \frac{1}{N}\sum G_{ii}(z).$$

Approximate subordination functions

$$\omega_A^c(z) := z - \frac{\mathrm{tr} AG(z)}{m_{\mu_{H_w}(z)}}, \qquad \omega_B^c(z) := z - \frac{\mathrm{tr}\mathcal{U}B\mathcal{U}^* G(z)}{m_{\mu_{H_w}(z)}}.$$

From $(A + UBU^* - z)G = I$, we have

$$-[m_{\mu_{H_w}(z)}]^{-1} = \omega_A^c(z) + \omega_B^c(z) - z.$$

Our aim: Show that

$$\left| m_{\mu_{H_w}(z)} - m_A(\omega_B^c(z)) \right| \prec \frac{1}{N\eta}, \qquad \left| m_{\mu_{H_w}(z)} - m_B(\omega_A^c(z)) \right| \prec \frac{1}{N\eta}.$$

Two steps for approximate SCE

Step 1 Green function subordination (entrywise local law)

$$\max_i \left| \Big(G(z) - G_A(\omega_B^c(z)) \Big)_{ii} \right| \prec \frac{1}{\sqrt{N\eta}}$$

Step 2 Fluctuation averaging (average improves the bound)

$$\left| m_{\mu_{H_w}(z)} - m_A\big(\omega_B^c(z)\big) \right| \prec \frac{1}{N\eta}$$

Remark Proof is similar to the local law of additive model [B-Erdős-Schnelli '17], but with new difficulties: no full Haar unitary for block model, uniform control on the parameter w, etc.

We briefly explain the approach with the simpler additive model $A + UBU^*$.

Step 1: Green function subordination

Non-optimal way: Using the full randomness of U at one time

Full expectation $\mathbb{E}[G_{ii}]$ + Gromov-Milman Concentration for $G_{ii} - \mathbb{E}[G_{ii}]$

G.-M.:	$\mathbb{P}(\vert f(U) - \mathbb{E}[f(U)]\vert \leqslant \delta) \geqslant 1 - \exp(-c\frac{N\delta^2}{\mathcal{L}_f^2})$,	$\mathcal{L}_f$: Lip.

E.g. $f(U) = G_{ii}$: $\mathcal{L}_f = 1/\eta^2 \Longrightarrow \delta \gg 1/\sqrt{N\eta^4} \Longrightarrow \eta \gg N^{-\frac{1}{4}}$

Optimal way: Separating some partial randomness $\mathbf{u}_i$ from U

Partial expectation $\mathbb{E}_{\mathbf{u}_i}[G_{ii}]$ + Concentration for $G_{ii} - \mathbb{E}_{\mathbf{u}_i}[G_{ii}]$

Remark: In general, to identify $\mathbb{E}[\cdot]$ is easier than $\mathbb{E}_{\mathbf{u}_i}[\cdot]$, to estimate $(\mathrm{Id} - \mathbb{E})[\cdot]$ is harder than $(\mathrm{Id} - \mathbb{E}_{\mathbf{u}_i})[\cdot]$.

Householder reflection as partial randomness

Proposition [Diaconis-Shahshahani '87] U: Haar on $\mathcal{U}(N)$,

$$U = -\mathrm{e}^{\mathrm{i}\theta_1}(I - \mathbf{r}_1\mathbf{r}_1^*)\begin{pmatrix} 1 & \\ & U_1 \end{pmatrix}, \qquad \mathbf{r}_1 := \sqrt{2}\frac{\mathbf{e}_1 + \mathrm{e}^{-\mathrm{i}\theta_1}\mathbf{u}_1}{\|\mathbf{e}_1 + \mathrm{e}^{-\mathrm{i}\theta_1}\mathbf{u}_1\|_2}$$

$\mathbf{u}_1 \in \mathcal{S}_{\mathbb{C}}^{N-1}$: uniform, $U_1 \in \mathcal{U}(N-1)$: Haar, $\mathbf{u}_1$, U_1 independent.

Remark 1 Analogously, we have independent pair $\mathbf{u}_i$ and U_i for all i. Actually, $-\mathrm{e}^{\mathrm{i}\theta_i}(I - \mathbf{r}_i\mathbf{r}_i^*)$ is the Householder reflection sending $\mathbf{e}_i$ to $\mathbf{u}_i$. Actually, $\mathbf{u}_i$ is the i-th column of U.

