Two-periodic Aztec diamond and matrix valued orthogonal polynomials

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Random Matrices and their Applications

Kyoto University

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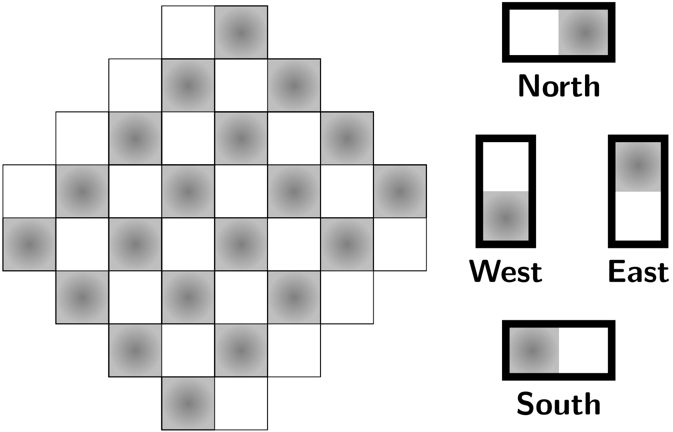
Outline

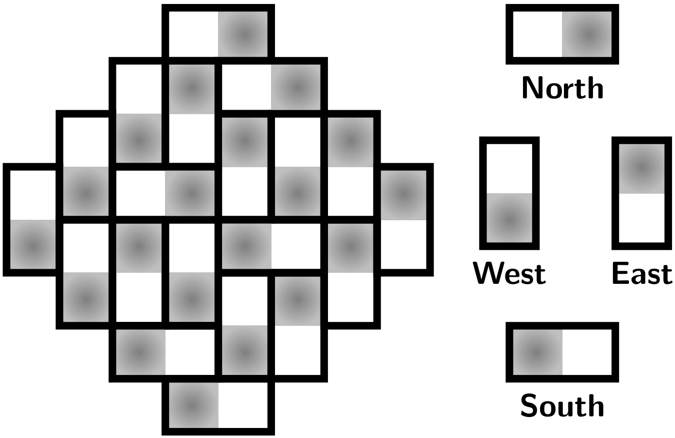
1. **Aztec diamond**
2. **Hexagon tilings**
3. **The two periodic model**
4. **Non-intersecting paths**
5. **Determinantal point processes**
6. **New result for periodic** T_m
7. **Matrix Valued Orthogonal Polynomials (MVOP)**
8. **Results for the Aztec diamond**
9. **Results for the hexagon**

