Localization of the continuous Anderson Hamiltonian in 1-d

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Random matrices and applications, Kyoto, May 2018

Works in collaboration with Romain Allez and Cyril Labbé

Continuous Anderson Hamiltonian

Continuous Anderson Hamiltonian :

$$\mathcal{H}_L \; : \; u \in L^2([0,L]) \mapsto -u'' + \xi \cdot u.$$

ξ : white noise.

Dirichlet boundary conditions: $u(0) = u(L) = 0$ or Neumann $u'(0) = u'(L) = 0$.

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$\mathcal{H}_L$: self adjoint operator on $L^2([0,L])$, pure point spectrum $\lambda_1 < \lambda_2 < \cdots$, associated eigenvectors $(\varphi_k)_k$ form an orthonormal basis of $L^2([0,L])$ and are Hölder $3/2-$.

Goals

Study the **spectrum** of this operator when $L \to \infty$:

- Eigenvalue distribution. Microscopic level: repulsion?
- Eigenvectors: localized? Shape?

Laplacian

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Eigenvalues $\lambda_1 < \lambda_2 < \cdots$ satisfy:

$$\lambda_k = (\pi k/L)^2.$$

And the associated eigenvectors are:

$$x \in [0, L] \mapsto \sin(\frac{\pi k}{L} x).$$

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Eigenvectors are completely **delocalized**!

Density of states

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Eigenvalues of the Laplacian:

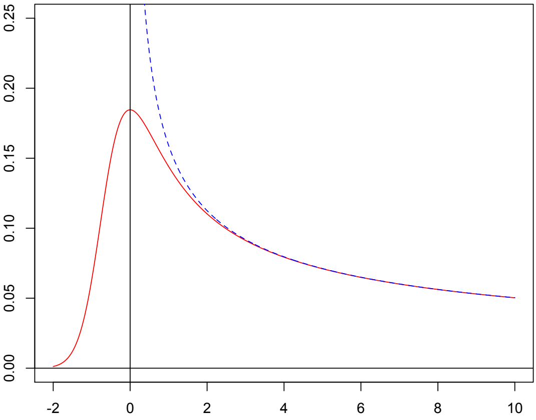
$$\lambda_k = (\pi k/L)^2.$$

$\to$ Density of states:

$$x \in \mathbb{R}_+ \mapsto \frac{1}{2\pi\sqrt{x}}.$$

Density of states for $\mathcal{H}_L$

Frisch and Lloyd ('60), Halperin ('65) and then Fukushima, Nakao ('77): Explicit formula for the density of states of $\mathcal{H}_L$:



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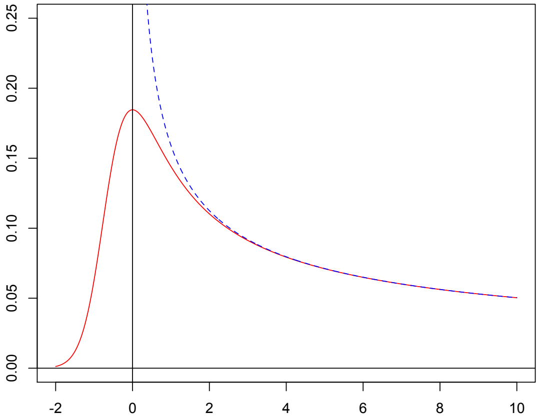

$\rightarrow$ We are interested in the localization **at the edge**.

Motivations and links with literature

Brief historical review

McKean ('94) : Convergence of the **smallest eigenvalue** λ_1 (recentred and rescaled) for Dirichlet, Neumann and periodic b.c.:

$$-4\sqrt{a_L}\,(\lambda_1 + a_L) \Rightarrow_{L\to\infty} e^{-e^{-x}}dx,$$

where

$$a_L = \left(\frac{3}{8}\ln L\right)^{2/3} + 3^{-1/3}2^{-1}(\ln L)^{-1/3}\left(\frac{1}{3}\ln\ln L + \ln\frac{3^{1/3}}{2\pi} + o(1)\right).$$

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Cambronero, McKean ('99) : Integral expression for the distribution of λ_1 when L **is fixed**, with periodic b.c..

Cambronero, Rider, Ramírez ('06) : Precise asymptotical behavior of the left tail of λ_1 when$L$ **is fixed** with periodic b.c.

