On the largest root of random Kac polynomials.

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Kac polynomials

$a_0, \ldots, a_n$ i.i.d. random variables such that $\mathbb{P}(a_0 = 0) = 0$

$$P_n(z) = a_0 + a_1 z + \cdots + a_n z^n = a_n \prod_{k=1}^{n} \left(z - z_k^{(n)}\right)$$

Empirical measure of the zeros

$$\mu_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k^{(n)}} \in \mathcal{M}_1(\mathbb{C}).$$

Limit of the empirical measures

Theorem (Šparo-Šur, Arnold, Ibragimov-Zaporozhets)

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{z_k^{(n)}} \xrightarrow[n\to\infty]{\mathbb{P}} \nu_{\mathbb{S}^1} \Leftrightarrow \mathbb{E}(\log(1+|a_0|)) < \infty$$

where the weak convergence in probability means

$$\forall f \in \mathcal{C}_b^0(\mathbb{C}), \quad \frac{1}{n}\sum_{k=1}^{n} f(z_k^{(n)}) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^{2\pi} f(e^{i\theta})\frac{1}{2\pi}d\theta.$$

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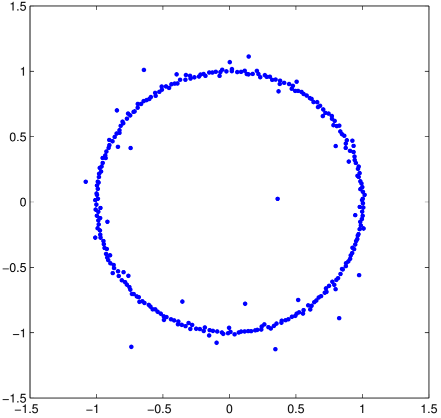
- 1962: Šparo Šur
- 1965: Arnold $\Leftarrow$
- 2013: Ibragimov-Zaporozhets $\Leftrightarrow$

Simulations



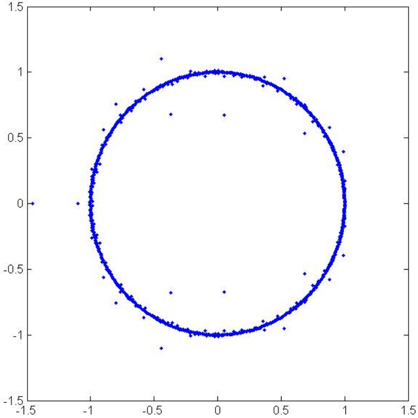


Figure: Coefficients $\mathcal{N}_{\mathbb{C}}(0, 1)$ and $\mathcal{E}(1)$

One question

Considering that

- Universal convergence: $\mu_n \to \nu_{\mathbb{S}^1}$
- When the coefficients are $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables,

$$(z_1, \ldots, z_n) \sim \frac{1}{Z_n} \prod_{i<j} |z_i - z_j|^2 \exp\left(-(n+1)\log \int \prod_{k=1}^{n} |z - z_k|^2 d\nu_{\mathbb{S}^1}(z)\right)$$

- (Universal) Large deviations for the empirical measures

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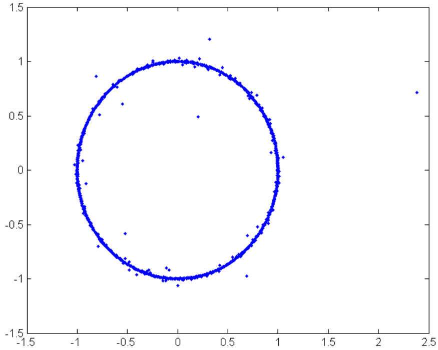
- (Universal) Large deviations for the empirical measures

What can we say on the root of largest modulus?

Can we expect $\max |z_k^{(n)}| \to 1$?
Gumble fluctuations?
Large deviations for $\max |z_k^{(n)}|$?

Can we expect convergence towards 1?



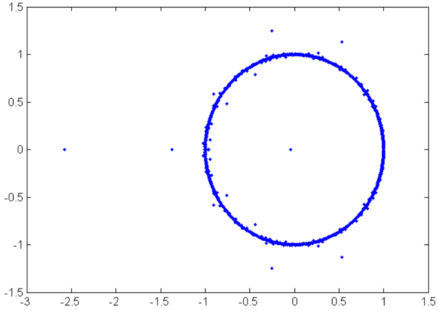


Figure: Convergence seems unlikely

The largest root is heavy tailed.

Theorem (B.)

