NEW RESULTS on SQUARED BESSEL PARTICLE SYSTEMS and WISHART PROCESSES

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Random matrices and their applications
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Articles

P. Graczyk, J. Małecki
On Squared Bessel particle systems,
to appear in Bernoulli, 2018

K. Bogus, P. Graczyk, J. Małecki
Properties of β–Squared Bessel particle systems,
preprint, 2018

P. Graczyk, J. Małecki, E. Mayerhofer
A Characterization of Wishart Processes and Wishart Distributions,
Stoch.Proc. Appl., 128 (2018), 1347–1385.

P. Graczyk, J. Małecki,
Strong solutions of non-colliding particle systems.
Electron. J. Probab. 19 (2014), pp. 1–21.

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We prove complete results on the existence, unicity and positivity properties of the solutions of the BESQ particle system.

Squared Bessel particle systems

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Göing-Jaeschke, Yor [Bernoulli 2003]: $\alpha \in \mathbb{R}$, $x \in \mathbb{R}$, $X \in \mathbb{R}$

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If N_t= Brownian Motion on $p \times \alpha$ matrices ($\alpha \in \mathbb{N}$), define
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$$d\mathbb{Y}_t = \sqrt{\mathbb{Y}_t} d\mathbb{W}_t + d\mathbb{W}_t^T \sqrt{\mathbb{Y}_t} + \alpha \mathbb{I} dt,$$

where $\mathbb{W}_t$ is a Brownian $p \times p$ matrix(Bru 1991). Bru showed that this matrix SDE has solutions for $\alpha \in]p-1, \infty) \cup \{1, \ldots, p-1\}$.

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Difficulties in solving SDEs forBESQ particles

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- non-Lipschitz functions $\sqrt{x}$ in martingale parts (Yamada-Watanabe th. is 1-dimensional!)
- The drift part contains singularities $(X_i - X_j)^{-1}$ (physicists want to start from $(0, \ldots, 0)$!)

Existence and uniqueness for BESQ particle systems $dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i - X_j} 1_{\{X_i \neq X_j\}}\right) dt$

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Proof (and proofs of other reported results) based on:
- SDEs for symmetric polynomials (explained now)
- "Glueing" solutions for subsystems (explained later on examples)

Methodology of proofs: SDEs for symmetric polynomials of $dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i - X_j} 1_{\{X_i \neq X_j\}}\right) dt$

The elementary symmetric polynomials of $X = (X_1, \ldots, X_p)$ are defined,for $n = 1, \ldots, p$ by $\boxed{e_n(X) = \sum_{i_1 < i_2 < \ldots < i_n} X_{i_1} X_{i_2} \cdot \ldots \cdot X_{i_n}}$

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The SDEs for symmetric polynomials are non-singular!!!

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Methodology introduced for **general particle systems** in:

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In this research we use extensively main results and methods of EJP(2014).

Consider a system of SDEs on the cone
$\overline{C_+} = \{(x_1, \ldots, x_p) \in \mathbb{R}^p : x_1 \leq x_2 \leq \ldots \leq x_p\}$

$$d\lambda_i = \sigma_i(\lambda_i) dB_i + \left(b_i(\lambda_i) + \sum_{j \neq i} \frac{H_{ij}(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt$$

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Such systems contain:
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– eigenvalue processes of matrix processes on $Sym_{p \times p}$ verifying the matrix SDE

$$dX_t = h(X_t)dW_t g(X_t) + g(X_t)dW_t^T h(X_t) + b(X_t)dt$$

where the functions $g, h, b : \mathbb{R} \to \mathbb{R}$ act spectrally on $Sym_{p \times p}$.

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- **non-colliding of solutions of this system**

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we prove, when starting from $\lambda_1(0) \leq \ldots \leq \lambda_p(0)$ and under **natural conditions on the coefficients** σ_i, H_{ij}, b_i **(symmetry of** H**, regularity, non-explosion, non-collision conditions)**

- **strong existence and pathwise unicity**
- **non-colliding of solutions of this system**
- **by methods of classical Itô calculus**

Consider $e_1 = \lambda_1 + \ldots + \lambda_p$.

