Macdonald denominators for affine root systems, orthogonal theta functions, and elliptic determinantal point processes

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Random matrices and their applications

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Plan

1. GUE DPP and its Elliptic Extensions
2. Orthogonal Theta Functions
3. Elliptic Determinantal Point Processes
4. Realization as Systems of Noncolliding Brownian Bridges
5. Concluding Remarks

1. GUE Determinantal Point Process and its Elliptic Extensions

- Random N-point process, $N \in \mathbb{N} \equiv \{1, 2, \dots\}$, on a space $S \in \mathbb{R}^d$ is a statistical ensemble of nonnegative integer-valued Radon measures

$$\Xi(\cdot) = \sum_{j=1}^{N} \delta_{X_j}(\cdot),$$

provided that the distribution of points $\{X_j\}_{j=1}^N$ on S is governed by a probability measure P.

- We assume that P has density p with respect to the Lebesgue measure $d\boldsymbol{x} = \prod_{j=1}^N dx_j$, *i.e.*,

$$\mathbf{P}(\boldsymbol{X} \in d\boldsymbol{x}) = \mathrm{p}(\boldsymbol{x}) d\boldsymbol{x}, \quad \boldsymbol{x} \in S^N.$$

- **For the point process $(\Xi, \mathbf{P})$, n-point correlation function of a set $\{x_1, \dots, x_n\} \in S^n$, $1 \le n \le N$, is defined by**

$$\rho(\{x_1, \dots, x_n\}) = \frac{1}{(N-n)!} \int_{S^{N-n}} \prod_{j=n+1}^{N} dx_j \, \mathbf{p}(x_1, \dots, x_n, x_{n+1}, \dots, x_N).$$

- **Then, for any set of observables $\chi_\ell, \ell = 1, 2, \dots, N$, we have the following useful formulas for expectations,**

$$\mathbf{E}\left[\int_{S^n} \prod_{\ell=1}^{n} \chi_\ell(x_\ell) \Xi(dx_\ell)\right] = \int_{S^n} \prod_{\ell=1}^{n} \Big\{dx_\ell \chi_\ell(x_\ell)\Big\} \rho(\{x_1, \dots, x_n\}), \quad n = 1, 2, \dots, N.$$

- **If any correlation function is expressed by a determinant in the form**

$$\rho(\{x_1, \dots, x_n\}) = \det_{1 \le j,k \le n} [K(x_j, x_k)]$$

with a two-point continuous function $K(x, y)$, $x, y \in S$, **then the point process is said to be determinantal and** K **is called the correlation kernel.**

- **A typical example of determinantal point process is the eigenvalue distribution on** $S = \mathbb{R}$ **of Hermitian random matrices in the Gaussian unitary ensemble (GUE) studied in random matrix theory. The probability measure is given as**

$$\mathbf{P}^{\mathrm{GUE}_N}(\boldsymbol{X} \in d\boldsymbol{x}) = \mathrm{p}^{\mathrm{GUE}_N}(\boldsymbol{x})d\boldsymbol{x} = \frac{1}{C^{\mathrm{GUE}_N}} \prod_{\ell=1}^{N} e^{-x_\ell^2} \prod_{1 \le j < k \le N} (x_k - x_j)^2 d\boldsymbol{x},$$

which is normalized as $(1/N!) \int_{\mathbb{R}^N} \mathrm{p}^{\mathrm{GUE}_N}(\boldsymbol{x})d\boldsymbol{x} = 1$.

- **It is** <u>not obvious</u> **that one can perform integrations**

$$\rho(\{x_1, \dots, x_n\}) = \frac{1}{(N-n)!} \int_{S^{N-n}} \prod_{j=n+1}^{N} dx_j \, \mathbf{p}^{\mathrm{GUE}_N}(x_1, \dots, x_n, x_{n+1}, \dots, x_N).$$

for

$$\mathbf{p}^{\mathrm{GUE}_N}(\boldsymbol{x}) = \frac{1}{C^{\mathrm{GUE}_N}} \prod_{\ell=1}^{N} e^{-x_\ell^2} \prod_{1 \le j < k \le N} (x_k - x_j)^2,$$

and obtained results are generally expressed by determinants as

$$\rho(\{x_1, \dots, x_n\}) = \det_{1 \le j,k \le n} [K(x_j, x_k)].$$

- **The verification is, however, not difficult, if we have the following preliminaries.**

[P1] **The factor** $\prod_{1 \le j < k \le N}(x_k - x_j)$ **in** $\mathrm{p}^{\mathrm{GUE}_N}(\boldsymbol{x})$ **obeys the Weyl denominator formula for the classical root system** A_{N-1}**,**

$$\det_{1 \le j,k \le N}[x_k^{j-1}] = \prod_{1 \le j < k \le N}(x_k - x_j).$$

[P2] **By a basic property of determinant, without change of value, we can replace the entries** x_k^{j-1} **in LHS by any monic polynomials of** x_k **with order** $j-1$**. Here we choose them as the monic Hermite polynomials** $2^{-(j-1)}H_{j-1}(x) \equiv 2^{-(j-1)}e^{x^2}(-d/dx)^{j-1}e^{-x^2}$**, and obtain the following equality including the square roots of Gaussian weights in** $\mathrm{p}^{\mathrm{GUE}_N}(\boldsymbol{x})$**,**

$$\prod_{\ell=1}^{N} e^{-x_\ell^2/2} \det_{1 \le j,k \le N}[x_k^{j-1}] = \det_{1 \le j,k \le N}\left[2^{-(j-1)}e^{-x_k^2/2}H_{j-1}(x_k)\right].$$

The reason of this choice is that they satisfy the orthogonal relation,

$$\int_{\mathbb{R}} \left\{ 2^{-(j-1)} e^{-x^2/2} H_{j-1}(x) \right\} \left\{ 2^{-(k-1)} e^{-x^2/2} H_{k-1}(x) \right\} dx = h_j \delta_{jk}, \quad j,k \in \mathbb{N},$$

where $h_j = 2^{-(j-1)}(j-1)!\sqrt{\pi}$.

- Then integrals are given by determinants with the correlation kernel,

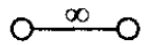
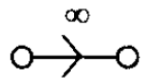
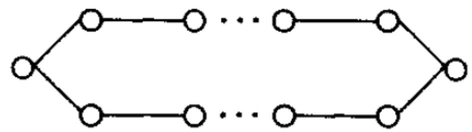
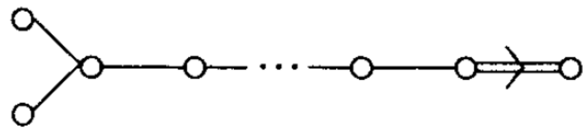
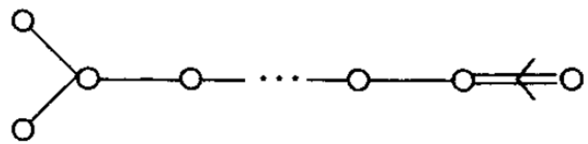
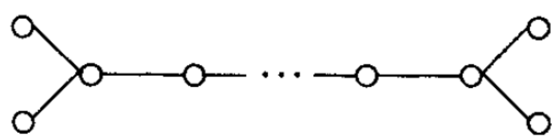
$$K^{\mathrm{GUE}_N}(x,y) = \sum_{n=1}^{N} \frac{1}{h_n} \left\{ 2^{-(n-1)} e^{-x^2/4} H_{n-1}(x) \right\} \left\{ 2^{-(n-1)} e^{-y^2/4} H_{n-1}(y) \right\}, \quad x,y \in \mathbb{R}.$$

- In [RS06], Rosengren and Schlosser extended the Weyl denominator formulas for classical root systems to the Macdonald denominator formulas for seven types of irreducible reduced affine root systems,

$$R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N.$$

- They expressed the result using the theta functions and stated that they are elliptic extensions of the classical results.
- In this talk, we use their result as an elliptic extension of the preliminary [P1].

[RS06] Rosengren, H. and Schlosser, M., "Elliptic determinant evaluations and the Macdonald identities for affine root systems," Compositio Math. 142, 937–961 (2006).

Type	
$A_1 = A_1^\vee$	
$BC_1 = BC_1^\vee$	
$A_l = A_l^\vee \quad (l \geqq 2)$	
$B_l \quad (l \geqq 3)$	
$B_l^\vee \quad (l \geqq 3)$	
$C_l \quad (l \geqq 2)$	
$C_l^\vee \quad (l \geqq 2)$	
$BC_l = BC_l^\vee \quad (l \geqq 2)$	
$D_l = D_l^\vee \quad (l \geqq 4)$	

Macdonald I.G., Affine root systems and Dedekind's η-function, *Invent. Math.* **15** (1972), 91–143.

- **We report in this talk an elliptic extension of the preliminary [P2], and then construct seven new types of determinantal point processes in the elliptic level,**

$$(\Xi^{R_N}, \mathbf{P}_t^{R_N}, t \in (0, t_*)), R_N = A_{N-1}, B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N, D_N, N \in \mathbb{N}.$$

- **This means that their correlation kernels are expressed by the orthogonal theta functions and, if we take appropriate limits of parameters, they are reduced to the classical ones expressed by trigonometric and rational functions.**

- Once the N-point systems have been proved to be determinantal, by taking proper scaling limit associated with $N \to \infty$ limit of the correlation kernels, we can define the determinantal point processes with an infinite number of points.

- Remark that any $N \to \infty$ limit of the probability measure $\mathbf{P}^{\mathrm{GUE}_N}$ is meaningless, since as shown by

$$\mathbf{P}^{\mathrm{GUE}_N}(\boldsymbol{X} \in d\boldsymbol{x}) = \mathbf{p}^{\mathrm{GUE}_N}(\boldsymbol{x})d\boldsymbol{x} = \frac{1}{C^{\mathrm{GUE}_N}} \prod_{\ell=1}^{N} e^{-x_\ell^2} \prod_{1 \le j < k \le N} (x_k - x_j)^2 d\boldsymbol{x},$$

it is absolutely continuous to the Lebesgue measure of N dimensions, $d\boldsymbol{x} = \prod_{j=1}^{N} dx_j$, and $N \to \infty$ limit of $d\boldsymbol{x}$ cannot be mathematically defined.

- **From**

$$K^{\mathrm{GUE}_N}(x,y) = \sum_{n=1}^{N} \frac{1}{h_n} \left\{ 2^{-(n-1)} e^{-x^2/4} H_{n-1}(x) \right\} \left\{ 2^{-(n-1)} e^{-y^2/4} H_{n-1}(y) \right\}, \quad x, y \in \mathbb{R},$$

by taking the scaling limit called the bulk scaling limit, we obtain the following kernel from

$$K^{\sin}(x,y) = \frac{\sin\{\pi\rho(x-y)\}}{\pi(x-y)}, \quad x, y \in \mathbb{R}.$$

- **This is called the sine kernel and it governs the determinantal point process on $\mathbb{R}$ with an infinite number of points, which is spatially homogeneous on $\mathbb{R}$ with constant density of points $\rho > 0$.**

- Our elliptic determinantal point processes have two positive parameters

$$t_* \qquad \text{and} \qquad r.$$

- We demonstrate that in the limit

$$t_* \to \infty,$$

our seven types of determinantal point processes in the elliptic level are reduced to the four types of determinantal point processes in the trigonometric level, in which the correlation kernels are expressed by sine functions.

- If we take the further limit

$$r \to \infty,$$

they are reduced to the three types of sine kernels.

- The bulk scaling limit is realized in our systems by taking the double limit

$$N \to \infty, r \to \infty \text{ with ratio } \frac{N}{r} \text{ fixed.}$$

- We construct four types of determinantal point processes in the elliptic level with an infinite number of particles.

- The reductions of them in the limit $t_* \to \infty$ to the classical infinite determinantal point processes are also shown.

- **The determinantal point process of GUE, $(\Xi^{\mathrm{GUE}_N}, \mathbf{P}^{\mathrm{GUE}_N})$, is related with an interacting particle system consisting of N Brownian motions (BMs) on $\mathbb{R}$, $N \in \mathbb{N}$.**

- **It is a system of BMs conditioned never to collide with each other (noncolliding BMs).**

- **The transition probability density of the one-dimensional standard Brownian motion (BM) from a point v at time s to a point x at time t is given by**

$$\mathrm{p}^{\mathrm{BM}}(s, v; t, x) = \frac{e^{-(x-v)^2/\{2(t-s)\}}}{\sqrt{2\pi(t-s)}}, \quad 0 \leq s < t, v, x \in \mathbb{R}.$$

- **Consider the Weyl chamber,**

$$\mathbb{W}_N = \{\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\}.$$

- **For $\boldsymbol{v}, \boldsymbol{x} \in \mathbb{W}_N$, the total probability mass of N-tuple of non-intersecting Brownian paths, in which the j-th path starts from v_j at time s and arrives at x_j at time $t > s$, $j = 1, 2, \dots, N$, is given by a determinant**

$$\det \Big[\mathsf{p}(s, \boldsymbol{v}; t, \boldsymbol{x}) \Big].$$

Here $\mathsf{p}(s, \boldsymbol{v}; t, \boldsymbol{x})$ is the $N \times N$ matrix whose (j, k)-entry is given by $\mathrm{p}^{\mathrm{BM}}(s, v_j; t, x_k)$;

$$\Big(\mathsf{p}(s, \boldsymbol{v}; t, \boldsymbol{x}) \Big)_{jk} = \mathrm{p}^{\mathrm{BM}}(s, v_j; t, x_k), \quad j, k \in \{1, 2, \dots, N\}.$$

This is known as the Karlin–McGregor–Lindström–Gessel–Viennot (KMLGV) formula.