Remark 2 Independence between $\mathbf{u}_i$ and U^i enables us to work on the partial expectation $\mathbb{E}_{\mathbf{u}_i}[G_{ii}]$ and the concentration of $G_{ii} - \mathbb{E}_{\mathbf{u}_i}[G_{ii}]$.

Step 2: Fluctuation averaging

We use a method inspired by [Khorunzhy-Khoruzhenko-Pastur '96]. Let $\mathcal{P}_i$ be certain variant of $G_{ii} - (G_A(\omega_B^c))_{ii}$ and let

$$\mathfrak{m}^{(k,\ell)} = \Big(\frac{1}{N}\sum \mathcal{P}_i\Big)^k \Big(\frac{1}{N}\sum \overline{\mathcal{P}}_i\Big)^\ell.$$

Claim: (Recursive moment estimate) For all $k \geqslant 2$, we have

$$\mathbb{E}\Big[\mathfrak{m}^{(k,k)}\Big] = \mathbb{E}\Big[O_\prec(\frac{1}{N\eta})\mathfrak{m}^{(k-1,k)}\Big] + \mathbb{E}\Big[O_\prec(\frac{1}{(N\eta)^2})\mathfrak{m}^{(k-2,k)}\Big] + \mathbb{E}\Big[O_\prec(\frac{1}{(N\eta)^2})\mathfrak{m}^{(k-1,k-1)}\Big].$$

Then using Young or Hölder, we get, for any k,

$$\mathbb{E}[\mathfrak{m}^{(k,k)}] \prec \frac{1}{(N\eta)^{2k}}$$

which will lead to the fluctuation averaging estimate by Markov.

Proof of recursive moment estimate

The proof of the recursive moment estimate again relies on the partial randomness decomposition. Write $\mathbf{u}_i = (u_{ij})$. Roughly, we can write

$$\begin{aligned}\mathbb{E}[\mathfrak{m}^{(k,k)}] &= \frac{1}{N}\sum_{i,j}\mathbb{E}\Big[\bar{u}_{ij}h_{ij}(U,U^*)\mathfrak{m}^{(k-1,k)}\Big] - \mathbb{E}\Big[\mathfrak{cm}^{(k-1,k)}\Big] \\ &= \frac{1}{N}\sum_{i,j}\mathbb{E}\Big[\bar{u}_{ij}h_{ij}(U,U^*)\Big(\frac{1}{N}\sum\mathcal{P}_i\Big)^{k-1}\Big(\frac{1}{N}\sum\overline{\mathcal{P}}_i\Big)^{k}\Big] - \mathbb{E}\Big[\mathfrak{cm}^{(k-1,k)}\Big]\end{aligned}$$

Observe that $u_{ij} \approx N_{\mathbb{C}}(0, \frac{1}{N})$. Using the integration by parts

$$\int_{\mathbb{C}} \bar{g} f(g,\bar{g}) \mathrm{e}^{-\frac{|g|^2}{\sigma^2}} \mathrm{d}g \wedge \mathrm{d}\bar{g} = \sigma^2 \int_{\mathbb{C}} \partial_g f(g,\bar{g}) \mathrm{e}^{-\frac{|g|^2}{\sigma^2}} \mathrm{d}g \wedge \mathrm{d}\bar{g}.$$

Taking derivative w.r.t. u_{ij} for $h_{ij}(U,U^*)$, $\big(\frac{1}{N}\sum\mathcal{P}_i\big)^{k-1}$ and $\big(\frac{1}{N}\sum\overline{\mathcal{P}}_i\big)^{k}$ gives

$$\mathbb{E}[\mathfrak{m}^{(k,k)}] = \mathbb{E}[\delta_1\mathfrak{m}^{(k-1,k)}] + \mathbb{E}[\delta_2\mathfrak{m}^{(k-2,k)}] + \mathbb{E}[\delta_3\mathfrak{m}^{(k-1,k-1)}].$$

Estimating δ_i's gives the answer.

THANK YOU!