Using polynomials to prove existence for particles SDEs

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- Analogous phenomenon occurs for other basic symmetric polynomials of $(\lambda_1, \ldots, \lambda_p)$

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(Brownian motions U_i are not independent, but their brackets $< U_i, U_j >$ are determined)

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  ($(-1)^k e_k(\Lambda)$ is the coefficient of x^{p-k} in $P(x) = \prod_{i=1}^p (x - \lambda_i)$)
- we define $\lambda = \lambda(e_1, \ldots, e_p)$ and show that they are solutions of the SDEs system for Λ

$$dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i - X_j} 1_{\{X_i \neq X_j\}}\right) dt$$

EJP(2014) implies

Corollary

Let $|\alpha| \in \mathbb{R}^+ \setminus \{0, 1, \ldots, p-2\}$. Then the BESQ particle system has a unique non-colliding solution for $t > 0$. If $\alpha \geq p - 1$ and $X_1(0) \geq 0$, then the solution is non-negative, i.e. $X_1(t) \geq 0$.

Application of EJP(2014) to BESQ particles $dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i-X_j} 1_{\{X_i\neq X_j\}}\right) dt$

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One of assumptions of EJP(2014) fails if $|\alpha| \in \{0, 1, \ldots, p-2\}$

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We prove it by glueing non-negative and non-positive solutions.
Let us explain it on the simplest example.

Existence of non-colliding BESQ particle systems for $\alpha \in \{0, 1, \dots, p-2\}$ $dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i - X_j} 1_{\{X_i\neq X_j\}}\right) dt$ Example: $p = 2$, $\alpha = 0$, start from $(0,0)$

The system is

$$dX_1 = 2\sqrt{|X_1|}dB_1 + \frac{|X_1| + |X_2|}{X_1 - X_2} 1_{\{X_1\neq X_2\}} dt,$$
$$dX_2 = 2\sqrt{|X_2|}dB_2 + \frac{|X_1| + |X_2|}{X_2 - X_1} 1_{\{X_1\neq X_2\}} dt,$$

where $X_1(t) \leq X_2(t)$, $t > 0$ and $X(0) = (x_1, x_2)$.

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where $X_1(t) \leq X_2(t)$, $t > 0$ and $X(0) = (x_1, x_2)$.

The process $\boxed{X_1(t) = X_2(t) \equiv 0}$ is a strong solution of the system and it is colliding.

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Example: $p = 2$, $\alpha = 0$, start from $(0, 0)$

We find another solution $\tilde{X}$ by supposing that $\boxed{\tilde{X}_1(t) \leq 0 \leq \tilde{X}_2(t)}$.
The system becomes

$$\begin{aligned} d\tilde{X}_1 &= 2\sqrt{|\tilde{X}_1|}dB_1 - dt, \quad \tilde{X}_1(0) = 0, \\ d\tilde{X}_2 &= 2\sqrt{|\tilde{X}_2|}dB_2 + dt, \quad \tilde{X}_2(0) = 0. \end{aligned}$$

The processes $\tilde{X}_1$, $\tilde{X}_2$ are two independent squared Bessel processes of dimension -1 and $+1$ respectively.

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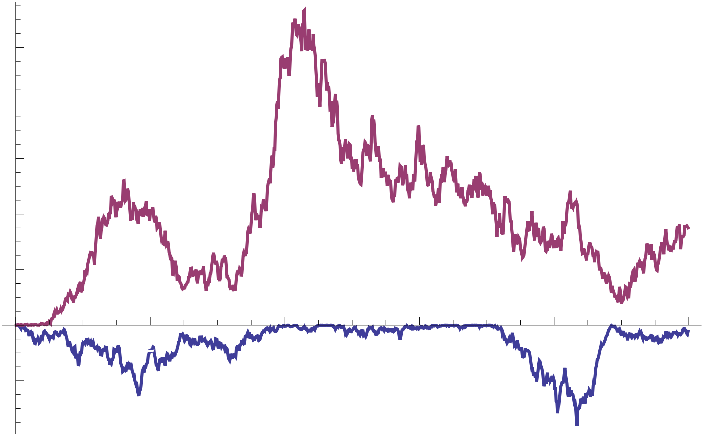
This illustrates our method of constructing of a non-colliding strong solution for $\alpha \in \{0, \ldots, p-2\}$, by glueing solutions in lower dimensions.