- **Here we consider the situation such that N BMs start from a given configuration $\boldsymbol{v} \in \mathbb{W}_N$ at time 0, execute noncolliding process, and then return to the configuration $\boldsymbol{v}$ at time $t_* > 0$.**

- **Such a process is called the N-particle system of noncolliding Brownian bridges from $\boldsymbol{v}$ to $\boldsymbol{v}$ in time duration t_***

- **The probability density at time t of this N-particle process is then given by**

$$\mathbf{p}_t^{\boldsymbol{v}\to\boldsymbol{v}}(\boldsymbol{x}; t_*) = \frac{\det[\mathsf{p}^{\mathrm{BM}}(0, \boldsymbol{v}; t, \boldsymbol{x})] \det[\mathsf{p}^{\mathrm{BM}}(t, \boldsymbol{x}; t_*, \boldsymbol{v})]}{\det[\mathsf{p}^{\mathrm{BM}}(0, \boldsymbol{v}; t_*, \boldsymbol{v})]}, \quad \boldsymbol{x} \in \mathbb{W}_N, \quad t \in (0, t_*).$$

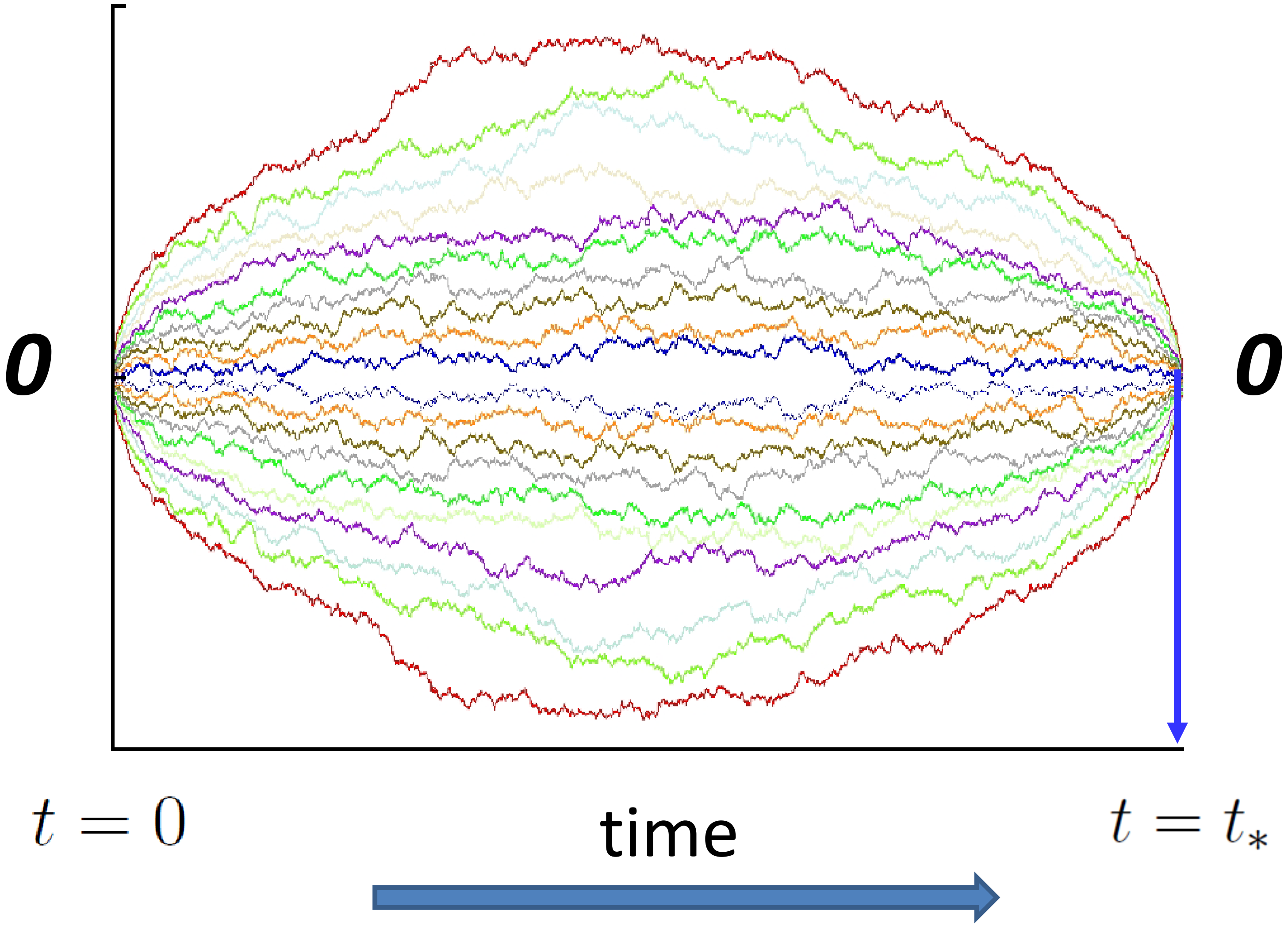
- **We can prove that the limit $\boldsymbol{v} \to \mathbf{0} \equiv (0, \dots, 0) \in \mathbb{R}^N$ exists (see [Katori–Tanemura, JMP (2004)]), and we obtain**

$$\mathrm{p}_t^{\mathbf{0}\to\mathbf{0}}(\boldsymbol{x}; t_*) = \frac{1}{C(N, t, t_*)} \prod_{\ell=1}^{N} e^{-x_\ell^2 t_*/\{2t(t_*-t)\}} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2, \quad \boldsymbol{x} \in \mathbb{W}_N, \quad t \in (0, t_*),$$

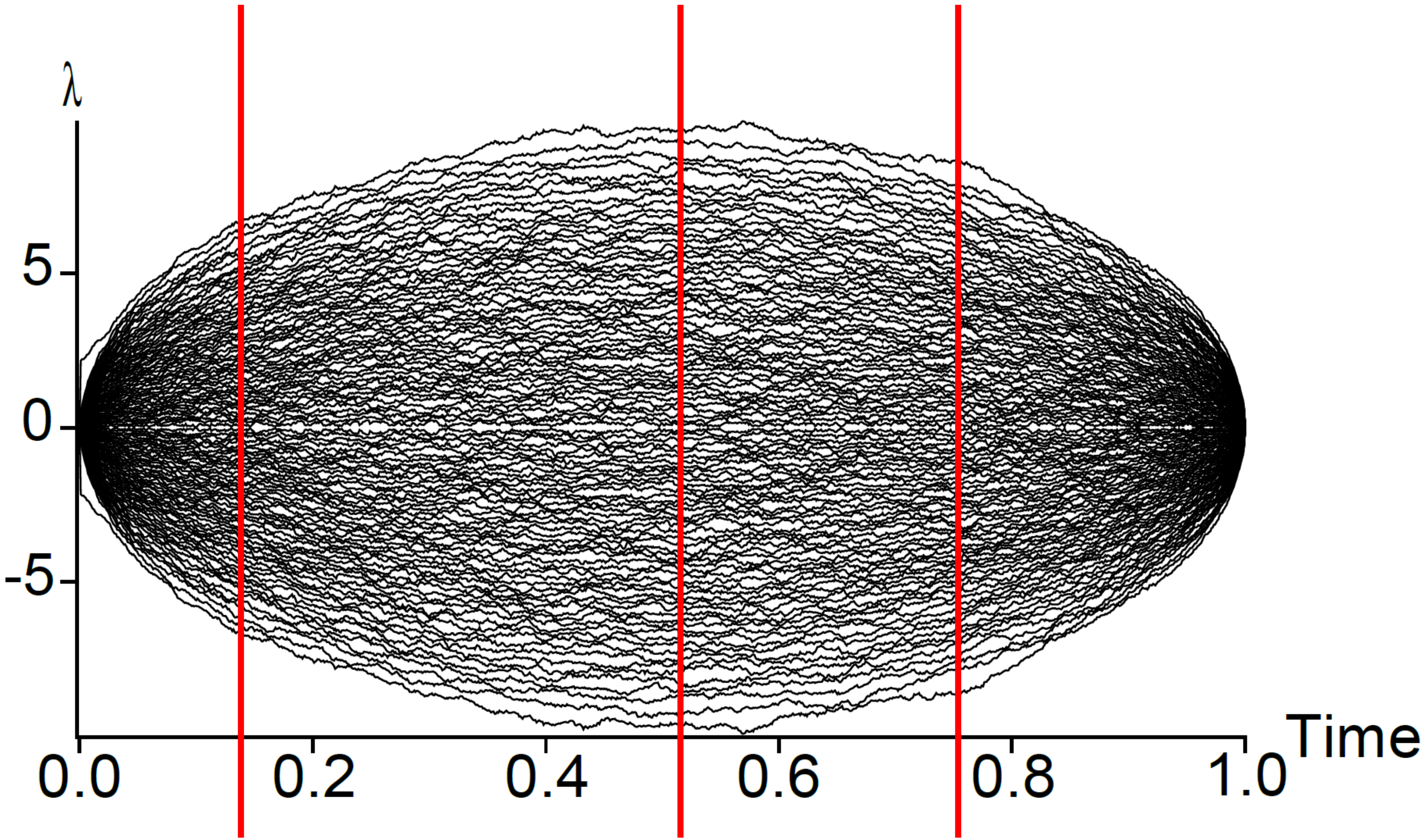
  with a normalization factor $C(N, t, t_*)$ which does not depend on $\boldsymbol{x}$.

- **If we put $t_* = 2$ and $t = t_*/2 = 1$, $\mathrm{p}_t^{\mathbf{0}\to\mathbf{0}}(\boldsymbol{x}; t_*)$ coincides with $\mathrm{p}^{\mathrm{GUE}_N}(\boldsymbol{x})$.**

- **In other words, the N-particle system of noncolliding Brownian bridges from 0 to 0 with time duration t_* realizes a one-parameter extension of determinantal point process of GUE.**



0
0
$t = 0$
time
$t = t_*$



λ
5
0
-5
0.0
0.2
0.4
0.6
0.8
1.0
Time

- **Each type of elliptic determinantal point processes reported in this talk makes a family with one continuous parameter $t \in (0, t_*)$ (in addition to a discrete parameter $N \in \mathbb{N}$).**

- **We can show that, $\underline{(\Xi^{A_{N-1}}, \mathbf{P}_t^{A_{N-1}}, t \in (0, t_*))}$ is realized as an N-particle system of noncolliding Brownian bridges on a circle with radius r.**

- **For $R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N$, $\underline{(\Xi^{R_N}, \mathbf{P}_t^{R_N}, t \in (0, t_*))}$ are realized as N-particle systems of noncolliding Brownian bridges in an interval $[0, \pi r]$ with absorbing boundary conditions at both edges.**

- **And $\underline{(\Xi^{D_N}, \mathbf{P}_t^{D_N}, t \in (0, t_*))}$ is realized as noncolliding N-Brownian bridges in $[0, \pi r]$ with reflecting boundary conditions at both edges.**

- **These Brownian bridges are specified by the pinned configurations $\boldsymbol{v}^{R_N}$ at the initial time $t = 0$ and at the final time $t = t_*$.**