The parabolic Anderson model (PAM)

PAM:

$$\begin{aligned}\partial_t u(t,x) &= \partial_x^2 u(t,x) + \xi(x) u(t,x), \quad t > 0,\ x \in [0,L] \\ &= -\mathcal{H}_L\, u(t,x)\end{aligned}$$

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Solution: with eigenvalues and eigenvectors of $\mathcal{H}_L$:

$$\begin{aligned}u(t,x) &= \sum_{k=1}^{\infty} e^{-\lambda_k t}\, \varphi_k(x) < \varphi_k, u(0,\cdot) >_{L_2} \\ &\simeq e^{-\lambda_1 t}\, \varphi_1(x) < \varphi_1, u(0,\cdot) >_{L_2} \quad \text{si } t \gg 1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{et} < \varphi_1, u(0,\cdot) >_{L_2} \neq 0\end{aligned}$$

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$\rightarrow$ Large time limit of the solution can be understood thanks to the study of the smallest eigenvectors of $\mathcal{H}_L$.

Discrete Schrödinger operator

$$\begin{pmatrix} \sigma\, g_1 & 1 & & \\ 1 & \sigma\, g_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & \sigma\, g_N \end{pmatrix}$$

where $g_k \sim \mathcal{N}(0, 1)$ i.i.d.

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- $\sigma \gg 1/\sqrt{N}$: localized, $\sigma \ll 1/\sqrt{N}$: deterministic.
- $\sigma \sim 1/\sqrt{N}$: transition between the two regimes.

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Heuristically: study of the largest eigenvalues of $\mathcal{H}_L$.

Random matrices: β-ensembles

Other discrete Schrödinger model (tridiagonal):

$$H_N^\beta := \frac{1}{\sqrt{\beta}} \begin{pmatrix} \sqrt{2}\,g_1 & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & \sqrt{2}\,g_2 & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \sqrt{2}\,g_{N-1} & \chi_\beta \\ & & & \chi_\beta & \sqrt{2}\,g_N \end{pmatrix}$$

g_k: indep $\mathcal{N}(0,1)$, $\chi_{k\beta}$: indep χ distribution.

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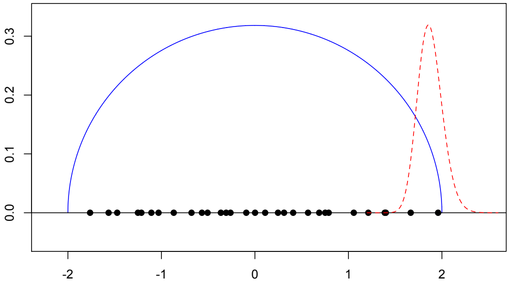

Edge: Airy Hamiltonian

Airy Hamiltonian:

$$\mathcal{A}_\beta : u \in L^2(\mathbb{R}_+) \mapsto -\partial_x^2 u + (x + \frac{2}{\sqrt{\beta}}\,\xi)u.$$

where ξ white noise.

Ramírez, Rider, Virág ('06): Edge limit.

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Ramírez, Rider, Virág ('06): Edge limit.

Allez, D. ('13): Convergence of the smallest eigenvalue TW_β when $\beta \to 0$ towards a Gumbel distribution.

Bulk: Sine_β

Valkó and Virág ('16): **Sine**$_\beta$ operator. Self adjoint random **Dirac operator**.

Spectrum = Sine_β process characterized by Brownian carousel and a family of coupled SDEs.

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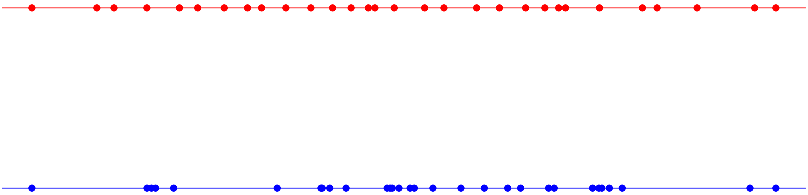
Allez, D. ('14): Convergence of Sine_β when $\beta \to 0$ to a Poisson point process. Sine_β : transition between Wigner ($\beta = 2$) and Poisson.