Example: $p = 2$, $\alpha = 0$, start from $(0, 0)$



Unique non-colliding solution starting from $x = (0, 0)$.
(Another (non-negative) solution: $X_1 = X_2 = 0$.*)*

Non–Uniqueness for BESQ particle systems with $\alpha \in \{0, 1, \ldots, p-2\}$ $dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i - X_j} 1_{\{X_i \neq X_j\}}\right) dt$

Our Thm says that there exists the unique $BESQ_{nc}^{(\alpha)}(x_1, \ldots, x_p)$

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Let $\mathrm{rank}^+(x) = \sum_{i=1}^p 1_{(0,\infty)}(x_i), \quad \mathrm{rank}^-(x) = \sum_{i=1}^p 1_{(-\infty,0)}(x_i)$

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Let $\mathrm{rank}^+(x) = \sum_{i=1}^p 1_{(0,\infty)}(x_i)$, $\mathrm{rank}^-(x) = \sum_{i=1}^p 1_{(-\infty,0)}(x_i)$

Theorem

Pathwise uniqueness with the initial condition $X(0) = x$, where $x = (x_1, \dots, x_p)$, holds if and only if one of the following conditions holds

(a) $|\alpha| \notin \{0, 1, \dots, p-2\}$

(b) $|\alpha| \in \{0, 1, \dots, p-2\}$ *and* $(rank^+(x) \geq \frac{p+\alpha-1}{2}$ *or* $rank^-(x) \geq \frac{p-\alpha-1}{2})$.

Then there exists unique strong solution, which is non-colliding.

Non–uniqueness for BESQ particle systems $dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i - X_j} 1_{\{X_i \neq X_j\}}\right) dt$

When $|\alpha| \in \{0, 1, \ldots, p-2\}$, $\mathrm{rank}^+(x) < (p+\alpha-1)/2$ and $\mathrm{rank}^-(x) < (p-\alpha-1)/2$,
then the uniqueness of the strong solutions is violated, i.e. there exist at least two solutions.

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The solutions are all colliding, except one.

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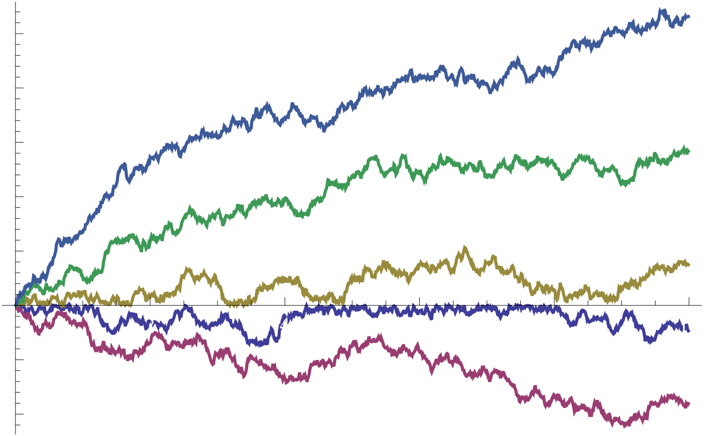
$p = 5$, $\alpha = 1$ start from $x = (0, \ldots, 0)$



Unique non-colliding solution $(BESQ_{nc}^{(-2)}(0,0), BESQ_{nc}^{(3)}(0,0,0))$

$$dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i - X_j} 1_{\{X_i \neq X_j\}}\right) dt$$