$N = 5$

$[A_N]$ $\mathcal{N}^{A_N} = N = 5$

$[B_N]$ $\mathcal{N}^{B_N} = 2N - 1 = 9$

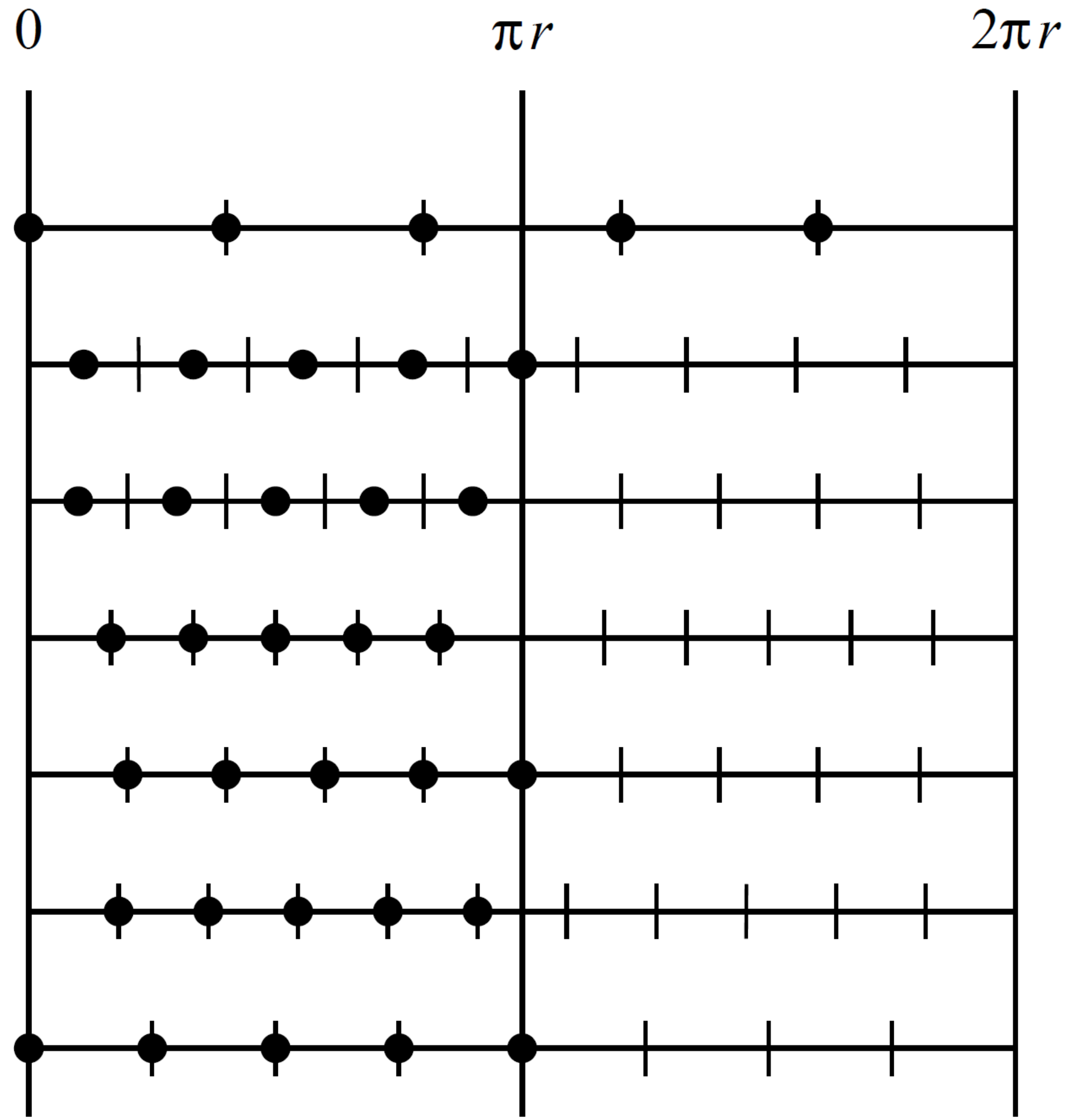
$[B_N^\vee]$ $\mathcal{N}^{B_N^\vee} = 2N = 10$

$[C_N]$ $\mathcal{N}^{C_N} = 2(N+1) = 12$

$[C_N^\vee]$ $\mathcal{N}^{C_N^\vee} = 2N = 10$

$[BC_N]$ $\mathcal{N}^{BC_N} = 2N + 1 = 11$

$[D_N]$ $\mathcal{N}^{D_N} = 2(N-1) = 8$



2. Orthogonal Theta Functions

- **Let**

$$z = e^{v\pi i}, \qquad q = e^{\tau\pi i},$$

where $v, \tau \in \mathbb{C}$ **and** $\Im\tau > 0$. **The Jacobi theta functions are defined as follows,**

$$\vartheta_0(v;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} z^{2n} = 1 + 2\sum_{n=1}^{\infty} (-1)^n e^{\tau\pi i n^2} \cos(2n\pi v),$$

$$\vartheta_1(v;\tau) = i\sum_{n\in\mathbb{Z}} (-1)^n q^{(n-1/2)^2} z^{2n-1} = 2\sum_{n=1}^{\infty} (-1)^{n-1} e^{\tau\pi i(n-1/2)^2} \sin\{(2n-1)\pi v\},$$

$$\vartheta_2(v;\tau) = \sum_{n\in\mathbb{Z}} q^{(n-1/2)^2} z^{2n-1} = 2\sum_{n=1}^{\infty} e^{\tau\pi i(n-1/2)^2} \cos\{(2n-1)\pi v\},$$

$$\vartheta_3(v;\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} z^{2n} = 1 + 2\sum_{n=1}^{\infty} e^{\tau\pi i n^2} \cos(2n\pi v).$$

- **We see the asymptotics**

$$\vartheta_0(v;\tau) \sim 1, \quad \vartheta_1(v;\tau) \sim 2e^{\tau\pi i/4}\sin(\pi v), \quad \vartheta_2(v;\tau) \sim 2e^{\tau\pi i/4}\cos(\pi v), \quad \vartheta_3(v;\tau) \sim 1,$$

$$\textbf{in} \quad \Im\tau \to +\infty \quad (i.e., \quad q = e^{\tau\pi i} \to 0).$$

- **The Macdonald denominators for the seven types of irreducible reduced affine root systems, $W^{R_N}(\boldsymbol{\xi})$, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{C}^N$, are written using the Jacobi theta functions as follows.**

$$\det_{1 \le j,k \le N} [x_k^{j-1}] = \prod_{1 \le j < k \le N} (x_k - x_j).$$

$$W^{A_{N-1}}(\boldsymbol{\xi};\tau) = \prod_{1 \le j < k \le N} \vartheta_1(\xi_k - \xi_j;\tau),$$

$$W^{B_N}(\boldsymbol{\xi};\tau) = \prod_{\ell=1}^{N} \vartheta_1(\xi_\ell;\tau) \prod_{1 \le j < k \le N} \Big\{ \vartheta_1(\xi_k - \xi_j;\tau)\vartheta_1(\xi_k + \xi_j;\tau) \Big\},$$

$$W^{B_N^\vee}(\boldsymbol{\xi};\tau) = \prod_{\ell=1}^{N} \vartheta_1(2\xi_\ell;2\tau) \prod_{1 \le j < k \le N} \Big\{ \vartheta_1(\xi_k - \xi_j;\tau)\vartheta_1(\xi_k + \xi_j;\tau) \Big\},$$

$$W^{C_N}(\boldsymbol{\xi};\tau) = \prod_{\ell=1}^{N} \vartheta_1(2\xi_\ell;\tau) \prod_{1 \le j < k \le N} \Big\{ \vartheta_1(\xi_k - \xi_j;\tau)\vartheta_1(\xi_k + \xi_j;\tau) \Big\},$$

$$W^{C_N^\vee}(\boldsymbol{\xi};\tau) = \prod_{\ell=1}^{N} \vartheta_1\left(\xi_\ell;\frac{\tau}{2}\right) \prod_{1 \le j < k \le N} \Big\{ \vartheta_1(\xi_k - \xi_j;\tau)\vartheta_1(\xi_k + \xi_j;\tau) \Big\},$$

$$W^{BC_N}(\boldsymbol{\xi};\tau) = \prod_{\ell=1}^{N} \Big\{ \vartheta_1(\xi_\ell;\tau)\vartheta_0(2\xi_\ell;2\tau) \Big\} \prod_{1 \le j < k \le N} \Big\{ \vartheta_1(\xi_k - \xi_j;\tau)\vartheta_1(\xi_k + \xi_j;\tau) \Big\},$$

$$W^{D_N}(\boldsymbol{\xi};\tau) = \prod_{1 \le j < k \le N} \Big\{ \vartheta_1(\xi_k - \xi_j;\tau)\vartheta_1(\xi_k + \xi_j;\tau) \Big\},$$

where $\tau \in \mathbb{C}, \Im\tau > 0$.

- **Rosengren and Schlosser [RS06] introduced the notions of A_{N-1}-theta function of norm α and R_N-theta function for $R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$.**
- **Then they proved that, if $f_j^{A_{N-1}}, j = 1, 2, \dots, N$ are A_{N-1}-theta function of norm α, then**

$$\det_{1 \le j,k \le N} \left[f_j^{A_{N-1}}(\xi_k; \tau) \right] = C^{A_{N-1}}(\tau) \vartheta_1 \left(\sum_{\ell=1}^{N} \xi_\ell + \widetilde{\alpha} \right) W^{A_{N-1}}(\boldsymbol{\xi}; \tau)$$

with $\alpha = e^{2\pi i \widetilde{\alpha}}$, and if $f_j^{R_N}, j = 1, 2, \dots, N$, are R_N-theta functions, ,

$$\det_{1 \le j,k \le N} \left[f_j^{R_N}(\xi_k; \tau) \right] = C^{R_N}(\tau) W^{R_N}(\boldsymbol{\xi}; \tau), \quad R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N,$$

where $C^{R_N}(\tau)$ depend on τ and N but not on $\boldsymbol{\xi}$.

- **The factors $C^{R_N}(\tau)$ are explicitly determined in Proposition 6.1 in [RS06] and the above equalities are called the Macdonald denominator formulas.**

[RS06] Rosengren, H. and Schlosser, M., "Elliptic determinant evaluations and the Macdonald identities for affine root systems," Compositio Math. 142, 937–961 (2006).

- **Assume that $0 < r < \infty$.**
- **Let $i = \sqrt{-1}$, and put**

$$\xi(x) = \xi(x; r) = \frac{x}{2\pi r}, \qquad \tau(t) = \tau(t; r) = \frac{it}{2\pi r^2},$$

and

$$\mathcal{N}^{R_N} = \begin{cases} N, & R_N = A_{N-1}, \\ 2N-1, & R_N = B_N, \\ 2N, & R_N = B_N^{\vee}, C_N^{\vee}, \\ 2(N+1), & R_N = C_N, \\ 2N+1, & R_N = BC_N, \\ 2(N-1), & R_N = D_N. \end{cases}$$

- **We consider the following seven sets of functions of $(x,t) \in \mathbb{R} \times [0,\infty)$, $\{M_j^{R_N}(x,t)\}_{j=1}^{N}$, which are defined using the A_{N-1}-theta function of norm $\alpha = e^{2\pi i \widetilde{\alpha}_N}$ with**

$$\widetilde{\alpha}_N = \begin{cases} N\tau(t)/2, & \textbf{if } N \textbf{ is even}, \\ (1+N\tau(t))/2, & \textbf{if } N \textbf{ is odd}, \end{cases}$$

and the R_N-theta functions, $R_N = B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N, D_N$, of Rosengren and Schlosser as

$$M_j^{R_N}(x,t) = f_j^{R_N}\Big(\mathcal{N}^{R_N}\xi(x); \mathcal{N}^{R_N}\tau(t)\Big), \quad j = 1, 2, \dots, N.$$

- **The explicit expressions of the Rosengren-Schlosser's theta functions in our version are given by follows,**

$$
\begin{aligned}
M_j^{A_{N-1}}(x,t) &= M_j^{A_{N-1}}(x,t;r) \\
&= e^{2\pi i J^{A_{N-1}}(j)\xi(x)}\vartheta_2\Big(\mathcal{N}^{A_{N-1}}\{J^{A_{N-1}}(j)\tau(t)+\xi(x)\};(\mathcal{N}^{A_{N-1}})^2\tau(t)\Big), \\
M_j^{R_N}(x,t) &= M_j^{R_N}(x,t;r) \\
&= e^{2\pi i J^{R_N}(j)\xi(x)}\vartheta_1\Big(\mathcal{N}^{R_N}\{J^{R_N}(j)\tau(t)+\xi(x)\};(\mathcal{N}^{R_N})^2\tau(t)\Big) \\
&\quad - e^{-2\pi i J^{R_N}(j)\xi(x)}\vartheta_1\Big(\mathcal{N}^{R_N}\{J^{R_N}(j)\tau(t)-\xi(x)\};(\mathcal{N}^{R_N})^2\tau(t)\Big), \quad \textbf{for } R_N = B_N, B_N^\vee, \\
M_j^{R_N}(x,t) &= M_j^{R_N}(x,t;r) \\
&= e^{2\pi i J^{R_N}(j)\xi(x)}\vartheta_2\Big(\mathcal{N}^{R_N}\{J^{R_N}(j)\tau(t)+\xi(x)\};(\mathcal{N}^{R_N})^2\tau(t)\Big) \\
&\quad - e^{-2\pi i J^{R_N}(j)\xi(x)}\vartheta_2\Big(\mathcal{N}^{R_N}\{J^{R_N}(j)\tau(t)-\xi(x)\};(\mathcal{N}^{R_N})^2\tau(t)\Big), \quad \textbf{for } R_N = C_N, C_N^\vee, BC_N, \\
M_j^{D_N}(x,t) &= M_j^{D_N}(x,t;r) \\
&= e^{2\pi i J^{D_N}(j)\xi(x)}\vartheta_2\Big(\mathcal{N}^{D_N}\{J^{D_N}(j)\tau(t)+\xi(x)\};(\mathcal{N}^{D_N})^2\tau(t)\Big) \\
&\quad + e^{-2\pi J^{D_N}(j)\xi(x)}\vartheta_2\Big(\mathcal{N}^{D_N}\{J^{D_N}(j)\tau(t)-\xi(x)\};(\mathcal{N}^{D_N})^2\tau(t)\Big),
\end{aligned}
$$

where

$$
J^{R_N}(j) = \begin{cases} j-1/2, & R_N = A_{N-1}, C_N^\vee, \\ j-1, & R_N = B_N, B_N^\vee, D_N, \\ j, & R_N = C_N, BC_N. \end{cases}
$$

- In our setting, the Macdonald denominator formulas of Rosengren and Schlosser are written as follows.

