Logarithmic energy of the Coulomb gas on the sphere at low temperature

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Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^3$ and

$$\mathbb{S} := \left\{x \in \mathbb{R}^3 : \ \|x\| \leq 1\right\}.$$

The logarithmic energy of a configuration $x_1, \ldots, x_N \in \mathbb{S}$ is

$$\mathscr{H}_N(x_1, \ldots, x_N) := \sum_{i \neq j} \log \frac{1}{\|x_i - x_j\|}.$$

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7th Smale's problem: For any N, provide a configuration $x_1, \ldots, x_N \in \mathbb{S}$ such that, for a universal constant $c > 0$,

$$\mathscr{H}_N(x_1, \ldots, x_N) - \min_{\mathbb{S}^N} \mathscr{H}_N \leq c \log N. \qquad \text{(Smale)}$$

"For a precise version one could ask for a real number algorithm in the sense of Blum, Cucker, Shub, and Smale which on input N produces as output distinct $x_1, \ldots, x_N$ on the 2-sphere satisfying (Smale) *with halting time polynomial in N."*

Asymptotic expansion

$$\min_{\mathbb{S}^N} \mathscr{H}_N = \mathbf{C}_{\log} N^2 - \frac{1}{2} N \log N + \mathbf{C}_* N + o(N)$$

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- Explicit leading order constant:

$$\mathbf{C}_{\log} := \min_{\mu \in \mathcal{P}(\mathbb{S})} \iint \log \frac{1}{\|x - y\|} \, \mu(\mathrm{d}x)\mu(\mathrm{d}y) = \frac{1}{2} - \log 2.$$

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- ◇ $\mathbf{C}_*$ exists [Betermin, Sandier 2018] and satisfies

$$\mathbf{C}_* \leq 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)} = -0.056...$$

$$\mathbf{C}_* \geq -0.223...$$

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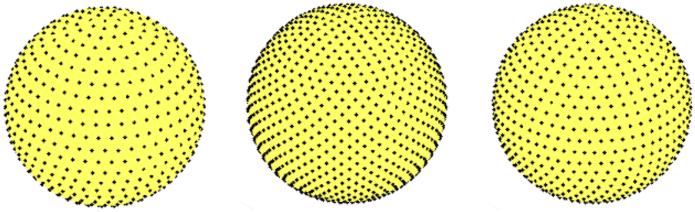
- **The precision $\log N$ is not yet reached...**

Deterministic contructions

Numerical simulations: No deterministic algorithm seems to reach the precision N [Hardin, Michaels, Saff 2016]



Independent configurations

If $x_1, \ldots, x_N$ are independent and uniformly distributed on $\mathbb{S}$,

$$\mathbb{E}_{\text{i.i.d}}\Big[\mathscr{H}_N(x_1, \ldots, x_N)\Big] = \mathbf{C}_{\log} N(N-1) = \mathbf{C}_{\log} N^2 + \text{wrong}.$$

Zeros of random polynomials

If $x_1, \ldots, x_N$ are the zeros of the spherical GAF,

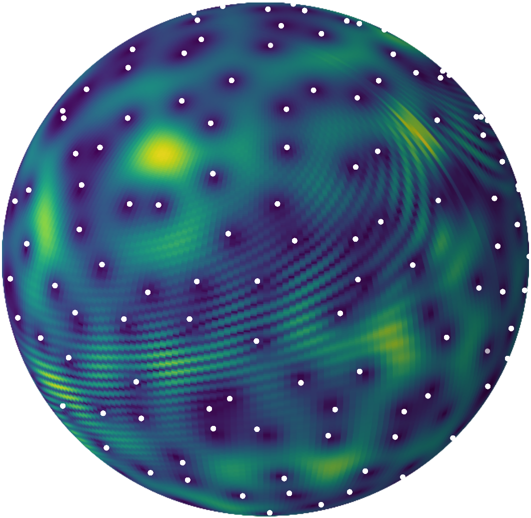
$$f_N(z) := \sum_{k=0}^{N} \boldsymbol{\xi}_k \sqrt{\binom{N}{k}}\, z^k, \qquad (\boldsymbol{\xi}_k)_{k=0}^{N} \text{ i.i.d } \mathcal{N}_{\mathbb{C}}(0,1),$$

up to a stereographic projection, then

$$\mathbb{E}_{\mathrm{GAF}}\Big[\mathscr{H}_N(x_1, \ldots, x_N)\Big] = \mathbf{C}_{\log} N^2 - \frac{1}{2} N \log N + \text{wrong}.$$

[Armentano, Beltrán, Shub 2011]

Zeros of random polynomials



Taken from [Bardenet, H. 2018?]