1. Aztec diamond

Aztec diamond

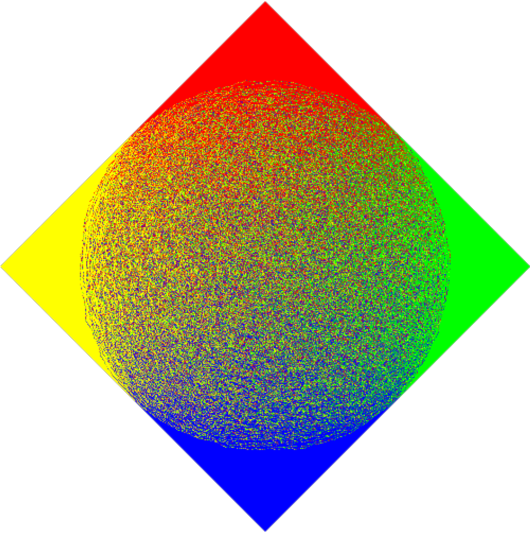


Tiling of an Aztec diamond



- Tiling with 2×1 and 1×2 rectangles (dominos)
- Four types of dominos

Large random tiling



Deterministic pattern near corners

Solid region

or

Frozen region

Disorder in the middle

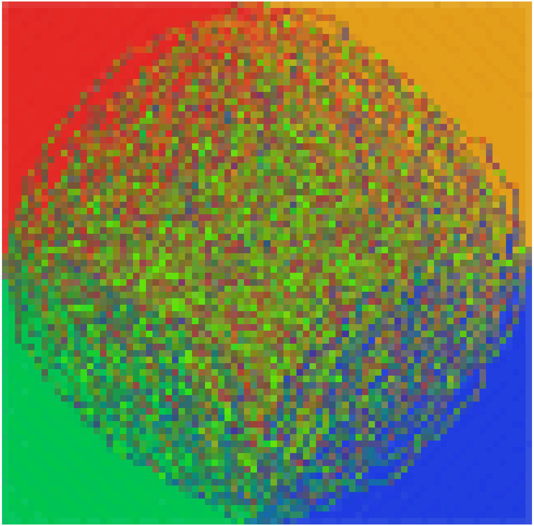
Liquid region

Boundary curve

Arctic circle

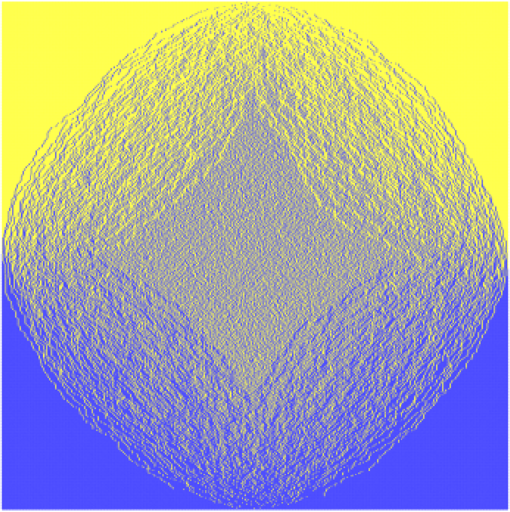
Recent development

- **Two-periodic weighting** Chhita, Johansson (2016) Beffara, Chhita, Johansson (2018 to appear)