Our results on the Hamiltonian of Anderson

Localization of the smallest eigenvectors

Notons

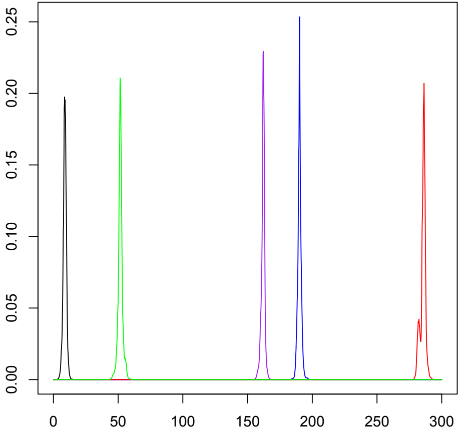
$$\mathcal{Q}_L := \sum_{k\geq 1} \delta_{4\sqrt{a_L}(\lambda_k + a_L)},$$
$$m_L(dt) := (L\,\varphi_k(L\,t)^2 dt)_{k\geq 1}.$$

Théorème (D., Labbé ('17))

$(\mathcal{Q}_L, m_{L,k}(dt))$ converges in distribution towards $(\mathcal{Q}_\infty, m_\infty)$ where:

- *$\mathcal{Q}_\infty$: Poisson point process of intensity $e^x\,dx$,*
- *$m_\infty = (\delta_{U_k})_{k\geq 1}$: $(U_k)_{k\geq 1}$ i.i.d, uniform on $[0,1]$, independent of $\mathcal{Q}_\infty$.*

Simulation of the first eigenvectors



The first 5 eigenvectors φ_k^2 in order: black, blue, purple, red, green.

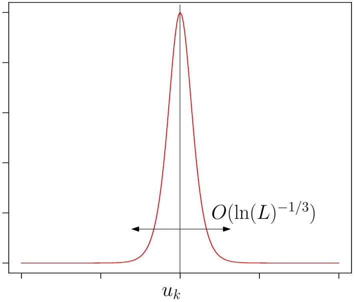
Shape of the eigenvectors

Théorème (D., Labbé ('17))

Let u_k be the point where φ_k reaches its maximum.

$$h_k(t) := \sqrt{a_L}\,\varphi_k^2(u_k + \sqrt{a_L}\,t)$$

converges towards $h(t) := 1/\cosh(t)^2$ *uniformly over compact subsets of* $\mathbb{R}$.



Eigenvalue equation

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For all $\lambda \in \mathbb{R}$, there is an **unique solution** u_λ (up to a scaling).

The couple (λ, u_λ) is an eigenvalue/eigenvector when

$$u_\lambda(L) = 0.$$

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Riccati transform: transforms a linear equation of the second order into a non linear one of the first order:

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$$X := \frac{u'_\lambda}{u_\lambda}$$

satisfies the SDE:

$$\begin{aligned} &dX(t) = (-\lambda - X^2(t))dt + dB(t) \\ &X(0) = +\infty, \end{aligned}$$

where $B(t) = \int \xi(s) 1_{[0,t]}(s) dt$ is a Brownian motion.

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λ is an **eigenvalue** when $X(L) = -\infty$.

Sturm-Liouville

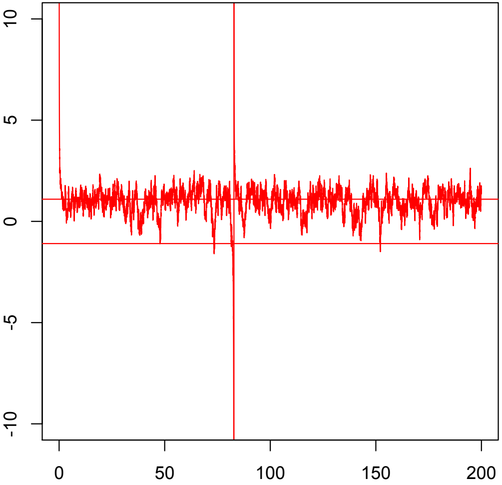
$$\begin{aligned} dX(t) &= (-\lambda - X^2(t))dt + dB(t) \\ X(0) &= +\infty. \end{aligned}$$

We have:

$$\# \{\text{eigenvalues } \leq \lambda\} = \# \{\text{blows up to } -\infty \text{ of } X_\lambda \text{ on } [0, L]\}.$$

