Dynamical universality for the Airy random point fields

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Today's talk

- The universality for random matrices, which is envisioned by E. Wigner, is a central topic in random matrix theory as the CLT for strongly correlated particle systems.
- We consider dynamical universality for random matrices, which is finite particle approximation for infinite dimensional stochastic differential equations (ISDE) as a counterpart of universality for random matrices.

Log-gases with $\beta = 1, 2, 4$ and its macroscopic limit

- For $\beta = 1, 2, 4$, consider β-log-gas on $\mathbb{R}$ with N particles:
$$\mu_{\beta,V}^N(d\mathbf{x}_N) \propto \prod_{i<j}^N |x_i - x_j|^\beta \prod_{k=1}^N e^{-\beta N V(x_k)} d\mathbf{x}_N.$$
Here, $V : \mathbb{R} \to \mathbb{R}$ is free potential of some suitable class.
- Let ρ_V be an equilibrium meas. for V, that is, for $\mathsf{x}^N = \sum_{1\le i\le N} \delta_{x_i}$,
$$\lim_{N\to\infty} \mathbb{E}_{\mu_{\beta,V}^N}[\frac{1}{N}\mathsf{x}^N((-\infty, s])] = \int_{-\infty}^s \rho_V(x)\, dx.$$
- When V is quadratic, $\mu_{\beta,V}^N$ is eigenvalue distribution of G(O/U/S)E for $\beta = 1, 2, 4$ respectively, and ρ_V is nothing but the Wigner semicircle law $\rho_{\rm sc}(x) = \frac{2}{\pi}\sqrt{1-x^2}\mathbf{1}_{\{|x|<1\}}$.
- This convergence is a macroscopic regime for log-gas. Next, we consider a thermodynamical limit for and obtain a random configuration with infinitely many particles as a microscopic regime.

Soft-edge scaling limit and the Airy random point fields

- First we consider the case V is quadratic.
- We take the soft-edge scaling as $x \mapsto \frac{s}{2N^{\frac{2}{3}}} + 1$ and let $\mu^N_{\mathrm{Ai},\beta,V}$ be the prob. meas. w.r.t. s:

$$\mu^N_{\mathrm{Ai},\beta,V}(d\mathbf{s}^N) \propto \prod_{i<j}^{N} |s_i - s_j|^\beta \prod_{k=1}^{N} \exp\Big\{ -\beta N \Big| \frac{s_k}{2N^{\frac{2}{3}}} + 1 \Big|^2 \Big\} d\mathbf{s}^N.$$

- Then it is well known that for $\beta = 1, 2, 4$,

$$\lim_{N\to\infty} \mu^N_{\mathrm{Ai},\beta,V} = \mu_{\mathrm{Ai},\beta} \text{ in law .}$$

Here, $\mu_{\mathrm{Ai},\beta}$ is the Airy$_\beta$ random point field (RPF).
For $\beta = 2$, n-corr. func. $\rho^n_{\mathrm{Ai},2}$ for $\mu_{\mathrm{Ai},2}$ is given by

$$\rho^n_{\mathrm{Ai},2}(\mathbf{x}^n) = \det\Big[\frac{\mathrm{Ai}(x_i)\mathrm{Ai}'(x_j) - \mathrm{Ai}'(x_i)\mathrm{Ai}(x_j)}{x_i - x_j} \Big]_{1\le i,j\le n}.$$

For $\beta = 1, 4$, corr. func. for $\mu_{\mathrm{Ai},\beta}$ have similar expressions.

ISDEs associated with the Airy$_\beta$ RPFs

- We'd like to find a ISDE related to the Airy$_\beta$ RPF.
- Recall that when V is quadratic, then

$$\mu^N_{\mathrm{Ai},\beta,V}(d\mathbf{s}^N) \propto \prod_{i<j}^{N} |s_i - s_j|^\beta \prod_{k=1}^{N} \exp\Big\{ -\beta N \Big| \frac{s_k}{2N^{\frac{2}{3}}} + 1 \Big|^2 \Big\} \, d\mathbf{s}^N,$$

  and consider the Dirichlet form on $L^2(\mathbb{R}^N, \mu^N_{\mathrm{Ai},\beta,x^2})$ given by

$$\mathcal{E}^N(f,g) = \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i=1}^{N} \nabla_i f \cdot \nabla_i g \, d\mu^N_{\mathrm{Ai},\beta,V}.$$

- By integration by parts for this Dirichlet integral, we obtain a generator and deduce the associated SDE: for $1 \le i \le N$,

$$dX^{N,i}_t = dB^i_t + \frac{\beta}{2} \Big\{ \sum_{1 \le j \ne i \le N} \frac{1}{X^{N,i}_t - X^{N,j}_t} - \frac{X^{N,i}_t}{2N^{\frac{1}{3}}} - N^{\frac{1}{3}} \Big\} dt.$$

- A limit formula $N \to \infty$ is supposed to be an ISDE related to the Airy$_\beta$ RPF.
  What is the limit ISDE?