The spherical ensemble

Let $\mathbf{A}, \mathbf{B}$ be independent $N \times N$ Ginibre matrices.
If $x_1, \ldots, x_N$ are the eigenvalues of $\mathbf{AB}^{-1}$ *up to a stereographic projection*, then

$$\mathbb{E}_{\mathbb{SE}}\Big[\mathscr{H}_N(x_1, \ldots, x_N)\Big] = \mathbf{C}_{\log} N^2 - \frac{1}{2} N \log N + \text{wrong},$$

where "wrong":="more wrong than GAF's wrong"

[Alishahi and Zamani 2015]

The Coulomb gas

For any $\beta > 0$, consider the probability distribution on $\mathbb{S}^N$,

$$\mathrm{d}\mathbb{P}_\beta(x_1, \dots, x_N) := \frac{1}{Z_\beta}\,\mathrm{e}^{-\beta \mathscr{H}_N(x_1,\dots,x_N)} \prod_{j=1}^{N} \mathrm{d}\sigma(x_j)$$

$$= \frac{1}{Z_\beta} \prod_{i \neq j} \|x_i - x_j\|^\beta \prod_{j=1}^{N} \mathrm{d}\sigma(x_j).$$

- The partition function reads

$$Z_\beta := \int \mathrm{e}^{-\beta \mathscr{H}_N}\,\mathrm{d}\sigma^{\otimes N}.$$

- σ is the uniform measure on $\mathbb{S}$ normalized so that $\sigma(\mathbb{S}) = 1$.

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- σ is the uniform measure on $\mathbb{S}$ normalized so that $\sigma(\mathbb{S}) = 1$.

Remark: $\beta = 1$ yields the spherical ensemble [Krishnapur 2006]

Theorem (Beltrán, H. 2018)

The Coulomb gas at $\beta := N$ *satisfies* (Smale) *with high probability*

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The Coulomb gas at $\beta := N$ satisfies (Smale) *with high probability:*

$$\mathbb{P}_N\left(\mathscr{H}_N(x_1,\dots,x_N) - \min_{\mathbb{S}^N} \mathscr{H}_N \leq 10\log N \right) \geq 1 - \mathrm{e}^{-N\log N}.$$

Moreover, the expected energy satisfies

$$\mathbb{E}_N\left[\mathscr{H}_N(x_1,\dots,x_N)\right] - \min_{\mathbb{S}^N} \mathscr{H}_N \leq 9\log N.$$

Open problem

The precise version of the 7th Smale's problem yields the natural problem:

Problem: *Can we sample configurations from the Coulomb gas* $\mathbb{P}_N$ *with a polynomial time algorithm?*

NB: Having in mind MCMC type methods, it's not even required to sample *exactly* from $\mathbb{P}_N$ but only approximately within the required precision range.

The strategy

Laplace's method heuristics: we expect that

$$\log Z_\beta = \log \int \mathrm{e}^{-\beta \mathscr{H}_N} \,\mathrm{d}\sigma^{\otimes N} \simeq -\beta \min_{\mathbb{S}^N} \mathscr{H}_N \quad \text{as } \beta \to \infty.$$

Trivial upper bound: for any $\beta > 0$,

$$\log Z_\beta \leq -\beta \min_{\mathbb{S}^N} \mathscr{H}_N.$$

Problem: *What about a lower bound?*

1st key of the proof

If one can find $C_\beta > 0$ *such that*

$$\log Z_\beta \geq -\beta \min_{\mathbb{S}^N} \mathscr{H}_N - C_\beta,$$

- *For any* $\delta > 0$,

$$\mathbb{P}_\beta\Big(\mathscr{H}_N(x_1, \ldots, x_N) - \min_{\mathbb{S}^N} \mathscr{H}_N > \delta \Big) \leq \mathrm{e}^{-\beta\delta + C_\beta}.$$

- *Moreover,*

$$\mathbb{E}_\beta\Big[\mathscr{H}_N(x_1, \ldots, x_N)\Big] - \min_{\mathbb{S}^N} \mathscr{H}_N \leq \frac{C_\beta}{\beta}.$$

2nd key of the proof

Let $(x_1^*, \ldots, x_N^*) \in \mathbb{S}^N$ *be any minimizer of* $\mathscr{H}_N$*.*
Let $(x_1, \ldots, x_N) \in \mathbb{S}^N$ *satisfying*

$$\max_{j=1}^{N} d_{\mathbb{S}}(x_j, x_j^*) \leq \arcsin\left(\frac{s}{\sqrt{5}N^{3/2}}\right)$$

for some $0 \leq s \leq \sqrt{5N}/2$*. Then,*

$$\mathscr{H}_N(x_1, \ldots, x_N) \leq \min_{\mathbb{S}^N} \mathscr{H}_N + s^2.$$

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Proof:

- Componentwise subharmonicity of $\mathscr{H}_N \Rightarrow$ Maximum principle
- Explicit computations in spherical geometry
- Elementary inequalities (Cauchy–Schwarz and 1st year analysis)

The lower bound

Now, pick $(x_1^*, \ldots, x_N^*) \in \mathbb{S}^N$ any minimizer of $\mathscr{H}_N$ and set

$$\Omega_s := \left\{ (x_1, \ldots, x_N) \in \mathbb{S}^N : \max_{j=1}^{N} d_{\mathbb{S}}(x_j, x_j^*) \leq \arcsin\left(\frac{s}{\sqrt{5} N^{3/2}} \right) \right\}.$$

Then, using the 2nd key

$$\begin{aligned} \log Z_\beta &\geq \log \int_{\Omega_s} \mathrm{e}^{-\beta \mathscr{H}_N} \mathrm{d}\sigma^{\otimes N} \\ &\geq -\beta \min_{\mathbb{S}^N} \mathscr{H}_N - \beta s^2 + \log \sigma^{\otimes N}(\Omega_s) \end{aligned}$$

and optimizing in s yields C_β.

Thank you
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