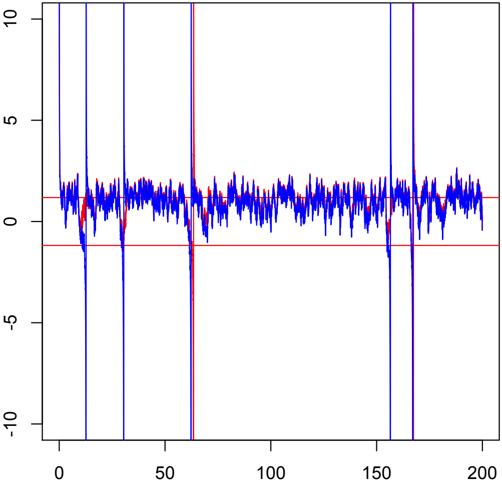
Sturm-Liouville

Simulation of the diffusion X_λ for $\lambda = -1.4$. $\to$ 1 explosions so we have 1 eigenvalue ≤ -1.4.



Sturm-Liouville

Simulation of the diffusion X_λ for $\lambda = -1.2$ (blue) and $\lambda = -1.4$ (red). $\rightarrow$ 5 explosions so 5 eigenvalues ≤ -1.2.



Eigenstate when $L \to \infty$

$$P_L(\lambda) = \# \{\text{explosions of } X_\lambda\} = \# \{\text{eigenvalues of } \mathcal{H}_L \leq \lambda\}.$$

$$P_L(\lambda) \sim_{L\to\infty} \frac{L}{\mathbb{E}_{+\infty}[\text{First explosion time of } X]}.$$

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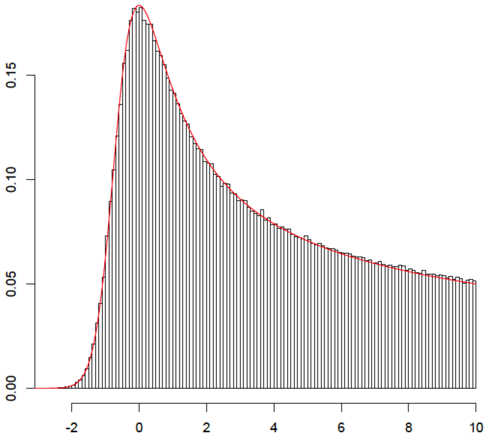
Eigenstate density $\lambda \mapsto \lim_{L\to\infty} \frac{1}{L}\frac{dP_L(\lambda)}{d\lambda}$.



Study of the diffusion

Let $a = -\lambda$.

Coupled SDEs $(X_a)_{a\in\mathbb{R}}$:

$$dX_a(t) = (a - X_a(t)^2)dt + dB(t).$$
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Barrier: $\Delta V = \frac{4}{3}\, a^{3/2}$.

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Explosion time denoted $(\zeta_1, \zeta_2, \cdots)$.

Trap the diffusion in a stationary well

Let

$$m(a) := \mathbb{E}_{+\infty}[\zeta_1]$$

We have

$$\begin{aligned} m(a) &= \sqrt{2\pi}\int_0^{\infty} \frac{dv}{\sqrt{v}} \exp\left(2av - \frac{1}{6}v^3\right) \\ &\sim_{a\to\infty} \frac{\pi}{a^{1/2}} \exp\left(\frac{8}{3}a^{3/2}\right). \end{aligned}$$

Proposition (First explosion of X_a)

$$\frac{\zeta_1}{m(a)} \Rightarrow_{a\to\infty} \mathcal{E}(1) \quad \textit{exponential of parameter } 1.$$

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Proof: Convergence of the Laplace transform (EDP).

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Corollary: $\sum_{k\in\mathbb{N}} \delta_{\zeta_k}$ converges to a Poisson point process of intensity 1.

Other corollary:
$\#\,\{\text{rescaled and recentred eigenvalues} \leq \lambda\}$ converges to a Poisson distribution $\exp(\lambda)$.

Typical behavior of the diffusion X_a when $a \gg 1$

1. X_a goes down from $+\infty$ to $\sqrt{a}$ deterministically in a short time $O(\ln(a)/\sqrt{a})$.

2. X_a spends time of order $m(a) \approx \exp(\frac{8}{3}a^{3/2})$ around $\sqrt{a}$.
   $\rightarrow$ Many short excursions out of the bottom of the well.