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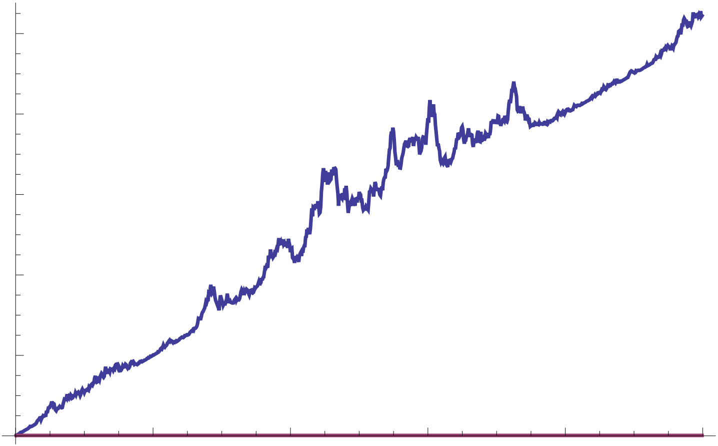

Unique non-negative solution $(0, 0, 0, 0, BESQ^{(5)}(0))$

$$dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i - X_j} 1_{\{X_i \neq X_j\}}\right) dt$$

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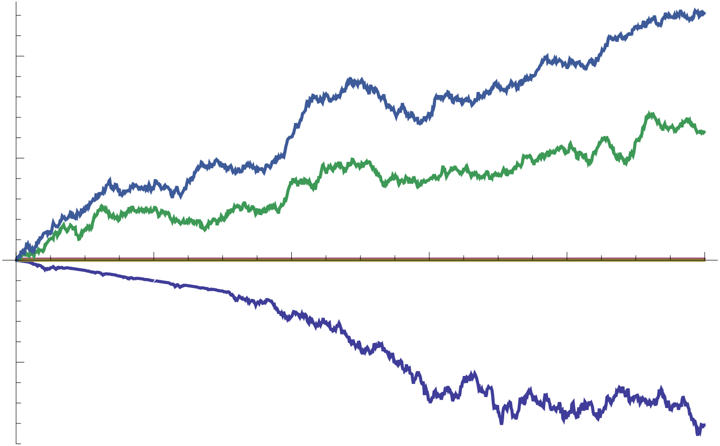

Solution $(BESQ^{(-3)}(0), 0, 0, BESQ^{(4)}_{nc}(0,0))$,

neither non-colliding nor non-negative.

Existence and uniqueness of non-negative solutions $dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i-X_j} 1_{\{X_i\neq X_j\}}\right) dt$ $0 \leq X_1(t) \leq \ldots \leq X_p(t)$

Theorem

There exists a ***non-negative*** *solution with the initial condition $X(0) = x$, where $x = (x_1, \ldots, x_p)$ and $x_1 \geq 0$, if and only if one of the following conditions holds*

(a) $\alpha \geq p - 1$

(b) $\alpha \in \{0, 1, \ldots, p-2\}$ *and* $rk(x) \leq \alpha$.

Then pathwise uniqueness among non-negative solutions holds and there exists unique non-negative strong solution.

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The proof uses drifts of symmetric polynomials and is the same as for an analogous **non-negativity problem for the BESQ matrix** (Wishart) process: **Stochastic Gindikin set**

Stochastic and Non-Central Gindikin Sets

Consider $\mathbb{Y}_t$, solution of

$$d\mathbb{Y}_t = \sqrt{|\mathbb{Y}_t|}\,d\mathbb{W}_t + d\mathbb{W}_t^T\sqrt{|\mathbb{Y}_t|} + \alpha\mathbb{I}dt, \quad \alpha \in \mathbb{R}$$

Problem. **For which α and $\mathbb{Y}_0$ does the process $\mathbb{Y}_t$ stay in $\overline{\mathcal{S}_p^+}$** ?
(this is the **Stochastic Gindikin Set**)