Assume that $\mathbb{E}(\log(1 + |a_0|)) < \infty$, a_0 *is non deterministic and* $\mathbb{P}(a_0 = 0) = 0$.
Let $n \in \mathbb{N}$, *we define*

$$x_n = \max_{0 \le k \le n} |z_k^{(n)}|.$$

1. *If there exists* $k \ge 0$, $a > 0$ *and* $\delta > 0$ *such that*

$$\forall 0 \le t \le \delta, \quad \mathbb{P}(|a_0| \le t) \ge at^k \quad \textit{then} \quad \mathbb{E}((x_n)^k) = +\infty.$$

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2. *The sequence* $(x_n)_{n \in \mathbb{N}}$ *converges in distribution towards a limiting random variable* x_∞, *supported on* $[1, \infty)$, *satisfying the property* 1.

Limit distribution of the largest root

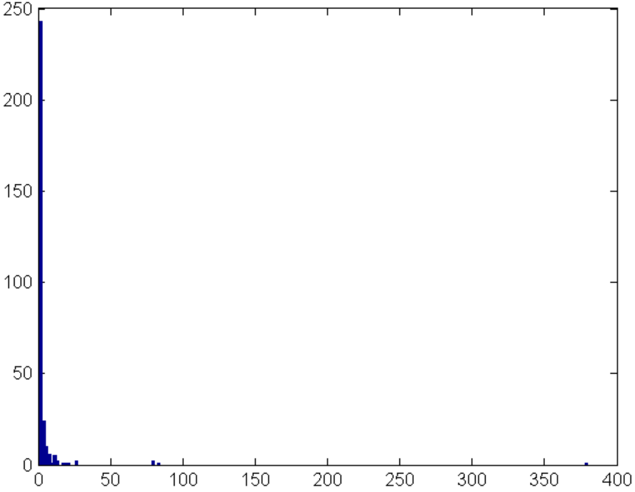
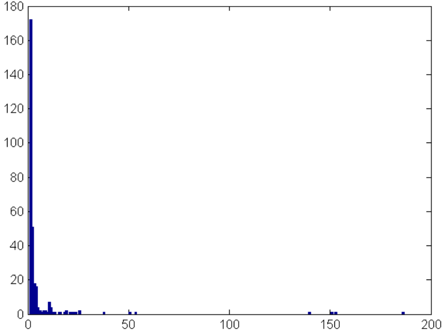


Figure: Histogram for x_{300} with $\mathcal{N}_{\mathbb{R}}(0, 1)$ coefficients.

Comparision with Cauchy random variables.



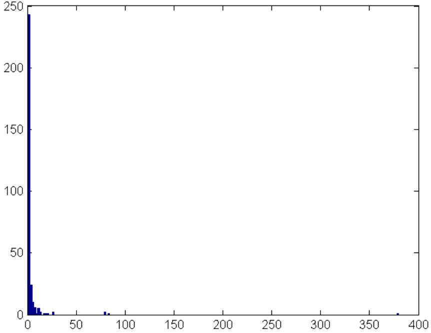


Figure: On the left, Cauchy variables, on the right x_{300} for $\mathcal{N}_{\mathbb{R}}(0,1)$ coefficients.

Link with the smallest root

The behavior of x_n is understood through the lowest modulus among the roots.

$$w_n = \min_{0 \le k \le n} |z_k^{(n)}|$$

Lemma

1. *For any* $n \in \mathbb{N}$, w_n *has the same distribution as* $1/x_n$.
2. *There exist constants* $\delta > 0$, $A > 0$, $r > 0$ *such that*

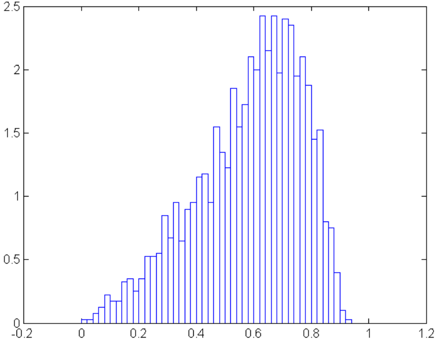
$$\forall 0 \le t \le \delta, \quad \mathbb{P}(w_n \le t) \ge A\mathbb{P}(|a_0| \le \frac{t}{r})$$

3. $(w_n)_{n \in \mathbb{N}}$ *converges almost surely towards* w_∞ *which is the smallest zero of the random analytic function*

$$P_\infty(z) = \sum_{k=0}^{\infty} a_k z^k$$

The result of Peres and Virág on GAF allows us to compute the distribution of w_∞

$$F_{w_\infty}(t) = \mathbb{P}(w_\infty \leq t) = 1 - \prod_{k=1}^{\infty}(1 - t^{2k}).$$



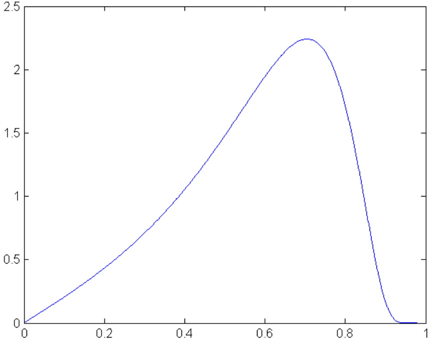


Figure: Histogram of w_{300} and density of w_∞.