$$\det_{1\le j,k\le N}[x_k^{j-1}] = \prod_{1\le j<k\le N}(x_k - x_j).$$

$$\det_{1\le j,k\le N}\left[M_j^{A_{N-1}}(x_k,t)\right] = \begin{cases} i^{-N(N+1)/2} a^{A_{N-1}}(t)\vartheta_0\left(\sum_{j=1}^{N}\xi(x_j);\mathcal{N}^{A_{N-1}}\tau(t)\right)W^{A_{N-1}}(\xi(\boldsymbol{x});\mathcal{N}^{A_{N-1}}\tau(t)), & \textbf{if } N \textbf{ is even},\\ i^{-(N-1)(N-2)/2} a^{A_{N-1}}(t)\vartheta_3\left(\sum_{j=1}^{N}\xi(x_j);\mathcal{N}^{A_{N-1}}\tau(t)\right)W^{A_{N-1}}(\xi(\boldsymbol{x});\mathcal{N}^{A_{N-1}}\tau(t)), & \textbf{if } N \textbf{ is odd},\end{cases}$$

$$\det_{1\le j,k\le N}\left[M_j^{R_N}(x_k,t)\right] = a^{R_N}(t)W^{R_N}(\xi(\boldsymbol{x});\mathcal{N}^{R_N}\tau(t)), \quad \textbf{for } R_N = B_N, B_N^\vee, D_N,$$

$$\det_{1\le j,k\le N}\left[M_j^{R_N}(x_k,t)\right] = i^{-N}a^{R_N}(t)W^{R_N}(\xi(\boldsymbol{x});\mathcal{N}^{R_N}\tau(t)), \quad \textbf{for } R_N = C_N, C_N^\vee, BC_N.$$

LHS are determinants

RHS are products of theta functions

- **Here**

$$
\begin{aligned}
a^{A_{N-1}}(t) &= q(\mathcal{N}^{A_{N-1}}\tau(t))^{-N(3N-1)/8} q_0(\mathcal{N}^{A_{N-1}}\tau(t))^{-(N-1)(N-2)/2},\\
a^{B_N}(t) &= 2q(\mathcal{N}^{B_N}\tau(t))^{-N(N-1)/4} q_0(\mathcal{N}^{BN}\tau(t))^{-N(N-1)},\\
a^{B_N^\vee}(t) &= 2q(\mathcal{N}^{B_N^\vee}\tau(t))^{-N(N-1)/4} q_0(\mathcal{N}^{B_N^\vee}\tau(t))^{-(N-1)^2} q_0(2\mathcal{N}^{B_N^\vee}\tau(t))^{-(N-1)},\\
a^{C_N}(t) &= q(\mathcal{N}^{C_N}\tau(t))^{-N^2/4} q_0(\mathcal{N}^{C_N}\tau(t))^{-N(N-1)},\\
a^{C_N^\vee}(t) &= q(\mathcal{N}^{C_N^\vee}\tau(t))^{-N(2N-1)/8} q_0(\mathcal{N}^{C_N^\vee}\tau(t))^{-(N-1)^2} q_0(\mathcal{N}^{C_N^\vee}\tau(t)/2)^{-(N-1)},\\
a^{BC_N}(t) &= q(\mathcal{N}^{BC_N}\tau(t))^{-N(N+1)/4} q_0(\mathcal{N}^{BC_N}\tau(t))^{-N(N-1)} q_0(2\mathcal{N}^{BC_N}\tau(t))^{-N},\\
a^{D_N}(t) &= 4q(\mathcal{N}^{D_N}\tau(t))^{-N(N-1)/4} q_0(\mathcal{N}^{D_N}\tau(t))^{-N(N-2)},
\end{aligned}
$$

and $\xi(\boldsymbol{x}) \equiv (\xi(x_1), \xi(x_2), \ldots, \xi(x_N))$**, here**

$$
q(\tau) = e^{\tau\pi i}, \qquad q_0(\tau) = \prod_{n=1}^{\infty}(1 - q(\tau)^{2n}).
$$

Bi-orthogonality

Lemma 2.1 *Assume* $0 < \tau_* < \infty$. *For any* $t \in (0, t_*)$, *if* $j, k \in \{1, 2, \ldots, N\}$, *then*

$$\int_0^{2\pi r} \overline{M_j^{A_{N-1}}}(x, t_* - t) M_k^{A_{N-1}}(x, t) dx = m_j^{A_{N-1}}(t_*)\delta_{jk},$$

$$\int_0^{\pi r} \overline{M_j^{R_N}}(x, t_* - t) M_k^{R_N}(x, t) dx = m_j^{R_N}(t_*)\delta_{jk}, \quad \text{for } R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N,$$

where

$$m_j^{R_N}(t_*) = 2\pi r \vartheta_2\Big(\mathcal{N}^{R_N} J^{R_N}(j)\tau(t_*); (\mathcal{N}^{R_N})^2\tau(t_*)\Big), \quad j \in \{1, 2, \ldots, N\},$$
$$\text{for } R_N = A_{N-1}, C_N, C_N^\vee, BC_N,$$

$$m_j^{R_N}(t_*) = \begin{cases} 4\pi r \vartheta_2\Big(0; (\mathcal{N}^{R_N})^2\tau(t_*)\Big), & j = 1, \\ 2\pi r \vartheta_2\Big(\mathcal{N}^{R_N} J^{R_N}(j)\tau(t_*); (\mathcal{N}^{R_N})^2\tau(t_*)\Big), & j \in \{2, 3, \ldots, N\}, \end{cases} \quad \text{for } R_N = B_N, B_N^\vee,$$

$$m_j^{D_N}(t_*) = \begin{cases} 4\pi r \vartheta_2\Big(0; (\mathcal{N}^{D_N})^2\tau(t_*)\Big), & j = 1, \\ 2\pi r \vartheta_2\Big(\mathcal{N}^{D_N} J^{D_N}(j)\tau(t_*); (\mathcal{N}^{D_N})^2\tau(t_*)\Big), & j \in \{2, 3, \ldots, N-1\}, \\ 4\pi r \vartheta_2\Big(\mathcal{N}^{D_N}(N-1)\tau(t_*); (\mathcal{N}^{D_N})^2\tau(t_*)\Big), & j = N. \end{cases}$$

Remark 1. In RHS of

$$M_j^{R_N}(x,t) = f_j^{R_N}\Big(\mathcal{N}^{R_N}\xi(x); \mathcal{N}^{R_N}\tau(t)\Big), \qquad j = 1, 2, \ldots, N,$$

the setting of the first variable be $\mathcal{N}^{R_N}\xi(x)$ instead of $\xi(x)$ is essential for establishing the biorthogonal relations.

Remark 2. When $t = t_*/2$, the functions $\{M_j^{R_N}(x, t_*/2)\}_{j=1}^{N}$ form orthogonal sets with respect to the inner product

$$\langle f|g\rangle = \int_0^L \overline{f}(x)g(x)dx$$

with $L = 2\pi r$ for $R_N = A_{N-1}$ and $L = \pi r$ for $R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$. For the case $R_N = A_{N-1}$, this fact was announced on page 217 in [For10].

[For10] Forrester, P. J., *Log-Gases and Random Matrices* (Princeton University Press, Princeton, NJ, 2010).

3. Elliptic Determinantal Point Processes
3.1 Main results

- Consider the following Weyl alcoves,

$$\mathbb{W}_N^{[0,2\pi r)} \equiv \{\boldsymbol{x} = (x_1, x_2 \dots, x_N) \in \mathbb{R}^N : 0 \leq x_1 < x_2 < \cdots < x_N < 2\pi r\},$$
$$\mathbb{W}_N^{[0,\pi r]} \equiv \{\boldsymbol{x} = (x_1, x_2 \dots, x_N) \in \mathbb{R}^N : 0 \leq x_1 < x_2 < \cdots < x_N \leq \pi r\}.$$

- By the basic properties of the Jacobi theta functions,

$$\vartheta_s \left(\sum_{j=1}^{N} \xi(x_j); \mathcal{N}^{A_{N-1}} \tau(t) \right) > 0 \quad \textbf{for } s = 0, 3, \textbf{ if } 0 \leq t < \infty,$$

and the definitions of Macdonald denominators imply that

$$W^{A_{N-1}}(\xi(\boldsymbol{x}); \mathcal{N}^{A_{N-1}} \tau(t)) \geq 0, \quad \textbf{if } \boldsymbol{x} \in \mathbb{W}_N^{[0,2\pi r)}, \ 0 \leq t < \infty,$$
$$W^{R_N}(\xi(\boldsymbol{x}); \mathcal{N}^{R_N} \tau(t)) \geq 0, \quad \textbf{if } \boldsymbol{x} \in \mathbb{W}_N^{[0,\pi r]}, \ 0 \leq t < \infty, \textbf{ for } R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N.$$

- **Now we introduce**

$$q_t^{R_N}(\boldsymbol{x}) = \det_{1 \le j,k \le N} \left[\overline{M_j^{R_N}}(x_k, t_* - t) \right] \det_{1 \le \ell,m \le N} \left[M_\ell^{R_N}(x_m, t) \right], \quad t \in (0, t_*).$$

- **By the basic properties of the Jacobi theta functions, this product form guarantees the follows.**

Lemma 3.1 *If* $t \in (0, t_*)$, $q_t^{R_N}(\boldsymbol{x}) \ge 0$, $\boldsymbol{x} \in \mathbb{R}^N$, *for* $R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$.

- **Moreover, we can verify the following.**

Lemma 3.2 *For* $t \in (0, t_*)$,

$$\int_{\mathbb{W}_N^{[0,2\pi r)}} q_t^{A_{N-1}}(\boldsymbol{x}) d\boldsymbol{x} = \prod_{n=1}^{N} m_n^{A_{N-1}}(t_*),$$

$$\int_{\mathbb{W}_N^{[0,\pi r]}} q_t^{R_N}(\boldsymbol{x}) d\boldsymbol{x} = \prod_{n=1}^{N} m_n^{R_N}(t_*), \quad R_N = B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N, D_N.$$

<u>**Proof**</u> **Let** $S^{A_{N-1}} = \mathbb{W}_N^{[0,2\pi r)}$, $L^{A_{N-1}} = 2\pi r$, **and** $S^{R_N} = \mathbb{W}_N^{[0,\pi r]}$, $L^{R_N} = \pi r$ **for** $R_N = B_N$, $B_N^{\vee}$, C_N, $C_N^{\vee}$, BC_N, D_N. **By the Heine identity,**

$$\int_{S^{R_N}} q_t^{R_N}(\boldsymbol{x}) d\boldsymbol{x} = \det_{1 \leq j,k \leq N} \left[\int_0^{L^{R_N}} \overline{M_j^{R_N}}(x, t_* - t) M_k^{R_N}(x, t) dx \right].$$

By the biorthogonality given by Lemma 2.1, this is equal to $\det_{1 \leq j,k \leq N}[m_j^{R_N}(t_*)\delta_{jk}]$, **and hence the statements are proved.** ∎

- **Then the seven types of one-parameter** ($t \in (0, t_*)$) **families of probability measures** $\mathbf{P}_t^{R_N}$ **are defined as**

$$\mathbf{P}_t^{R_N}(\boldsymbol{X} \in d\boldsymbol{x}) = \mathbf{p}_t^{R_N}(\boldsymbol{x})d\boldsymbol{x} = \begin{cases} \dfrac{q_t^{A_{N-1}}(\boldsymbol{x})}{\prod_{n=1}^N m_n(t_*)} d\boldsymbol{x}, & \textbf{for } R_N = A_{N-1}, \\ \dfrac{q_t^{R_N}(\boldsymbol{x})}{\prod_{n=1}^N m_n(t_*)} d\boldsymbol{x}, & \textbf{for } R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N. \end{cases}$$

- **They are normalized as**

$$\int_{\mathbb{W}^{[0,2\pi r)}} \mathbf{p}^{A_{N-1}}(\boldsymbol{x})d\boldsymbol{x} = 1,$$
$$\int_{\mathbb{W}^{[0,\pi r]}} \mathbf{p}^{R_N}(\boldsymbol{x})d\boldsymbol{x} = 1, \quad R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N.$$

- **Under these probability measures $\mathbf{P}_t^{R_N}$ with one parameter $t \in (0, t_*)$, we consider seven types of point processes,**

$$\Xi^{A_{N-1}}(\cdot) = \sum_{j=1}^{N} \delta_{X_j^{A_{N-1}}}(\cdot) \quad \textbf{on } S = [0, 2\pi r),$$

and

$$\Xi^{R_N}(\cdot) = \sum_{j=1}^{N} \delta_{X_j^{R_N}}(\cdot) \quad \textbf{on } S = [0, \pi r], \quad \textbf{for } R_N = B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N, D_N.$$

- **Given the determinantal expressions for the probability measures associated with the biorthogonal functions, we can readily prove the following fact by the standard method in random matrix theory.**

Theorem 3.3 *The seven types of one-parameter families of point processes, $(\Xi^{R_N}, \mathbf{P}_t^{R_N}, t \in (0, t_*))$, $R_N = A_{N-1}$, B_N, $B_N^{\vee}$, C_N, $C_N^{\vee}$, BC_N, D_N, are determinantal with the correlation kernels,*

$$K_t^{R_N}(x, y; t_*, r) = \sum_{n=1}^{N} \frac{1}{m_n^{R_N}(t_*)} M_n^{R_N}(x, t) \overline{M_n^{R_N}}(y, t_* - t), \quad t \in (0, t_*),$$
$$x, y \in [0, 2\pi r), \quad \textit{for } R_N = A_{N-1},$$
$$x, y \in [0, \pi r], \quad \textit{for } R_N = B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N, D_N.$$

3.2 Temporally homogeneous limit

- **We consider the determinantal point processes at $t = t_*/2$. The correlation kernels become**

$$K^{R_N}_{t_*/2}(x, y; t_*, r) = \sum_{n=1}^{N} \frac{1}{m_n^{R_N}(t_*)} M_n^{R_N}(x, t_*/2) \overline{M_n^{R_N}}(y, t_*/2),$$

$x, y \in [0, 2\pi r)$ for $R_N = A_{N-1}$, and $x, y \in [0, \pi r]$ for $R_N = B_N$, $B_N^\vee$, C_N, $C_N^\vee$, BC_N, D_N.