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Solution in: P. Graczyk, J. Małecki, E. Mayerhofer
A Characterization of Wishart Processes and Wishart Distributions
Stoch.Proc. Appl., 2017

Answer: $\alpha \geq p-1$ or $\alpha \in \{0, 1, \ldots, p-2\}$ and $\mathrm{rank}(\mathbb{Y}_0) \leq \alpha$

Proof of the Stochastic Gindikin set

Recall that the symmetric polynomials $e_n = e_n(X)$, $n = 1, \dots, p$ are semimartingales satisfying the following system of SDEs

$$\begin{aligned} de_1 &= 2\sqrt{e_1}dV_1 + p\alpha dt, \\ de_n &= M_n(e_1, \dots, e_p)dV_n + (p-n+1)(\alpha-n+1)e_{n-1}dt, \quad , \\ de_p &= 2\sqrt{e_{p-1}e_p}dV_p + (\alpha-p+1)e_{p-1}dt, \end{aligned}$$

where V_n, $n = 1, \dots, p$ are one-dimensional Brownian motions and the functions M_n are continuous on $\mathbb{R}^p$.

Proof of the Stochastic Gindikin set for $\alpha \in \{0, 1, \ldots, p-2\}$

Suppose that a solution $X_i(t) \geq 0$. Consider $\mathbb{E}e_n(t)$ where $n = \alpha + 1$. Then

$$\begin{aligned} \mathbb{E}e_n(t) &= e_n(0) + (p-n+1)(\alpha - n + 1)\int_0^t \mathbb{E}e_{n-1}(s)ds = e_n(0) \\ \mathbb{E}e_{n+1}(t) &= e_{n+1}(0) + (p-n)(\alpha-n)e_n(0)t \end{aligned}$$

If $e_n(0) > 0$, then the leading term of $\mathbb{E}e_{n+1}(t)$ is negative and thus $\mathbb{E}e_{n+1}(t) < 0$ for large t. It implies $e_n(0) = 0$, i.e. $rank(x_0) \leq n - 1 = \alpha$.

Stochastic and Non-Central Gindikin Sets

Application of Stochastic Gindikin Set: **First proof** of the

characterization of the Non-Central Gindikin Set (NCGS)

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BESQ matrix processes are affine processes:
the exponent of the Laplace transform of $\mathbb{Y}_t$ is affine function of the starting point x_0

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THANK YOU FOR YOUR ATTENTION

ARIGATO GOZAIMASU!

Open questions: multi-indexed case

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Q1. How to define Multi-indexed BESQ Matrix (Wishart) processes for any admissible multi-index $\underline{\alpha}$?

Q2. Study the corresponding multi-indexed BESQ particle systems

Methodology of proofs: SDEs for symmetric polynomials of $dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j\neq i} \frac{|X_i|+|X_j|}{X_i-X_j} 1_{\{X_i\neq X_j\}}\right) dt$

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(We write $e_n^{\overline{j_1},\overline{j_2},\ldots,\overline{j_m}}(X)$ for an incomplete elementary symmetric polynomial of degree n, written without any of $X_{j_1},\ldots,X_{j_m}$.)

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Theorem

If X is a non-colliding solution of BESQ particle system, then $(e_1, \ldots, e_p)$ are semi-martingales described, for $n = 1, \ldots, p$, by

$$de_n = \sqrt{\sum_{i=1}^{p} |X_i|(e_{n-1}^{\bar{i}})^2 dV_n} + (\sum_{i=1}^{p} \alpha e_{n-1}^{\bar{i}} - \sum_{i<j}(|X_i| + |X_j|)e_{n-2}^{\bar{i},\bar{j}})dt$$

Here $(V_1, \ldots, V_p)$ is a collection of one-dimensional Brownian motions such that

$$d\langle e_n(X), e_m(X)\rangle = 4\sum_{i=1}^{p} |X_i| e_{n-1}^{\bar{i}}(X) e_{m-1}^{\bar{i}}(X) dt.$$