3. X_a crosses $[-\sqrt{a}, \sqrt{a}]$ then blows up to $-\infty$ in a time of order $O(\ln(a)/\sqrt{a})$.

4. X_a restarts from $+\infty$ right after its explosion.

Convergence to a Poisson point process

We want to show that

$$(\mathcal{Q}_L([\lambda_1, \lambda_2]), \cdots, \mathcal{Q}_L([\lambda_{n-1}, \lambda_n]))$$

converges towards n independent random variables variables with Poisson distributions of parameter $e^{\lambda_2} - e^{\lambda_1}$, $\cdots$, $e^{\lambda_n} - e^{\lambda_{n-1}}$.

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i.e.

$$(\#X_{a_L - \frac{\lambda_2}{4\sqrt{a_L}}} - \#X_{a_L - \frac{\lambda_1}{4\sqrt{a_L}}}, \cdots, \#X_{a_L - \frac{\lambda_n}{4\sqrt{a_L}}} - \#X_{a_L - \frac{\lambda_{n-1}}{4\sqrt{a_L}}})$$

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Main ideas

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  Next explosion time after $s > 0$ does not depend on the exact position $X_a(s)$ when $X_a(s)$ is in the well (memory loss).

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  Next explosion time after $s > 0$ does not depend on the exact position $X_a(s)$ when $X_a(s)$ is in the well (memory loss).
- **Coupling:** If $a' > a$ and $X_{a'}(s) \geq X_a(s)$ then $X_{a'}(t) \geq X_a(t)$ for all $t \geq s$ **before the next explosion**.

First eigenvector: back to Riccati transform

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We have seen that the **explosions** of the diffusions $(X_a)_{a\in\mathbb{R}}$ characterize the law of the eigenvalues.

If $a_1 := -\lambda_1$, X_{a_1} is the Riccati transform of the first eigenvector and $X_{a_1}(L) = -\infty$.

We find back the first eigenvector φ_1 performing the inverse of the Riccati transform:

$$\varphi_1(t) = \varphi_1(s)\exp\left(\int_s^t X_{a_1}(u)du\right)$$

First eigenvector: back to Riccati transform

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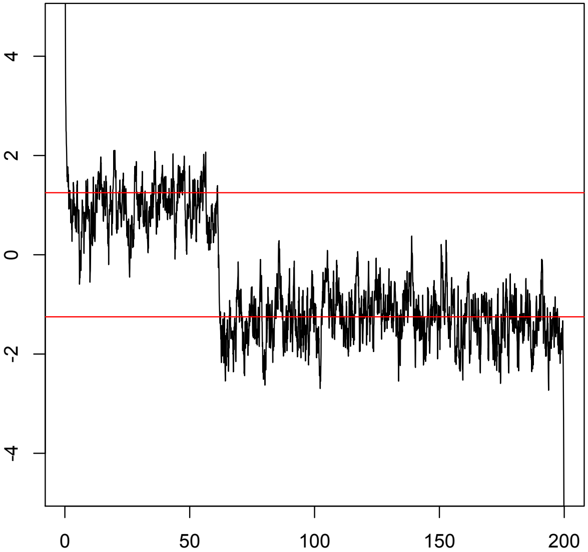
Strategy: We bound X_{a_1} with typical diffusions.

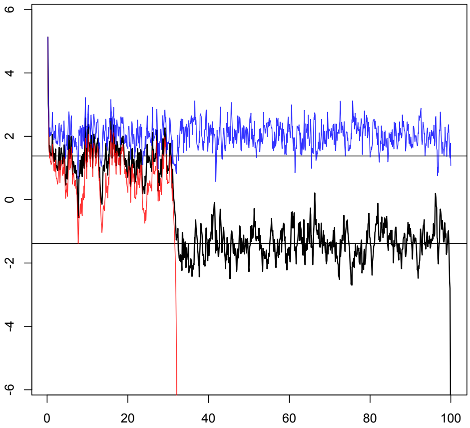
Riccati transform of the first eigenvector



First eigenvector

We have for $a \leq a_1 \leq a'$

$$X_a(t) \leq X_{a_1}(t) \quad \text{for all } t \leq \zeta_1(X_a)$$
$$X_{a_1}(t) \leq X_{a'}(t) \quad \text{for all } t \leq \zeta_1(X_{a_1}) = L.$$



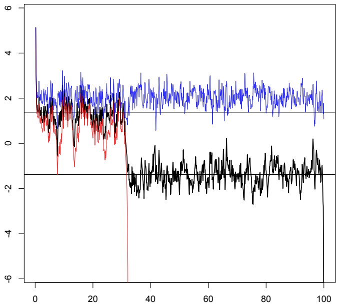
First eigenvector

We have $X_a \approx X_{a'} \approx \sqrt{a}$ most of the time.