- **Remark the asymptotics of the Jacobi theta functions,**

$$\vartheta_0(v;\tau) \sim 1, \quad \vartheta_1(v;\tau) \sim 2e^{\tau\pi i/4} \sin(\pi v), \quad \vartheta_2(v;\tau) \sim 2e^{\tau\pi i/4} \cos(\pi v), \quad \vartheta_3(v;\tau) \sim 1,$$
$$\textbf{in} \quad \Im\tau \to +\infty \quad (i.e., \quad q = e^{\tau\pi i} \to 0).$$

- **Then the temporally homogeneous limit $t_* \to \infty$ of are obtained as follows.**

(i) For $R_N = A_{N-1}$,

$$K^{A_{N-1}}(x,y;r) \equiv \lim_{t_*\to\infty} K^{A_{N-1}}_{t_*/2}(x,y;t_*,r)$$
$$= \frac{1}{2\pi r}\sum_{n=1}^{N} e^{2\pi i(n-1)(\xi(x)-\xi(y))} = \frac{1}{2\pi r}\frac{\sin\{N(x-y)/2r\}}{\sin\{(x-y)/2r\}}, \qquad x,y \in [0,2\pi r).$$

(ii) For $R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N$,

$$K^{R_N}(x,y;r) \equiv \lim_{t_*\to\infty} K^{R_N}_{t_*/2}(x,y;t_*,r)$$
$$= \begin{cases} \frac{1}{2\pi r}\left[\frac{\sin\{(\mathcal{N}^{R_N}+1)(x-y)/2r\}}{\sin\{(x-y)/2r\}} - \frac{\sin\{(\mathcal{N}^{R_N}+1)(x+y)/2r\}}{\sin\{(x+y)/2r\}}\right], & \textbf{if } R_N = B_N, B_N^\vee, \\ \frac{1}{2\pi r}\left[\frac{\sin\{(\mathcal{N}^{R_N}-1)(x-y)/2r\}}{\sin\{(x-y)/2r\}} - \frac{\sin\{(\mathcal{N}^{R_N}-1)(x+y)/2r\}}{\sin\{(x+y)/2r\}}\right], & \textbf{if } R_N = C_N, BC_N, \\ \frac{1}{2\pi r}\left[\frac{\sin\{\mathcal{N}^{R_N}(x-y)/2r\}}{\sin\{(x-y)/2r\}} - \frac{\sin\{\mathcal{N}^{R_N}(x+y)/2r\}}{\sin\{(x+y)/2r\}}\right], & \textbf{if } R_N = C_N^\vee, \end{cases}$$

$x,y \in [0,\pi r]$.

(iii) For $R_N = D_N$,

$$K^{D_N}(x,y;r) \equiv \lim_{t_*\to\infty} K^{D_N}_{t_*/2}(x,y;t_*,r)$$
$$= \frac{1}{2\pi r}\left[\frac{\sin\{(2N-1)(x-y)/2r\}}{\sin\{(x-y)/2r\}} + \frac{\sin\{(2N-1)(x+y)/2r\}}{\sin\{(x+y)/2r\}}\right], \qquad x,y \in [0,\pi r].$$

- **Since** $\mathcal{N}^{B_N}+1=\mathcal{N}^{BC_N}-1=\mathcal{N}^{C_N^\vee}=2N$, **and** $\mathcal{N}^{B_N^\vee}+1=\mathcal{N}^{C_N}-1=2N+1$,

$$
\begin{aligned}
&K^{B_N}(x,y;r)=K^{BC_N}(x,y;r)=K^{C_N^\vee}(x,y;r)\\
&\quad=\frac{1}{2\pi r}\left[\frac{\sin\{N(x-y)/r\}}{\sin\{(x-y)/2r\}}-\frac{\sin\{N(x+y)/r\}}{\sin\{(x+y)/2r\}}\right], \quad x,y\in[0,\pi r],\\
&K^{C_N}(x,y;r)=K^{B_N^\vee}(x,y;r)\\
&\quad=\frac{1}{2\pi r}\left[\frac{\sin\{(2N+1)(x-y)/2r\}}{\sin\{(x-y)/2r\}}-\frac{\sin\{(2N+1)(x+y)/2r\}}{\sin\{(x+y)/2r\}}\right], \quad x,y\in[0,\pi r].
\end{aligned}
$$

Corollary 3.4 *Put* $t=t_*/2$ *in Theorem 3.3. In the limit* $t_*\to\infty$, *the seven types of determinantal point processes* $(\Xi^{R_N},\mathbf{P}^{R_N}_{t_*/2})$ *are degenerated into the four types of determinantal point processes specified by the correlation kernels* $K^{A_{N-1}}(x,y;r)$, $K^{B_N}(x,y;r)$, $K^{C_N}(x,y;r)$, *and* $K^{D_N}(x,y;r)$.

3.3 Infinte determinantal point processes

- We fix the density of points as

$$\rho = \begin{cases} \dfrac{N}{2\pi r}, & R_N = A_{N-1}, \\ \\ \dfrac{N}{\pi r}, & R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N, \end{cases}$$

and take double limit $N \to \infty$, $r \to \infty$. Then we obtain the following limits of correlation kernels.

Lemma 3.5 *For $t \in (0, t_*)$, the following scaling limits are obtained for correlation kernels.*
(i) *For $R_N = A_{N-1}$,*

$$\mathcal{K}_t^A(x, y; t_*, \rho) \equiv \lim_{\substack{N\to\infty, r\to\infty,\\ N/2\pi r=\rho}} K_t^{A_{N-1}}(x, y; t_*, r)$$
$$= \int_0^\rho d\lambda \, e^{2\pi i(x-y)\lambda} \frac{\vartheta_2(\rho x + 2\pi i t \rho\lambda; 2\pi i t \rho^2)\vartheta_2(\rho y - 2\pi i(t_* - t)\rho\lambda; 2\pi i(t_* - t)\rho^2)}{\vartheta_2(2\pi i t_* \rho\lambda; 2\pi i t_* \rho^2)},$$

$x, y \in \mathbb{R}$.
(ii) *For $R_N = B_N, B_N^\vee$,*

$$\mathcal{K}_t^B(x, y; t_*, \rho) \equiv \lim_{\substack{N\to\infty, r\to\infty,\\ N/\pi r=\rho}} K_t^{R_N}(x, y; t_*, r)$$
$$= \frac{1}{2}\Bigg[\int_{-\rho}^\rho d\lambda \, e^{\pi i(x-y)\lambda} \frac{\vartheta_1(\rho x + \pi i t \rho\lambda; 2\pi i t \rho^2)\vartheta_1(\rho y - \pi i(t_* - t)\rho\lambda; 2\pi i(t_* - t)\rho^2)}{\vartheta_2(\pi i t_* \rho\lambda; 2\pi i t_* \rho^2)}$$
$$- \int_{-\rho}^\rho d\lambda \, e^{\pi i(x+y)\lambda} \frac{\vartheta_1(\rho x + \pi i t \rho\lambda; 2\pi i t \rho^2)\vartheta_1(-\rho y - \pi i(t_* - t)\rho\lambda; 2\pi i(t_* - t)\rho^2)}{\vartheta_2(\pi i t_* \rho\lambda; 2\pi i t_* \rho^2)}\Bigg],$$

$x, y \in [0, \infty)$.

(iii) *For* $R_N = C_N, C_N^\vee, BC_N$,

$$\mathcal{K}_t^C(x,y;t_*,\rho) \equiv \lim_{\substack{N\to\infty, r\to\infty,\\ N/\pi r=\rho}} K_t^{R_N}(x,y;t_*,r)$$

$$= \frac{1}{2}\Bigg[\int_{-\rho}^{\rho} d\lambda\, e^{\pi i(x-y)\lambda} \frac{\vartheta_2(\rho x + \pi i t\rho\lambda; 2\pi i t\rho^2)\vartheta_2(\rho y - \pi i(t_* - t)\rho\lambda; 2\pi i(t_* - t)\rho^2)}{\vartheta_2(\pi i t_*\rho\lambda; 2\pi i t_*\rho^2)}$$

$$- \int_{-\rho}^{\rho} d\lambda\, e^{\pi i(x+y)\lambda} \frac{\vartheta_2(\rho x + \pi i t\rho\lambda; 2\pi i t\rho^2)\vartheta_2(-\rho y - \pi i(t_* - t)\rho\lambda; 2\pi i(t_* - t)\rho^2)}{\vartheta_2(\pi i t_*\rho\lambda; 2\pi i t_*\rho^2)}\Bigg],$$

$x, y \in [0, \infty)$.

(iv) *For* $R_N = D_N$,

$$\mathcal{K}_t^D(x,y;t_*,\rho) = \lim_{\substack{N\to\infty, r\to\infty,\\ N/\pi r=\rho}} K_t^{D_N}(x,y;t_*,r)$$

$$= \frac{1}{2}\Bigg[\int_{-\rho}^{\rho} d\lambda\, e^{\pi i(x-y)\lambda} \frac{\vartheta_2(\rho x + \pi i t\rho\lambda; 2\pi i t\rho^2)\vartheta_2(\rho y - \pi i(t_* - t)\rho\lambda; 2\pi i(t_* - t)\rho^2)}{\vartheta_2(\pi i t_*\rho\lambda; 2\pi i t_*\rho^2)}$$

$$+ \int_{-\rho}^{\rho} d\lambda\, e^{\pi i(x+y)\lambda} \frac{\vartheta_2(\rho x + \pi i t\rho\lambda; 2\pi i t\rho^2)\vartheta_2(-\rho y - \pi i(t_* - t)\rho\lambda; 2\pi i(t_* - t)\rho^2)}{\vartheta_2(\pi i t_*\rho\lambda; 2\pi i t_*\rho^2)}\Bigg],$$

$x, y \in [0, \infty)$.

- **The uniform convergence of correlation kernels implies the convergence of all correlation kernels. Then we conclude the following.**

Theorem 3.6 *In the scaling limit $N \to \infty$, $r \to \infty$ with constant density of points, the seven types of one-parameter families of determinantal point processes,* $(\Xi^{R_N}, \mathbf{P}_t^{R_N}, t \in (0, t_*))$, $R_N = A_{N-1}, B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N, D_N$, *converge in the sense of distribution to the four types of infinite dimensional point processes as follows,*

$$(\Xi^{A_{N-1}}, \mathbf{P}_t^{A_{N-1}}, t \in (0, t_*)) \Longrightarrow (\Xi^{A}, \mathbf{P}^{A}, t \in (0, t_*)) \quad \text{as } N \to \infty, r \to \infty \text{ with } \frac{N}{2\pi r} = \rho \ ,$$

$$\left.\begin{array}{l} (\Xi^{B_N}, \mathbf{P}_t^{B_N}, t \in (0, t_*)) \\ (\Xi^{B_N^{\vee}}, \mathbf{P}_t^{B_N^{\vee}}, t \in (0, t_*)) \end{array}\right\} \Longrightarrow (\Xi^{B}, \mathbf{P}^{B}, t \in (0, t_*)) \quad \text{as } N \to \infty, r \to \infty \text{ with } \frac{N}{\pi r} = \rho \ ,$$

$$\left.\begin{array}{l} (\Xi^{C_N}, \mathbf{P}_t^{C_N}, t \in (0, t_*)) \\ (\Xi^{C_N^{\vee}}, \mathbf{P}_t^{C_N^{\vee}}, t \in (0, t_*)) \\ (\Xi^{BC_N}, \mathbf{P}_t^{BC_N}, t \in (0, t_*)) \end{array}\right\} \Longrightarrow (\Xi^{C}, \mathbf{P}^{C}, t \in (0, t_*)) \quad \text{as} \quad N \to \infty, r \to \infty, \frac{N}{\pi r} = \rho,$$

$$(\Xi^{D_N}, \mathbf{P}_t^{D_N}, t \in (0, t_*)) \Longrightarrow (\Xi^{D}, \mathbf{P}^{D}, t \in (0, t_*)) \quad \text{as } N \to \infty, r \to \infty \text{ with } \frac{N}{\pi r} = \rho \ ,$$

where $(\Xi^{A}, \mathbf{P}^{A}, t \in (0, t_*))$, $(\Xi^{B}, \mathbf{P}^{B}, t \in (0, t_*))$, $(\Xi^{C}, \mathbf{P}^{C}, t \in (0, t_*))$, *and* $(\Xi^{D}, \mathbf{P}^{D}, t \in (0, t_*))$ *are infinite determinantal point processes associated with the correlation kernels* $\mathcal{K}_t^{A}$, $\mathcal{K}_t^{B}$, $\mathcal{K}_t^{C}$, *and* $\mathcal{K}_t^{D}$, $t \in (0, t_*)$, *respectively.*

