The Stochastic Semigroup Approach to the Edge of β-ensembles

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Based on work by V. Gorin and M. Shkolnikov, and joint work with M. Shkolnikov.

Problem: Edge Fluctuations of β-ensembles

Given $\beta > 0$, let $\lambda_1^\beta \geq \lambda_2^\beta \geq \cdots \geq \lambda_N^\beta$ be sampled from

$$\frac{1}{\mathcal{Z}_\beta} \cdot \prod_{i<j} (x_j - x_i)^\beta \cdot \exp\left(-\frac{\beta}{4} \sum_{i=1}^N x_i^2\right), \quad x_1 \geq \cdots \geq x_N.$$

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Problem

Given $k \in \mathbb{N}$, understand the fluctuations of $(\lambda_1^\beta, \ldots, \lambda_k^\beta)$ as $N \to \infty$.

Operator Limit

Define the stochastic Airy opterator (SAO) with parameter $\beta > 0$ as

$$[\mathcal{H}^\beta f](x) := -f''(x) + xf(x) + \frac{2}{\sqrt{\beta}} W_x' f(x), \qquad f : \mathbb{R}_+ \to \mathbb{R}, f(0) = 0,$$

where $(W_x)_{x \geq 0}$ is a Brownian motion.

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Theorem (Dumitriu-Edelman (2002); Edelman-Sutton (2007); Ramírez-Rider-Virág (2011))

Let $\Lambda_1^\beta \leq \Lambda_2^\beta \leq \cdots$ be the eigenvalues of $\mathcal{H}^\beta$. For every $k \in \mathbb{N}$ fixed,

$$N^{1/6}(2\sqrt{N} - \lambda_i^\beta)_{1 \leq i \leq k} \Rightarrow (\Lambda_i^\beta)_{1 \leq i \leq k}.$$

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Advantages of operator limit approach.

1. Unified method (i.e., for all $\beta > 0$) of studying β-ensembles.
2. Study limiting fluctuations through functional analysis, as they arise as the spectrum of a differential operator.

Stochastic Semigroup Approach

Idea. Study the asymptotic extreme value fluctuations of Gaussian β-ensembles through the semigroups generated by the SAOs:

$$\mathcal{U}_T^\beta := \mathrm{e}^{-T\cdot\mathcal{H}^\beta/2}, \qquad T \geq 0.$$

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Theorem

Let $(e_t)_{t\in[0,1]}$ be a Brownian excursion, and let $(\ell^a)_{a\geq 0}$ be its local time process on $[0,1]$.

$$\int_0^1 e_t \,\mathrm{d}t - \frac{1}{2}\int_0^\infty (\ell^a)^2 \,\mathrm{d}a \sim N(0,1/12)$$