$\rightarrow$ we control φ_1 until the **first explosion of** X_a :

$$\varphi_1(t) \approx \varphi_1(t_0)\exp(\sqrt{a}(t - t_0)), \quad \text{if } t \leq \zeta_1(X_a)$$

where $a \approx a_1 \approx a_L$.



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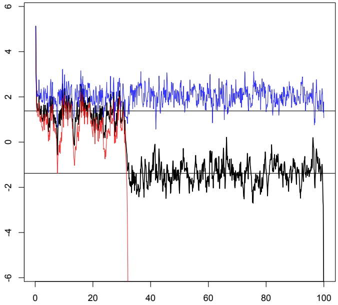

How can we control the eigenvector AFTER this explosion time?

Time reversal

Look at the problem:

$$-\hat{u}'' + \xi \cdot \hat{u} = \lambda \hat{u}$$
$$\hat{u}(L) = 0$$

λ eigenvalue iff $\hat{u}(0) = 0$.

Time reversal

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$$\hat{u}(L) = 0$$

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Riccati transform:

$$d\hat{X}_a(t) = (-a + \hat{X}_a(t)^2)dt + d\hat{B}(t)$$

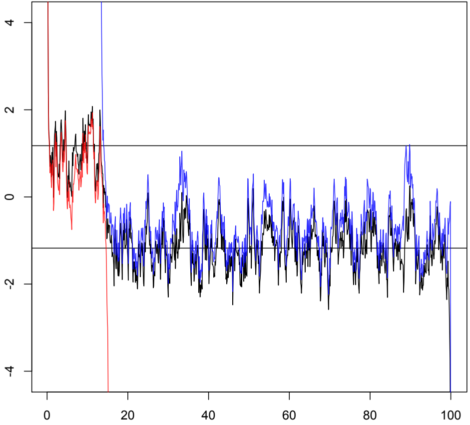
where $\hat{B}(\cdot) = B(L - \cdot)$.

First eigenvector

For all $a \leq a_1$

$$X_a(t) \leq X_{a_1}(t) \quad \text{for all } t \leq \zeta_1(X_a)$$

$$X_{a_1}(t) \leq \hat{X}_a(L-t) \quad \text{for all } t \geq L - \zeta_1(\hat{X}_a).$$



Time reversal

$\rightarrow$ We control φ_1 between the first explosion of $\hat{X}_a$ and time L :

$$\varphi_1(t) \approx \varphi_1(t_1) \exp(-\sqrt{a}(t - t_1)), \quad \text{si } t \geq t_1 \geq L - \zeta_1(\hat{X}_a)$$

Time reversal

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$$\varphi_1(t) \approx \varphi_1(t_1) \exp(-\sqrt{a}(t - t_1)), \quad \text{si } t \geq t_1 \geq L - \zeta_1(\hat{X}_a)$$

It suffices to prove that the first explosion of $\hat{X}_a(L - \cdot)$ is close to the first explosion time of X_a to deduce the convergence towards a Dirac mass.

Shape of the first eigenvector

Shape of X_a when it crosses the interval $[-\sqrt{a}, \sqrt{a}]$?

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Girsanov : Diffusion X_a conditioned to explode right away follows the diffusion:

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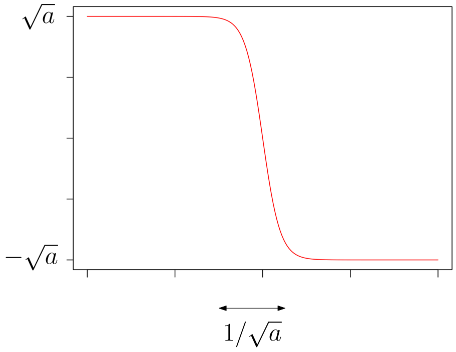
With large probability, this diffusion is close to $f(t) := -\sqrt{a}\tanh(\sqrt{a}\,(t - A))$ where $f(0) = \sqrt{a} - \delta$.



THANK YOU!