Ideas of proof: the theorem

Proof of the theorem: If X is a positive random variable, then

$$\forall k \in \mathbb{N}^* \quad \frac{1}{k}\mathbb{E}(X^k) = \int_0^{\infty} t^{k-1}\mathbb{P}(X \geq t)dt$$

Apply this to x_n along with

$$\forall t \geq \frac{1}{\delta}, \quad \mathbb{P}(x_n \geq t) = \mathbb{P}(w_n \leq \frac{1}{t}) \geq A\mathbb{P}(|a_0| \leq \frac{r}{t}) \geq B\frac{1}{t^k}$$

Where we used points 1 and 2 of the lemma.

Ideas of proof: lemma

Point 1 comes from

$$z^n P_n(1/z) = a_n + a_{n-1}z + \cdots + a_0 z^n = P_n(z) \quad \text{in distribution}$$

Ideas of proof: lemma

Point 1 comes from

$$z^n P_n(1/z) = a_n + a_{n-1}z + \cdots + a_0 z^n = P_n(z) \quad \text{in distribution}$$

Point 2 means:
If $|a_0|$ is sufficiently small, P_n has a small zero.

Theorem (Weak version of Rouché)

Let f and g be two holomorphic functions on a disk D, then if

$$\forall z \in \partial D \quad |f(z) - g(z)| < |g(z)|$$

then f and g have the same number of zeros inside D.

When $|a_0|$ is small, P_n and P_1 have the same number of zeros in a small disk.

We use Rouché 's theorem with P_n and P_1, hence

$$\begin{aligned}\mathbb{P}(w_n \leq t) &\geq \mathbb{P}(w_n \leq t \text{ and } w_1 \leq t)\\ &\geq \mathbb{P}(P_n \text{ and } P_1 \text{ have a root in } D(0,t))\\ &\geq \mathbb{P}(\sup|P_n - P_1| < inf|P_1| \text{ and } w_1 \leq t)\\ &\geq \mathbb{P}(\sup|\sum_{k=2}^{n} a_k z^k| < \inf|a_0 + a_1 z| \text{ and } \frac{|a_0|}{|a_1|} \leq t)\end{aligned}$$

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&\ge \mathbb{P}(\sup |\sum_{k=2}^{n} a_k z^k| < \inf |a_0 + a_1 z| \text{ and } \frac{|a_0|}{|a_1|} \le t)
\end{aligned}$$

Point 3 is just Hurwitz's theorem, which is a consequence of Rouché's theorem.

For many examples, when $a_0 \in \mathbb{R}$, $\mathbb{E}(x_n) = +\infty$ while when $a_0 \in \mathbb{C}$, we only have $\mathbb{E}(x_n^2) = +\infty$.

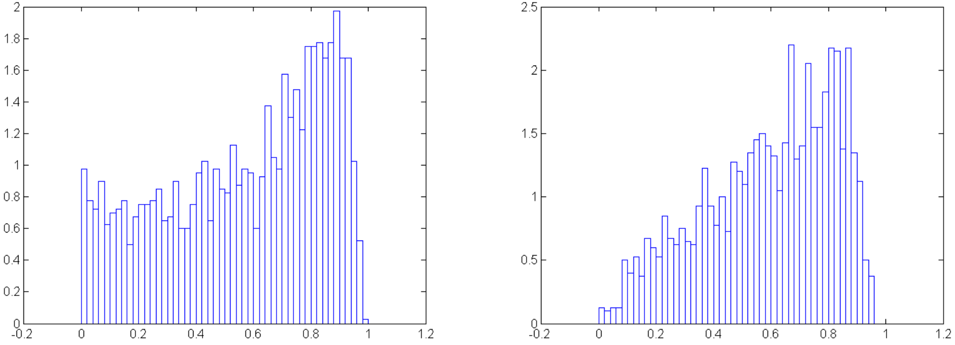

Figure: Histogram of w_{300} for coefficients $\mathcal{E}(1)$ et $\frac{1}{Z}e^{-|z|}d\ell_{\mathbb{C}}$

Perspectives

What can we do next?

1. Similar result for other polynomial models.
   The same limit distribution is expected in the Gaussian case when we have rotational symmetry.
2. Similar result for some Coulomb gases in dimension 2 (soon!)

Thank you for your attention!

ご清聴ありがとうございました