ISDEs associated with the Airy$_\beta$ RPFs

- For $\beta = 1, 2, 4$, [Osada-Tanemura '16 +] proved that the limit ISDE for

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\Big\{ \sum_{1\le j\ne i\le N} \frac{1}{X_t^{N,i} - X_t^{N,j}} - \frac{X_t^{N,i}}{2N^{\frac{1}{3}}} - N^{\frac{1}{3}} \Big\} dt$$

  is given by the following Airy$_\beta$ interacting ISDE

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{s\to\infty} \Big\{ \sum_{|X_t^j|<s, j\ne i} \frac{1}{X_t^i - X_t^j} - \int_{|x|<s} \frac{\hat{\rho}(x)}{-x} dx \Big\} dt,\ i \in \mathbb{N},$$

  where $\hat{\rho}(x) = \frac{\mathbf{1}_{(-\infty,0)}(x)}{\pi}\sqrt{-x}$.
- In other word, the (labeled) distorted Brownian motion w.r.t. the Airy$_\beta$ RPF solves the Airy$_\beta$ interacting ISDE.

Soft-edge universality for log-gases

- Let $\mu^N_{\beta,V}$ for $V(x)=\sum_{i=0}^{2l}\kappa_i x^i$ $(\kappa_{2l}>0)$ and the soft-edge scaling
$$x\mapsto N^{-\frac{1}{2l}}\Big\{c_N\Big(1+\frac{s}{\alpha_N N^{\frac{2}{3}}}\Big)+d_N\Big\},$$
then the prob. meas. $\mu^N_{\mathrm{Ai},\beta,V}$ is given by
$$\mu^N_{\mathrm{Ai},\beta,V}(d\mathbf{s}^N)\propto\prod_{i<j}^{N}|s_i-s_j|^\beta$$
$$\times\prod_{k=1}^{N}\exp\Big\{-\beta NV\Big(N^{-\frac{1}{2l}}\Big\{c_N\Big(1+\frac{s}{\alpha_N N^{\frac{2}{3}}}\Big)+d_N\Big\}\Big)\Big\}\,d\mathbf{s}^N.$$
Here, c_N,α_N,d_N are constants depend only on N and V.
- For $\beta=1,2,4$,
$$\lim_{N\to\infty}\mu^N_{\mathrm{Ai},\beta,V}=\mu_{\mathrm{Ai},\beta}\text{ in law.}$$
- $\mu_{\mathrm{Ai},\beta}$ is independent of V (the Airy$_\beta$ RPF is universal).
- The soft-edge universality was proven for more general V, but for a certain reason we consider even degree polynomial (explain later).

Dynamical universality

- We'd like to formulate dynamical version of universality for RM.
- Recalling

$$\mu^N_{\mathrm{Ai},\beta,V}(d\mathbf{s}^N) \propto \prod_{i<j}^{N} |s_i - s_j|^\beta \times \prod_{k=1}^{N} \exp\Big\{ - \frac{\beta}{2} NV\Big(N^{-\frac{1}{2l}}\Big\{c_N\Big(1+\frac{s}{\alpha_N N^{\frac{2}{3}}}\Big)+d_N\Big\}\Big)\Big\}\, d\mathbf{s}^N,$$

from the same procedure as $V(x) = x^2$, we deduce the associated SDE : for $1 \le i \le N$,

$$dX^{N,i}_t = dB^i_t + \frac{\beta}{2}\Bigg\{ \sum_{1\le j\ne i\le N} \frac{1}{X^{N,i}_t - X^{N,j}_t} dt - \frac{N^{\frac{1}{3}-\frac{1}{2l}} c_N}{2\alpha_N} V'\Big(N^{-\frac{1}{2l}}\Big\{c_N\Big(1+\frac{X^{N,i}_t}{\alpha_N N^{\frac{2}{3}}}\Big)+d_N\Big\}\Big)\Bigg\} dt, .$$

Dynamical universality

- It is supposed that the limit ISDE for

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\Bigg\{ \sum_{1\le j\ne i\le N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{N^{\frac{1}{3}-\frac{1}{2l}} c_N}{2\alpha_N} V'\Bigg(N^{-\frac{1}{2l}} \Big\{ c_N \Big(1 + \frac{X_t^{N,i}}{\alpha_N N^{\frac{2}{3}}} \Big) + d_N \Big\} \Bigg) \Bigg\} dt,$$

as $N \to \infty$ is the Airy$_\beta$ interacting ISDE given by

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{s\to\infty} \Bigg\{ \sum_{|X_t^j|<s, j\ne i} \frac{1}{X_t^i - X_t^j} - \int_{|x|<s} \frac{\hat{\rho}(x)}{-x} dx \Bigg\} dt.$$

Here, the limit is independent of V (dynamical universality).