- **Put** $t = t_*/2$**.**
- **By the asymptotics of the Jacobi theta functions, we obtain the following three types of limits,**

$$
\begin{aligned}
\mathcal{K}^A(x,y;\rho) &= \lim_{t_*\to\infty} \mathcal{K}^A_{t_*/2}(x,y;t_*,\rho) = e^{-i\pi\rho(x-y)} \int_0^\rho e^{2\pi i(x-y)\lambda} d\lambda \\
&= \frac{\sin\{\pi\rho(x-y)\}}{\pi(x-y)}, \qquad x,y \in \mathbb{R}, \\
\mathcal{K}^C(x,y;\rho) &= \lim_{t_*\to\infty} \mathcal{K}^R_{t_*/2}(x,y;t_*,\rho) \\
&= \frac{\sin\{\pi\rho(x-y)\}}{\pi(x-y)} - \frac{\sin\{\pi\rho(x+y)\}}{\pi(x+y)}, \qquad \textbf{for } R = B, C, \qquad x,y \in [0,\infty), \\
\mathcal{K}^D(x,y;\rho) &= \lim_{t_*\to\infty} \mathcal{K}^D_{t_*/2}(x,y;t_*,\rho) \\
&= \frac{\sin\{\pi\rho(x-y)\}}{\pi(x-y)} + \frac{\sin\{\pi\rho(x+y)\}}{\pi(x+y)}, \qquad x,y \in [0,\infty).
\end{aligned}
$$

particle density profiles

- **If we put $x = y$ in the correlation kernels, we can obtain the densities of particles ($i.e.$, the profiles of particle distributions),**

$$\rho^R(x) = \lim_{y \to x} K^R(x, y).$$

type A in the classical limit

$$\rho^A(x;\rho) = \lim_{y \to x} \frac{\sin\{\pi\rho(x-y)\}}{\pi(x-y)} = \rho, \qquad x \in \mathbb{R}.$$

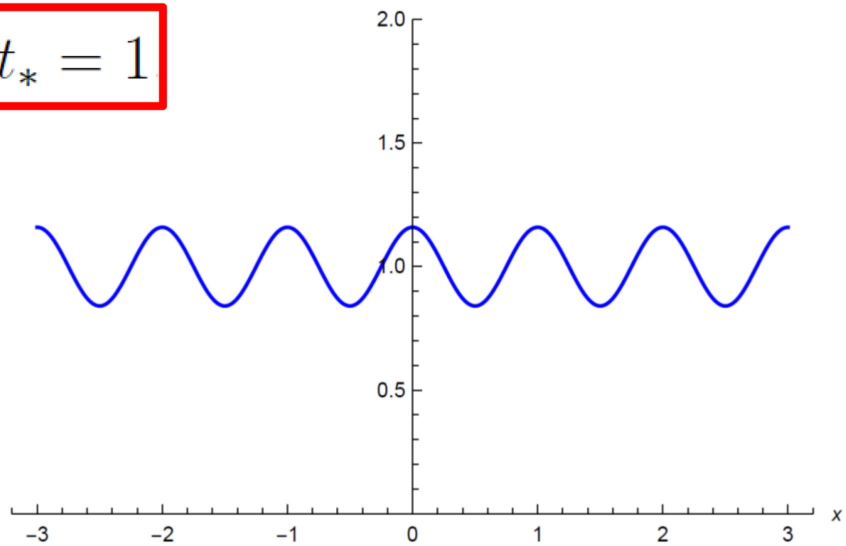
type A in the elliptic level with parameter $0 < t_* < \infty$.

$$\rho^A(x;t_*,\rho) = \int_0^\rho d\lambda \, \frac{\vartheta_2(\rho x + \pi i t_* \rho\lambda; \pi i t_* \rho^2)\vartheta_2(\rho x - \pi i t_* \rho\lambda; \pi i t_* \rho^2)}{\vartheta_2(2\pi i t_* \rho\lambda; 2\pi i t_* \rho^2)}, \qquad x \in \mathbb{R}.$$

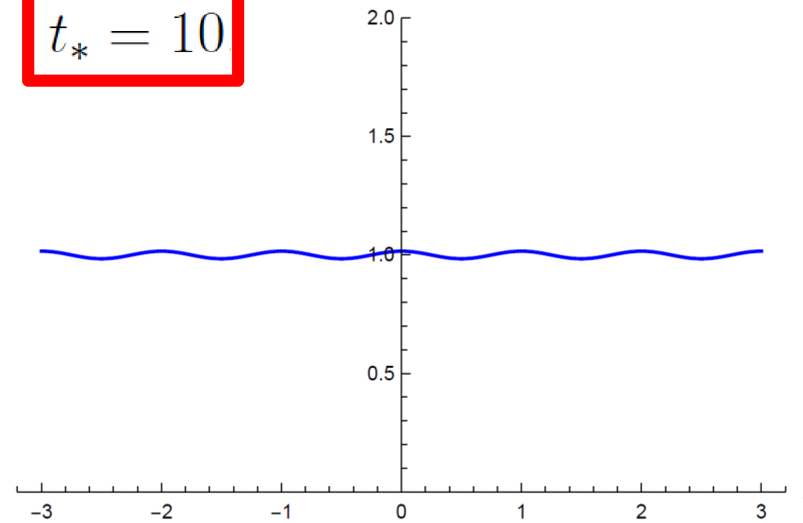
$\rho = 1$

Figures are given by **Hiroya Baba** (my graduate student).

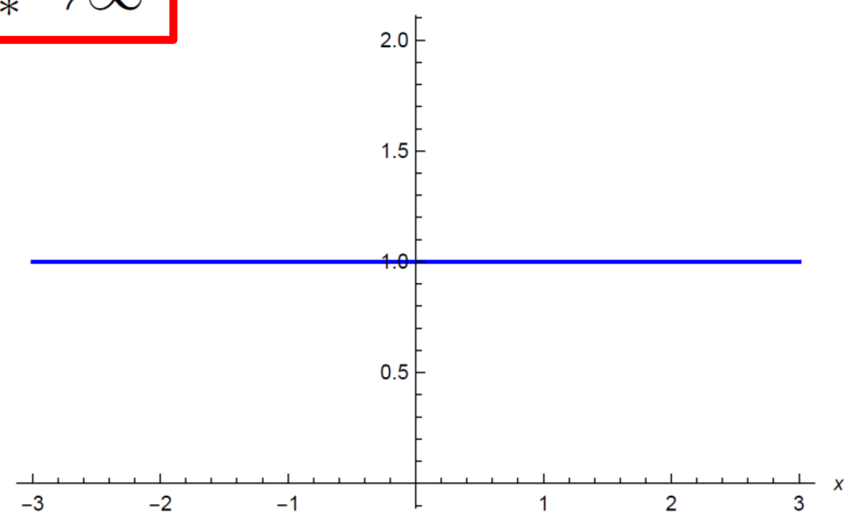
$t_* = 1$



$t_* = 10$



$t_* \to \infty$



4. Realization as Systems of Noncolliding Brownian Bridges

4.1 New expression of Macdonald denominators by KMLGV determinants

- **Denote the transition probability density of BM, starting from x at time s and arriving at y at time t, $x, y \in \mathbb{R}, 0 \leq s < t < \infty$, by**

$$\mathrm{p}^{\mathrm{BM}}(s, x; t, y) = \mathrm{p}^{\mathrm{BM}}(s, y; t, x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-(x-y)^2/\{2(t-s)\}}.$$

- **For** $0 \le s < t < \infty$**, define**

$$
\begin{aligned}
\mathrm{p}^{\mathrm{circ}}(s,x;t,y) &= \mathrm{p}^{\mathrm{circ}}(s,x;t,y;r) \\
&= \begin{cases} \displaystyle\sum_{w\in\mathbb{Z}} (-1)^w \mathrm{p}^{\mathrm{BM}}(s,x;t,y+2\pi r w), & \textbf{if } N \textbf{ is even}, \\ \displaystyle\sum_{w\in\mathbb{Z}} \mathrm{p}^{\mathrm{BM}}(s,x;t,y+2\pi r w), & \textbf{if } N \textbf{ is odd}, \end{cases} \\
&= \begin{cases} \mathrm{p}^{\mathrm{BM}}(s,x;t,y)\vartheta_0(i(x-y)r/(t-s); -1/\tau(t-s)), & \textbf{if } N \textbf{ is even}, \\ \mathrm{p}^{\mathrm{BM}}(s,x;t,y)\vartheta_3(i(x-y)r/(t-s); -1/\tau(t-s)), & \textbf{if } N \textbf{ is odd}, \end{cases} \\
&= \begin{cases} \dfrac{1}{2\pi r}\vartheta_2(\xi(x-y);\tau(t-s)), & \textbf{if } N \textbf{ is even}, \\[2ex] \dfrac{1}{2\pi r}\vartheta_3(\xi(x-y);\tau(t-s)), & \textbf{if } N \textbf{ is odd}, \end{cases}
\end{aligned}
$$

$x, y \in [0, 2\pi r)$**, where**

$$
\xi(x) = \xi(x;r) = \frac{x}{2\pi r}, \qquad \tau(t) = \tau(t;r) = \frac{it}{2\pi r^2}.
$$

- **And define**

$$\mathrm{p}^{\mathrm{abs}}(s,x;t,y) = \sum_{k\in\mathbb{Z}} \left\{ p^{\mathrm{BM}}(s,x;t,y+2\pi rk) - p^{\mathrm{BM}}(s,-x;t,y+2\pi rk) \right\}$$

$$= \frac{1}{2\pi r} \left\{ \vartheta_3(\xi(x-y);\tau(t-s)) - \vartheta_3(\xi(x+y);\tau(t-s)) \right\},$$

$$\mathrm{p}^{\mathrm{ref}}(s,x;t,y) = \sum_{k\in\mathbb{Z}} \left\{ p^{\mathrm{BM}}(s,x;t,y+2\pi rk) + p^{\mathrm{BM}}(s,-x;t,y+2\pi rk) \right\}$$

$$= \frac{1}{2\pi r} \left\{ \vartheta_3(\xi(x-y);\tau(t-s)) + \vartheta_3(\xi(x+y);\tau(t-s)) \right\}.$$

- **We introduce time dependent $N \times N$ matrices, $\mathsf{p}^\sharp(s, \boldsymbol{x}; t, \boldsymbol{y})$, $0 \leq s < t < \infty$, with entries**

$$(\mathsf{p}^\sharp(s, \boldsymbol{x}; t, \boldsymbol{y}))_{jk} = \mathrm{p}^\sharp(s, x_j; t, y_k), \quad \sharp = \mathrm{circ}, \mathrm{abs}, \mathrm{ref}, \quad j, k = 1, 2, \dots, N,$$

for $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$, $\boldsymbol{y} = (y_1, \dots, y_N) \in \mathbb{R}^N$.

- **The determinant $\det[\mathsf{p}^{\mathrm{circ}}(s, \boldsymbol{x}; t, \boldsymbol{y})]$ with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^{[0, 2\pi r)}$, $0 \leq s < t$, is the KMLGV determinant giving the total probability mass of N-tuples of noncolliding Brownian paths on a circle with radius $r > 0$, starting from the unlabeled configuration $\boldsymbol{x}$ at time s and arriving at the unlabeled configuration $\boldsymbol{y}$ at time $t > s$ (see Forrester (1990), Fulmek (2004), Liechty-Wang (2016)).**

- **The determinant $\det[\mathsf{p}^{\mathrm{abs}}(s, \boldsymbol{x}; t, \boldsymbol{y})]$ (resp. $\det[\mathsf{p}^{\mathrm{ref}}(s, \boldsymbol{x}; t, \boldsymbol{y})]$) can be regarded as the KMLGV determinant for the noncolliding BMs in the interval $[0, \pi r]$ with the absorbing (resp. reflecting) boundary conditions at both edges.**

- **We consider the following seven types of configurations of N points, $\boldsymbol{v}^{R_N} = \boldsymbol{v}^{R_N}(r)$ with the elements,**

$$
\begin{aligned}
v_j^{A_{N-1}} &= v_j^{A_{N-1}}(r) = \frac{2\pi r}{N}(j-1), \\
v_j^{R_N} &= v_j^{R_N}(r) = \frac{2\pi r}{\mathcal{N}^{R_N}}(j-1/2), \quad \textbf{for } R_N = B_N, B_N^{\vee}, \\
v_j^{R_N} &= v_j^{R_N}(r) = \frac{2\pi r}{\mathcal{N}^{R_N}} j, \quad \textbf{for } R_N = C_N, C_N^{\vee}, BC_N, \\
v_j^{D_N} &= v_j^{D_N}(r) = \frac{\pi r}{N-1}(j-1), \qquad j = 1, 2, \dots, N.
\end{aligned}
$$

- **The configurations $\boldsymbol{v}^{R_N}$ make equidistant series of points in $[0, 2\pi r)$ for $R_N = A_{N-1}$ and in $[0, \pi r]$ for others.**

- **We also consider $N \times N$ matrices whose entries are given by the biorthogonal theta functions,**

$$
\mathsf{M}^{R_N}(\boldsymbol{x}, t) = \Big(M_j^{R_N}(x_k, t) \Big)_{1 \leq j,k \leq N}.
$$