Methodology of proofs: SDEs for symmetric polynomials. Non-negative case $0 \le X_1(t) \le \dots X_p(t)$

Theorem

The elementary symmetric polynomials of the non-colliding solution of BESQ particle system starting from $0 \le x_1 \le \dots \le x_p$ *are semi-martingales described up to the first exit time* $\tau = \inf\{t > 0 : X_1(t) < 0\}$ *by the following system of p SDEs*

$$\begin{aligned} de_n &= 2\sqrt{\sum_{k=1}^{p}(2k-1)e_{n-k}e_{n+k-1}}dV_n \\ &+ (p-n+1)(\alpha-n+1)e_{n-1}dt, \end{aligned}$$

where V_n *are one-dimensional Brownian motions such that*

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We show that λ_i never collide for $t > 0$.

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McKean argument: non-explosion of $U = \ln V_N$ since the finite variation part is bounded

EJP(2014): Assumptions (A4-A4') on coefficients of

$$d\lambda_i = \sigma_i(\lambda_i)dB_i + \left(b_i(\lambda_i) + \sum_{j \neq i} \frac{H_{ij}(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt$$

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Conditions for non-collisions

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Otherwise, if $\sigma_k^2(x) + \sigma_l^2(x) + H_{kl}(x,x) = 0$, such critical points x are isolated and verify the following condition (A4')

(A4') **In each critical collision point x there is a force making the particles leave from it**

In points critical for (A4), the martingale part as well as the force coming from repulsive forces between particles can not move them from x. Then we require

$$\sum_{i=k}^{l} \left(b_i(x) + \sum_{j=1}^{p-2} \frac{H_{ij}(x, y_j)}{x - y_j} 1_{\mathbb{R}\setminus\{x\}}(y_j) \right) \neq 0,$$

for every $y_1, \ldots, y_{p-2} \in \mathbb{R}$. It guarantees that the force coming from the whole drift part will move particles from the critical point and cause instant diffraction of the particles.

Proof of the Stochastic Gindikin set

Suppose that $\alpha < p - 1$ and $\alpha \notin \mathbb{N}$. Suppose that the particles $(X_i(t))$ **are non-negative.**
We compute the expected value of the symmetric polynomials

$$\mathbb{E}e_1(t) = e_1(0) + p\alpha \int_0^t ds = e_1(0) + p\alpha t.$$

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and so on. Consequently, the coefficient of t^n in $\mathbb{E}e_n(t)$ is

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Take n the first integer greater than $\alpha+1$. Then $\mathbb{E}e_n(t)$ is a polynomial with the leading coefficient negative $\Rightarrow \mathbb{E}e_n(t)$ cannot stay positive for every $t>0$, contradiction!

Proof. Let a random variable Y on $\bar{\mathcal{S}}_p^+$ exist with law $\Gamma_{p,\Sigma}(\beta,\omega)$, i.e. $\mathcal{L}(Y)(u) = (\det(I + \Sigma u))^{-\beta}\, e^{-\mathbf{Tr}(u(I+\Sigma u)^{-1}\omega)}, \quad u \in \bar{\mathcal{S}}_p^+$.

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$\Rightarrow$ All the laws $\Gamma_{p,tI}(\beta,\omega')$, $t \geq 0$, $rank(\omega') \leq rank(\omega)$ exist

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Step 1.

$\Rightarrow$ All the laws $\Gamma_{p,tI}(\beta,\omega')$, $t \geq 0$, $rank(\omega') \leq rank(\omega)$ exist
(take exponential family of Y, use Fourier-Laplace transform and Lévy continuity theorem)

$\Rightarrow$ there exists an affine Markov process Y_t with state space $\bar{\mathcal{S}}_p^+ \cap \{M : \ rank(M) \leq \max\{rank(\omega), 2\beta\}\}$ and with law of Y_t equal to $\Gamma_{p,2tI}(\beta,\omega)$.

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Step 2. The affine Markov process Y_t is a weak solution of the BESQ Matrix SDE, with $\alpha = 2\beta$.