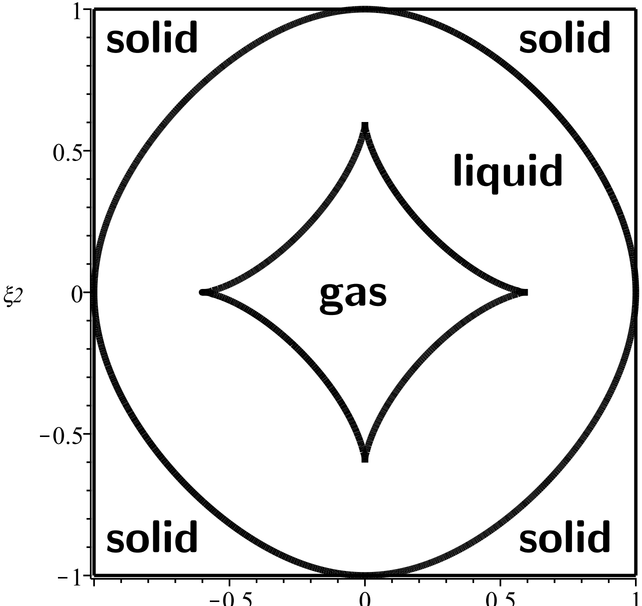


Two-periodic weights

- **A new phase within the liquid region: gas region**

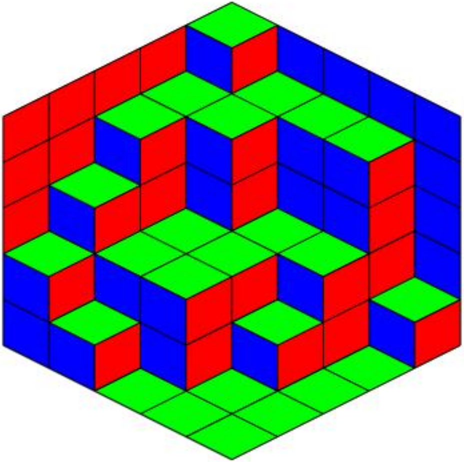


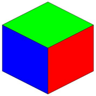
Phase diagram



2. Hexagon tilings

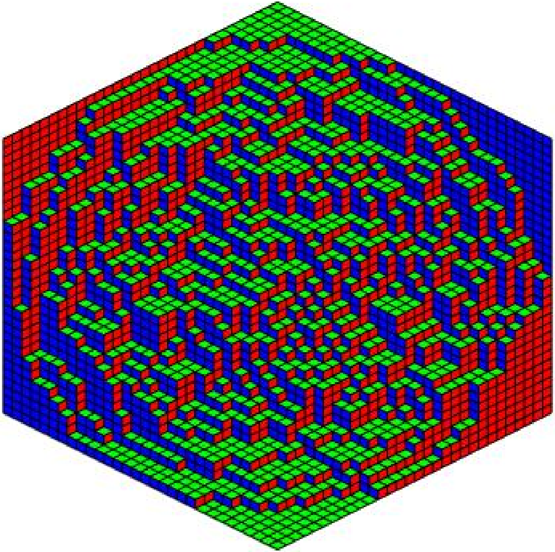
Lozenge tiling of a hexagon



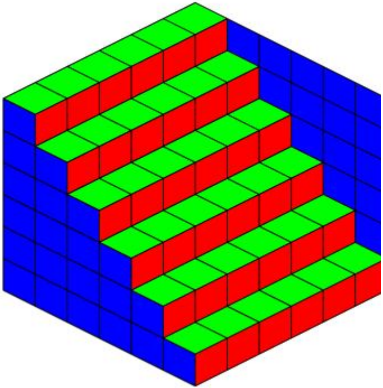


three types of lozenges

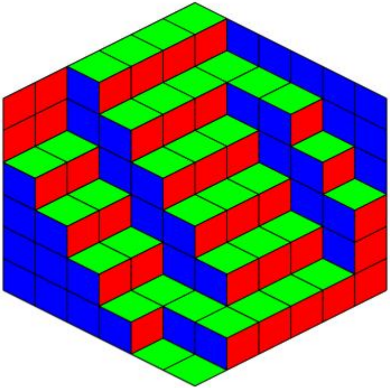
Arctic circle phenomenon



Two periodic hexagon (size 6)

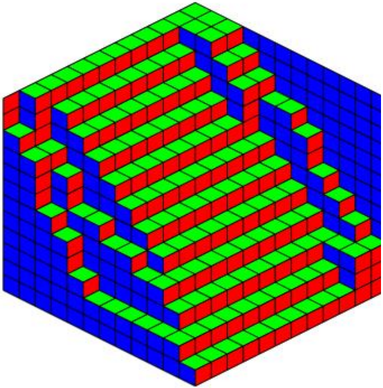


$\alpha = 0$

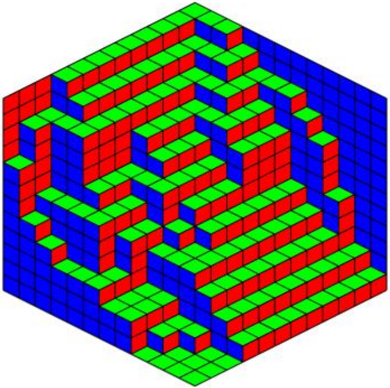


$\alpha = 0.1$

Two periodic hexagon (size 30)

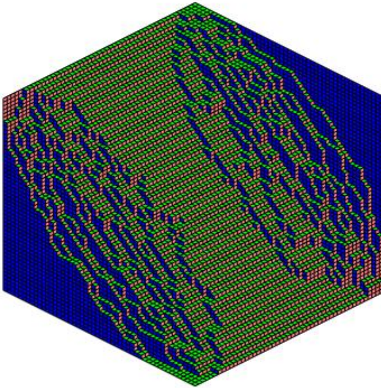


$\alpha = 0.1$

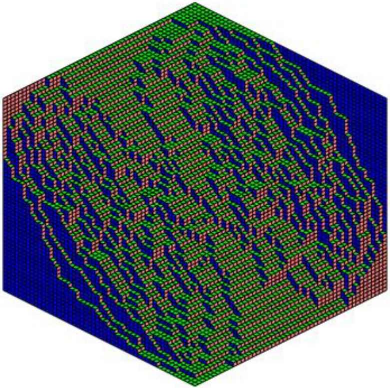


$\alpha = 0.18$

Two periodic hexagon (size 50)