$N = 5$

$[A_N]$ $\mathcal{N}^{A_N} = N = 5$

$[B_N]$ $\mathcal{N}^{B_N} = 2N{-}1 = 9$

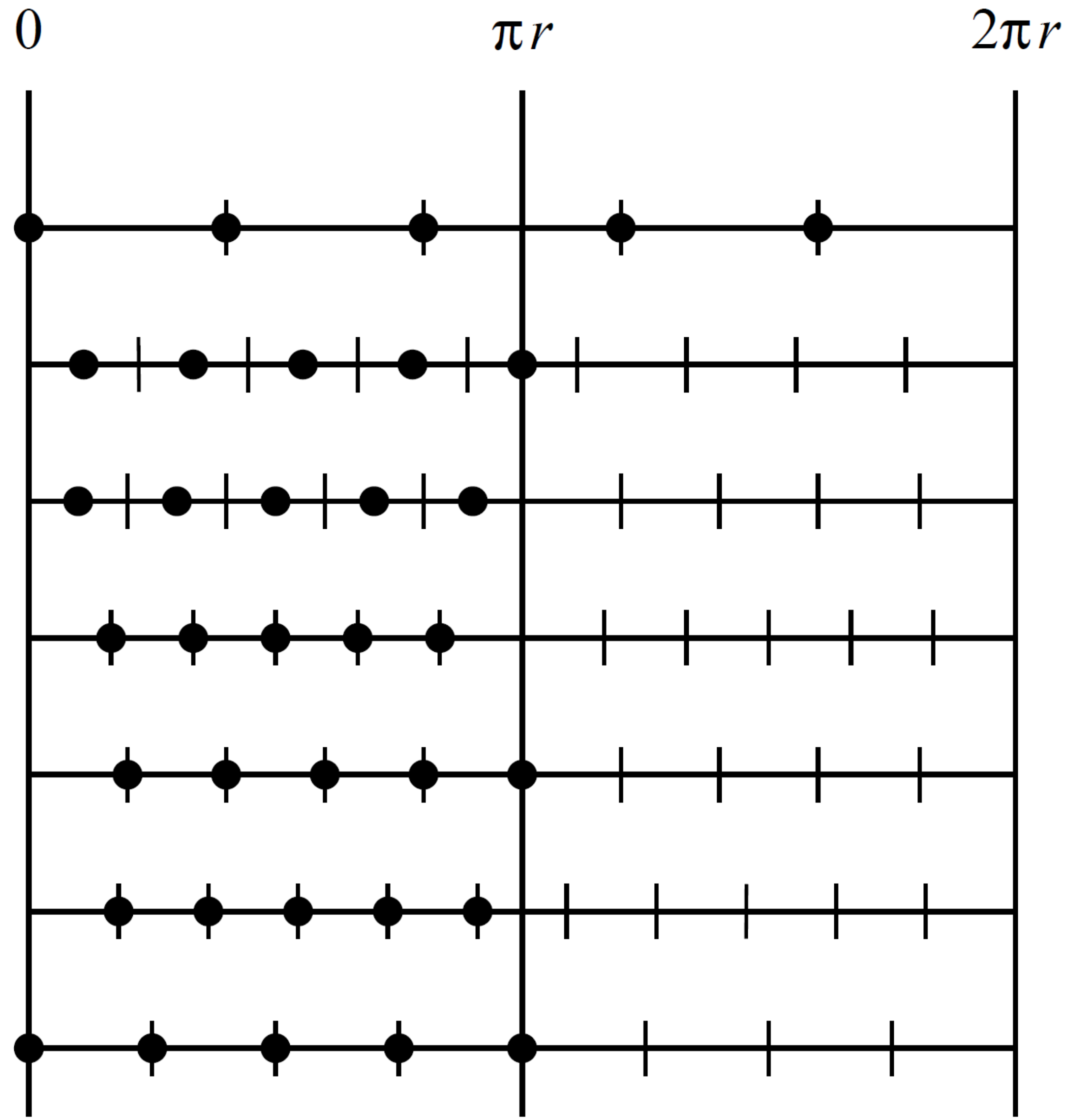
$[B_N^\vee]$ $\mathcal{N}^{B_N^\vee} = 2N = 10$

$[C_N]$ $\mathcal{N}^{C_N} = 2(N{+}1) = 12$

$[C_N^\vee]$ $\mathcal{N}^{C_N^\vee} = 2N = 10$

$[BC_N]$ $\mathcal{N}^{BC_N} = 2N{+}1 = 11$

$[D_N]$ $\mathcal{N}^{D_N} = 2(N{-}1) = 8$



- **Then the following relations hold between matrices.**

Lemma 4.1 *Consider the* $N \times N$ *matrices* $\mathsf{r}^{R_N}(t)$ *with the following entries; for* $j = 1, \dots, N$,

$$(\mathsf{r}^{A_{N-1}}(t))_{jk} = \frac{2\pi r}{N} e^{-\pi i (j-1/2)^2 \tau(t) - \pi i \{N-2(j-1/2)\}(k-1)/N}, \quad k = 1, \dots, N,$$

$$(\mathsf{r}^{R_N}(t))_{jk} = \frac{4\pi r}{i\mathcal{N}^{R_N}} e^{-\pi i (J^{R_N}(j))^2 \tau(t)} \sin\left[(\mathcal{N}^{R_N} - 2J^{R_N}(j)) \frac{v_k^{R_N}}{2r}\right], \quad k = 1, \dots, N,$$

$$\text{for } R_N = B_N^\vee, C_N, BC_N,$$

$$(\mathsf{r}^{R_N}(t))_{jk} = \begin{cases} \dfrac{4\pi r}{i\mathcal{N}^{R_N}} e^{-\pi i (J^{R_N}(j))^2 \tau(t)} \sin\left[(\mathcal{N}^{R_N} - 2J^{R_N}(j)) \dfrac{v_k^{R_N}}{2r}\right], & k = 1, \dots, N-1, \\ \dfrac{2\pi r}{i\mathcal{N}^{R_N}} e^{-\pi i (J^{R_N}(j))^2 \tau(t)} \sin[\pi(N + 1/2 - j)], & k = N. \end{cases}$$

$$\text{for } R_N = B_N, C_N^\vee,$$

$$(\mathsf{r}^{D_N}(t))_{jk} = \begin{cases} \dfrac{\pi r}{N-1} e^{-\pi i (j-1)^2 \tau(t)}, & k = 1, \\ \dfrac{2\pi r}{N-1} e^{-\pi i (j-1)^2 \tau(t)} \cos\left[\dfrac{\pi(N-j)(k-1)}{N-1}\right], & k = 2, \dots, N-1, \\ \dfrac{\pi r}{N-1} e^{-\pi i (j-1)^2 \tau(t)} \cos[\pi(N-j)], & k = N. \end{cases}$$

Then for $t \in [0, \infty)$,

$$\mathsf{r}^{A_{N-1}}(t) \mathsf{p}^{\mathrm{circ}}(0, \boldsymbol{v}^{A_{N-1}}; t, \boldsymbol{x}) = \mathsf{M}^{A_{N-1}}(t, \boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{W}_N^{[0,2\pi r)},$$

$$\mathsf{r}^{R_N}(t) \mathsf{p}^{\mathrm{abs}}(0, \boldsymbol{v}^{R_N}; t, \boldsymbol{x}) = \mathsf{M}^{R_N}(t, \boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{W}_N^{[0,\pi r]}, \quad R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N,$$

$$\mathsf{r}^{D_N}(t) \mathsf{p}^{\mathrm{ref}}(0, \boldsymbol{v}^{D_N}; t, \boldsymbol{x}) = \mathsf{M}^{D_N}(t, \boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{W}_N^{[0,\pi r]}.$$

- **If we take the determinants of both sides of the above equalities between matrices, we obtain the equalities**

$$\det[\mathsf{r}^{R_N}(t)]\det[\mathsf{p}^{\sharp}(0,\boldsymbol{v}^{R_N};t,\boldsymbol{x})]=\det[\mathsf{M}^{R_N}(t,\boldsymbol{x})],$$

where $\sharp =$ **circ for** $R_N = A_{N-1}$, $\sharp =$ **abs for** $R_N = B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N$, **and** $\sharp =$ **ref for** $R_N = D_N$.

- **Combine them with the Macdonald denominator formulas of Rosengren and Schlosser [RS06], we obtain new determinantal expressions for the Macdonald denominators.**

Proposition 4.2 *For the irreducible reduced affine root systems, $R_N = A_{N-1}$, B_N, $B_N^\vee$, C_N, $C_N^\vee$, BC_N, D_N, the Macdonald denominators W^{R_N} are proportional to the KMLGV determinants for noncolliding Brownian paths starting from the configurations $\boldsymbol{v}^{R_N}$ as follows. Let $s(N) = 0$ if N is even, and $s(N) = 3$ if N is odd, then*

$$\vartheta_{s(N)}\left(\sum_{j=1}^{N} \xi(x_j); N\tau(t)\right) W^{A_{N-1}}(\xi(\boldsymbol{x}); N\tau(t)) = b^{A_{N-1}}(t) \det[\mathsf{p}^{\mathrm{circ}}(0, \boldsymbol{v}^{A_{N-1}}; t, \boldsymbol{x})],$$

$$W^{R_N}(\xi(\boldsymbol{x}); \mathcal{N}^{R_N}\tau(t)) = \begin{cases} b^{R_N}(t) \det[\mathsf{p}^{\mathrm{abs}}(0, \boldsymbol{v}^{R_N}; t, \boldsymbol{x})], & \text{for } R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, \\ b^{D_N}(t) \det[\mathsf{p}^{\mathrm{ref}}(0, \boldsymbol{v}^{D_N}; t, \boldsymbol{x})], & \text{for } R_N = D_N, \end{cases}$$

with coefficients $b^{R_N}(t)$.

Remark 3.

- Forrester proved this equality for the type A_{N-1} independently of the Macdonald denominator formulas given by Rosengren and Schlosser [RS06]. The matrix relation was also used to prove the determinantal equality for $R_N = A_{N-1}$ in pages 216-217 in [For10].
- The above lemma and proposition are extensions of Forrester's results to other six types of matrices and their determinants.
- Here we identify LHS of the equations as the Macdonald denominators and the determinants in RHS of them as the KMLGV determinants of noncolliding Brownian paths.

4.2 Noncolliding Brownian bridges

The following is derived by Lemma 4.1.

Proposition 4.3 *The probability densities for the determinantal point processes,* $(\Xi^{R_N}, \mathbf{P}_t^{R_N}, t \in (0, t_*))$*, have the following expressions,*

$$\mathbf{p}_t^{A_{N-1}}(\boldsymbol{x}) = \frac{\det[\mathsf{p}^{\rm circ}(0, \boldsymbol{v}^{A_{N-1}}; t, \boldsymbol{x})] \det[\mathsf{p}^{\rm circ}(t, \boldsymbol{x}; t_*, \boldsymbol{v}^{A_{N-1}})]}{\det[\mathsf{p}^{\rm circ}(0, \boldsymbol{v}^{A_{N-1}}; t_*, \boldsymbol{v}^{A_{N-1}})]}, \quad \boldsymbol{x} \in \mathbb{W}^{[0,2\pi r)},$$

$$\mathbf{p}_t^{R_N}(\boldsymbol{x}) = \frac{\det[\mathsf{p}^{\rm abs}(0, \boldsymbol{v}^{R_N}; t, \boldsymbol{x})] \det[\mathsf{p}^{\rm abs}(t, \boldsymbol{x}; t_*, \boldsymbol{v}^{R_N})]}{\det[\mathsf{p}^{\rm abs}(0, \boldsymbol{v}^{R_N}; t_*, \boldsymbol{v}^{R_N})]}, \quad \boldsymbol{x} \in \mathbb{W}^{[0,\pi r]},$$

for $R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N$,

$$\mathbf{p}_t^{D_N}(\boldsymbol{x}) = \frac{\det[\mathsf{p}^{\rm ref}(0, \boldsymbol{v}^{D_N}; t, \boldsymbol{x})] \det[\mathsf{p}^{\rm ref}(t, \boldsymbol{x}; t_*, \boldsymbol{v}^{D_N})]}{\det[\mathsf{p}^{\rm ref}(0, \boldsymbol{v}^{D_N}; t_*, \boldsymbol{v}^{D_N})]}, \quad \boldsymbol{x} \in \mathbb{W}^{[0,\pi r]}.$$

- **From these expressions in the above Proposition, we can conclude the following.**

Theorem 4.4 (i) *The one-parameter family of determinantal point process,* $(\Xi^{A_{N-1}} \mathbf{P}_t^{A_{N-1}}, t \in (0, t_*))$, *is realized as the particle configuration at time* $t \in (0, t_*)$ *of the noncolliding Brownian bridges on a circle with radius* r, *starting from and returning to the configuration* $\boldsymbol{v}^{A_{N-1}} = (2\pi r(j-1)/N)_{j=1}^{N}$.

(ii) *For* $R_N = B_N, B_N^{\vee}, C_N, C_N^{\vee}, BC_N$, *each one-parameter family of determinantal point process,* $(\Xi^{R_N}, \mathbf{P}_t^{R_N}, t \in (0, t_*))$, *is realized as the particle configuration at time* $t \in (0, t_*)$ *of the noncolliding Brownian bridges starting from and returning to the configuration* $\boldsymbol{v}^{R_N}$ *in the interval* $[0, \pi r]$ *with the absorbing boundary conditions at both edges.*

(iii) *The one-parameter family of determinantal point process,* $(\Xi^{D_N}, \mathbf{P}_t^{D_N}, t \in (0, t_*))$, *is realized as the particle configuration at time* $t \in (0, t_*)$ *of the noncolliding bridges starting from and returning to the configuration* $\boldsymbol{v}^{D_N} = (\pi r(j-1)/(N-1))_{j=1}^{N}$ *in the interval* $[0, \pi r]$ *with the reflecting boundary conditions at both edges.*

5. Concluding Remarks

- In the present paper we have constructed seven types of one-parameter families of determinantal point processes, $(\Xi^{R_N}, \mathbf{P}_t^{R_N}, t \in (0, t_*))$, $R_N = A_{N-1}$, B_N, $B_N^\vee$, C_N, $C_N^\vee$, BC_N, D_N. These point processes can be interpreted as configurations at time $t \in (0, t_*)$ of the noncolliding Brownian bridges starting from and returning to the equidistant configurations $\boldsymbol{v}^{R_N}$.