- The Airy$_\beta$ interacting ISDE is a universal dynamical object.
- We expect geometrical universality derives dynamical universality.
- How to prove this limit transition?

How to prove the dynamical universality

- $$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\Bigg\{ \sum_{1\le j\ne i\le N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{N^{\frac{1}{3}-\frac{1}{2l}} c_N}{2\alpha_N} V'\left(N^{-\frac{1}{2l}}\left\{c_N\left(1+\frac{X_t^{N,i}}{\alpha_N N^{\frac{2}{3}}}\right)+d_N\right\}\right)\Bigg\}dt,$$
$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{s\to\infty}\left\{ \sum_{|X_t^j|<s, j\ne i} \frac{1}{X_t^i - X_t^j} - \int_{|x|<s} \frac{\hat\rho(x)}{-x} dx \right\} dt.$$
- One way to prove the limit transition is to calculate the drift term, like Osada-Tanemura's argument for quadratic V, but such argument is difficult (even for the simplest case $V(x) = x^2$, it involves hard analysis).
- To avoid such model dependent hard calculation, we constructed a general framework such that geometrical universality derives dynamical universality automatically.

Strong convergence for RPFs

- We saw that for $\beta = 1, 2, 4$,
$$\lim_{N\to\infty} \mu^N_{\mathrm{Ai},\beta,V} = \mu_{\mathrm{Ai},\beta} \text{ in law.}$$
- One requirement for the dynamical universality is strong convergence for RPFs in the following sense: for any $n \in \mathbb{N}$,
$$\lim_{N\to\infty} \rho^{N,n}_{\mathrm{Ai},\beta,V} = \rho^n_{\mathrm{Ai},\beta} \quad \text{compact uniformly,}$$
Here $\rho^{N,n}_{\mathrm{Ai},\beta,V}$ and $\rho^n_{\mathrm{Ai},\beta}$ are n-corr. func's for $\mu^N_{\mathrm{Ai},\beta,V}$ and $\mu_{\mathrm{Ai},\beta}$ resp.
- We quote the following strong convergence result:

Lemma 1 ('07 Deift-Gioev)

For $\beta = 1, 2, 4$ and $V(x) = \sum_{i=0}^{2l} \kappa_i x^i$ $(\kappa_{2l} > 0)$, then for any $n \in \mathbb{N}$,
$$\lim_{N\to\infty} \rho^{N,n}_{\mathrm{Ai},\beta,V} = \rho^n_{\mathrm{Ai},\beta} \quad \textit{compact uniformly.}$$

- This lemma (and non-colliding property for Airy$_\beta$ interacting ISDE) deduces the next result.

Dynamical soft-edge universality

Theorem 1 (K. -Osada 17+ for $\beta = 2$, K. 18+ for $\beta = 1, 4$)

For $\beta = 1, 2, 4$ *and* $V(x) = \sum_{i=0}^{2l} \kappa_i x^i$, *let* $(X^{N,1}, \dots, X^{N,N})$ *be a solution with equilibrium initial distribution for*

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\Bigg\{ \sum_{1 \le j \ne i \le N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{N^{\frac{1}{3}-\frac{1}{2l}} c_N}{2\alpha_N} V'\Big(N^{-\frac{1}{2l}} \Big\{ c_N \Big(1 + \frac{X_t^{N,i}}{\alpha_N N^{\frac{2}{3}}} \Big) + d_N \Big\} \Big) \Bigg\} dt.$$

Then there exists a stoch. proc. $(X^1, X^2, \dots) \in C([0,\infty], \mathbb{R}^{\mathbb{N}})$ *satisfying*

$$dX_t^i = dB_t^i + \frac{\beta}{2}\Bigg\{ \lim_{s\to\infty} \Big\{ \sum_{|X_t^j|<s, j\ne i} \frac{1}{X_t^i - X_t^j} - \int_{|x|<s} \frac{\hat\rho(x)}{-x} dx \Big\} dt \Bigg\},$$

with equilibrium initial distribution such that for any $m \in \mathbb{N}$

$$\lim_{N\to\infty} (X^{N,1}, \dots, X^{N,m}) = (X^1, \dots, X^m) \text{ in law in } C([0,\infty], \mathbb{R}^m).$$

Here $\hat\rho(x) = \mathbf{1}_{(-\infty,0)}(x) \pi^{-1} \sqrt{-x}$.

Concluding remarks & Summary

- We see that the "strong" universality for random matrices derives dynamical version.
- The universality for random matrices have been generalized intensively ([Bourgade-Erdös-Yau 2014], etc.), but many results show only weak convergence of correlation functions.
  If we improve their weak convergence results to strong convergence results, accordingly our approach can prove the dynamical universality.