- In this picture, the variety of elliptic determinantal processes is due to various choices of configurations pinned at the initial time $t = 0$ and at the final time $t = t_*$. If we regard these Brownian bridges on a circle with radius r, $\mathrm{P}^1(r)$, or in an interval $[0, \pi r]$ with time duration t_* as the statistical ensembles of noncolliding paths on the spatio-temporal cylinder $\mathrm{P}^1(r) \times (0, t_*)$ or on the spatio-temporal plane $[0, \pi r] \times (0, t_*)$, $\boldsymbol{v}^{R_N}$ gives a boundary condition to the paths.

- The degeneracy of types in the scaling limit $N \to \infty$, $r \to \infty$ with constant density ρ of paths shown by Theorem 3.6 is caused by vanishing of the boundary effect in this bulk limit.

- **Characterization of the present determinantal point processes $(\Xi^{R_N}, \mathbf{P}_t^{R_N}, t \in (0, t_*))$ in terms of SDEs should be further studied. The noncolliding Brownian bridges discussed in Section 4 are determinantal and the spatio-temporal correlation kernels should be determined.**

- **The corresponding SDEs will be the systems with the drift terms given by the logarithmic derivatives of Macdonald denominators,**

$$dX_j^{R_N}(t) = dB_j(t) + \frac{\partial}{\partial x_j} \log W^{R_N}(\xi(\boldsymbol{x}); \mathcal{N}^{R_N}\tau(t_* - t))\Big|_{\boldsymbol{x}=\boldsymbol{X}^{R_N}(t)} dt, \quad j = 1, 2, \dots, N,$$

where $B_j(t), j = 1, 2, \dots, N$ are independent copies of one-dimensional standard Brownian motions on $\mathbb{R}$ for each type R_N.

- **We write the logarithmic derivatives of Jacobi's theta functions $\theta_\mu(\xi; \tau)$ as**

$$d \log \vartheta_\mu(\xi; \tau) \equiv \frac{\partial}{\partial \xi} \log \vartheta_\mu(\xi; \tau) = \frac{\vartheta'_\mu(\xi; \tau)}{\vartheta_\mu(\xi; \tau)}, \quad \mu = 0, 1, 2, 3,$$

with $\vartheta'_\mu(\xi; \tau) = \partial \vartheta_\mu(\xi; \tau)/\partial \xi$.

- **Then the explicit expressions for the SDEs are given as follows;**

$$
\begin{aligned}
dX_j^{A_{N-1}}(t) &= dB_j(t) + d\log\vartheta_{s(N)}\left(\sum_{\ell=1}^{N} \xi(X_\ell^{R_N}(t)); \mathcal{N}^{A_{N-1}}\tau(t_*-t)\right) dt \\
&\quad + \sum_{\substack{1\le k\le N,\\ k\neq j}} d\log\vartheta_1\Big(\xi(X_j^{A_{N-1}}(t)) - \xi(X_k^{A_{N-1}}(t)); \mathcal{N}^{A_{N-1}}\tau(t_*-t)\Big)dt, \quad j=1,2,\dots,N, \\
&\qquad\qquad t\in(0,t_*), \quad \boldsymbol{X}(t)\in \mathrm{P}^1(r)^N, \\
dX_j^{B_N}(t) &= dB_j(t) + d\log\vartheta_1\Big(\xi(X_j^{B_N}(t)); \mathcal{N}^{B_N}\tau(t_*-t)\Big)dt \\
&\quad + \sum_{1\le k\le N, k\neq j} \Big\{ d\log\vartheta_1\Big(\xi(X_j^{B_N}(t)) - \xi(X_k^{B_N}(t)); \mathcal{N}^{B_N}\tau(t_*-t)\Big) \\
&\qquad + d\log\vartheta_1\Big(\xi(X_j^{B_N}(t)) + \xi(X_k^{B_N}(t)); \mathcal{N}^{B_N}\tau(t_*-t)\Big)\Big\}dt, \quad j=1,2,\dots,N, \\
dX_j^{B_N^\vee}(t) &= dB_j(t) + 2d\log\vartheta_1\Big(\xi(2X_j^{B_N^\vee}(t)); 2\mathcal{N}^{B_N^\vee}\tau(t_*-t)\Big)dt \\
&\quad + \sum_{1\le k\le N, k\neq j} \Big\{ d\log\vartheta_1\Big(\xi(X_j^{B_N^\vee}(t)) - \xi(X_k^{B_N^\vee}(t)); \mathcal{N}^{B_N^\vee}\tau(t_*-t)\Big) \\
&\qquad + d\log\vartheta_1\Big(\xi(X_j^{B_N^\vee}(t)) + \xi(X_k^{B_N^\vee}(t)); \mathcal{N}^{B_N^\vee}\tau(t_*-t)\Big)\Big\}dt, \quad j=1,2,\dots,N, \\
dX_j^{C_N}(t) &= dB_j(t) + 2d\log\vartheta_1\Big(2\xi(X_j^{C_N}(t)); \mathcal{N}^{C_N}\tau(t_*-t)\Big)dt \\
&\quad + \sum_{1\le k\le N, k\neq j} \Big\{ d\log\vartheta_1\Big(\xi(X_j^{C_N}(t)) - \xi(X_k^{C_N}(t)); \mathcal{N}^{C_N}\tau(t_*-t)\Big) \\
&\qquad + d\log\vartheta_1\Big(\xi(X_j^{C_N}(t)) + \xi(X_k^{C_N}(t)); \mathcal{N}^{C_N^\vee}\tau(t_*-t)\Big)\Big\}dt, \quad j=1,2,\dots,N, \\
dX_j^{C_N^\vee}(t) &= dB_j(t) + d\log\vartheta_1\Big(\xi(X_j^{C_N^\vee}(t)); \mathcal{N}^{C_N^\vee}\tau(t_*-t)/2\Big)dt \\
&\quad + \sum_{1\le k\le N, k\neq j} \Big\{ d\log\vartheta_1\Big(\xi(X_j^{C_N^\vee}(t)) - \xi(X_k^{C_N^\vee}(t)); \mathcal{N}^{C_N^\vee}\tau(t_*-t)\Big) \\
&\qquad + d\log\vartheta_1\Big(\xi(X_j^{C_N^\vee}(t)) + \xi(X_k^{C_N^\vee}(t)); \mathcal{N}^{C_N^\vee}\tau(t_*-t)\Big)\Big\}dt, \quad j=1,2,\dots,N, \\
dX_j^{BC_N}(t) &= dB_j(t) + d\log\vartheta_1\Big(\xi(X_j^{BC_N}(t)); \mathcal{N}^{BC_N}\tau(t_*-t)\Big)dt + 2d\log\vartheta_0\Big(2\xi(X_j^{BC_N}(t)); 2\mathcal{N}^{BC_N}\tau(t_*-t)\Big)dt \\
&\quad + \sum_{1\le k\le N, k\neq j} \Big\{ d\log\vartheta_1\Big(\xi(X_j^{BC_N}(t)) - \xi(X_k^{BC_N}(t)); \mathcal{N}^{BC_N}\tau(t_*-t)\Big) \\
&\qquad + d\log\vartheta_1\Big(\xi(X_j^{BC_N}(t)) + \xi(X_k^{BC_N}(t)); \mathcal{N}^{BC_N}\tau(t_*-t)\Big)\Big\}dt, \quad j=1,2,\dots,N, \\
dX_j^{D_N}(t) &= dB_j(t) + \sum_{1\le k\le N, k\neq j} \Big\{ d\log\vartheta_1\Big(\xi(X_j^{D_N}(t)) - \xi(X_k^{D_N}(t)); \mathcal{N}^{D_N}\tau(t_*-t)\Big) \\
&\qquad + d\log\vartheta_1\Big(\xi(X_j^{D_N}(t)) + \xi(X_k^{D_N}(t)); \mathcal{N}^{D_N}\tau(t_*-t)\Big)\Big\}dt, \quad j=1,2,\dots,N, \\
&\qquad t\in(0,t_*), \quad \boldsymbol{X}^{R_N}(t)\in[0,\pi r]^N, \quad R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N.
\end{aligned}
$$

- **An important observation is the fact that as $t \uparrow t_*$, these SDEs show the following asymptotics,**

$$dX_j^{R_N}(t) \sim dB_j(t) + \frac{v_j^{R_N} - X_j^{R_N}(t)}{t_* - t}, \quad j = 1, 2, \dots, N, \quad t \uparrow t_*.$$

- **This implies that these SDEs indeed represent the Brownian bridges pinned at the configurations $\boldsymbol{v}^{R_N}$ at $t = t_*$.**

- **Further study of these SDEs involving elliptic functions are now in progress in the joint work with P. Graczyk (Angers) and J. Małecki (Wroclaw).**

- In [For06] Forrester studied the quantum N-particle systems in two dimensions with doubly periodic boundary conditions, in which the N-body potentials and wave functions are described using the Jacobi theta functions. He constructed the doubly periodic probability measures on a complex plane and discussed solvability and universality of the obtained two-dimensional systems.

- From the view point of the present study, his systems are of type A_{N-1} and they are truly elliptic.

- Generalization of his study to the quantum systems associated with other six types of irreducible reduced affine root systems will be an important future problem.

[For06] Forrester, P. J., "Particles in a magnetic field and plasma analogies: doubly periodic boundary conditions," J. Phys. A: Math. Gen. 39, 13025–13036 (2006).

- **Consider the problem on $\mathbb{C}$ with the double periodicity with periods**

$$L \text{ and } iW, \qquad 0 < L, W < \infty.$$

- **Put**

$$\xi(z) = \xi(z; L) = \frac{z}{L}, \qquad \alpha = \alpha(L, W) = \frac{W}{L},$$

and consider the following versions of Rosengren-Schlosser's functions of $z \in \mathbb{C}$,

$$M_j^{R_N}(z) = M_j^{R_N}(z; L, W) = f_j^{R_N}\Big(\mathcal{N}^{R_N}\xi(z); i\mathcal{N}^{R_N}\alpha\Big), \quad j = 1, 2, \dots, N.$$

- **Then we can prove the following orthogonality with respect to the double integrals with respect to $x = \Re z$ and $y = \Im z$,**

Proposition 5.1 *Let $z = x+iy, x, y \in \mathbb{R}$. If $j, k \in \{1, 2, \dots, N\}$, then for $R_N = A_{N-1}$, B_N, $B_N^\vee$, C_N, $C_N^\vee$, BC_N, D_N,*

$$\int_0^L dx \int_0^W dy \exp\left(-\frac{2\pi \mathcal{N}^{R_N}}{LW} y^2\right) \overline{M_j^{R_N}(z)} M_k^{R_N}(z) = h_j^{R_N} \delta_{jk},$$

where

$$h_j^{A_{N-1}} = \frac{LW}{\sqrt{2N\alpha}} e^{2\pi\alpha(j-1/2)^2/N}, \quad j \in \{1, 2, \dots, N\},$$

$$h_j^{R_N} = \frac{2LW}{\sqrt{2\mathcal{N}^{R_N}\alpha}} e^{2\pi\alpha(J^{R_N}(j))^2/\mathcal{N}^{R_N}}, \quad j \in \{1, 2, \dots, N\}, \quad \textit{for } R_N = C_N, C_N^\vee, BC_N,$$

$$h_j^{R_N} = \begin{cases} \dfrac{4LW}{\sqrt{2\mathcal{N}^{R_N}\alpha}} e^{2\pi\alpha(J^{R_N}(j))^2/\mathcal{N}^{R_N}}, & j = 1, \\[2ex] \dfrac{2LW}{\sqrt{2\mathcal{N}^{R_N}\alpha}} e^{2\pi\alpha(J^{R_N}(j))^2/\mathcal{N}^{R_N}}, & j \in \{2, 3, \dots, N\}, \end{cases} \quad \textit{for } R_N = B_N, B_N^\vee,$$

$$h_j^{D_N} = \begin{cases} \dfrac{4LW}{\sqrt{2\mathcal{N}^{D_N}\alpha}} e^{2\pi\alpha(j-1)^2/\mathcal{N}^{D_N}}, & j \in \{1, N\}, \\[2ex] \dfrac{2LW}{\sqrt{2\mathcal{N}^{D_N}\alpha}} e^{2\pi\alpha(j-1)^2/\mathcal{N}^{D_N}}, & j \in \{2, 3, \dots, N-1\}. \end{cases}$$

- **We can obtain new six families of DPP on $\mathbb{C}$. (The type A_{N-1} was studied by Forrester (2006).) More detail will be reported somewhere else.**

- As mentioned in Section 1, the present determinantal point processes are elliptic extensions of the eigenvalue ensembles of Hermitian random matrices in GUE (and chiral GUE).

- The trigonometric reductions are related with the eigenvalue distributions of random matrices in circular ensembles.

- It will be an interesting future problem to find the statistical ensembles of random matrices in the elliptic level whose eigenvalues realize the present seven types of elliptic determinantal point processes.

Thank you very much
for your attention.