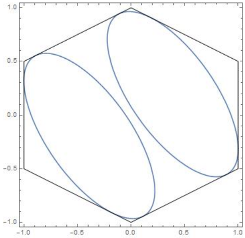


$\alpha = 0.1$

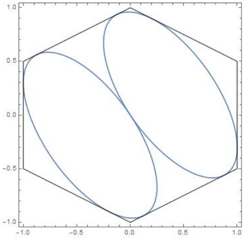


$\alpha = 0.15$

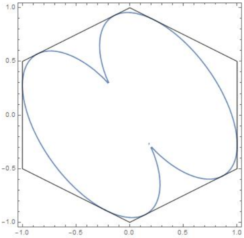
Phase Diagrams



$\alpha < 1/9$,



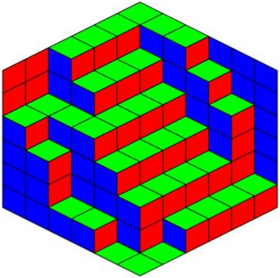
$\alpha = 1/9$,

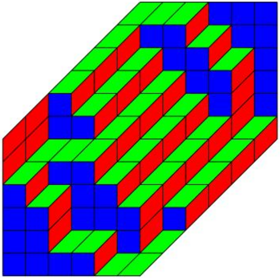


$\alpha > 1/9$

3. The two periodic model

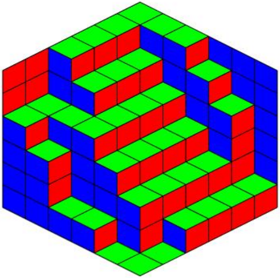
Oblique hexagon and weights

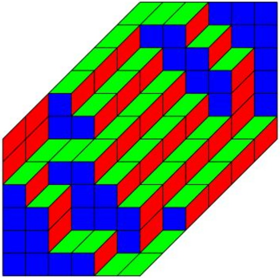




- **Vertices are on the integer lattice $\mathbb{Z}^2$**

Oblique hexagon and weights





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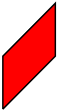
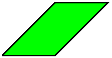


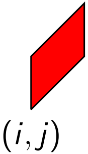
(i, j)

has weight $\begin{cases} \alpha & \textbf{if } i+j \textbf{ is even}, \\ 1 & \textbf{if } i+j \textbf{ is odd}, \end{cases}$

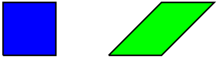
 

have weight 1

Weight



has weight $\begin{cases} \alpha & \textbf{if } i+j \textbf{ is even,} \\ 1 & \textbf{if } i+j \textbf{ is odd,} \end{cases}$



have weight 1

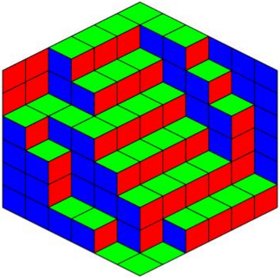
- **Weight of a tiling** T **is the product of the weights of the lozenges in the tiling.**
- **Probability is proportional to the weight**

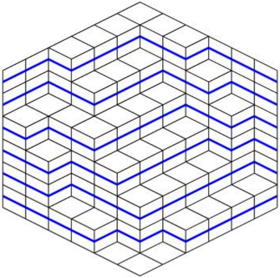
$$\textbf{Prob}(T) = \frac{w(T)}{Z_N}$$

where $Z_N = \sum_T w(T)$ **is the normalizing constant (partition function)**

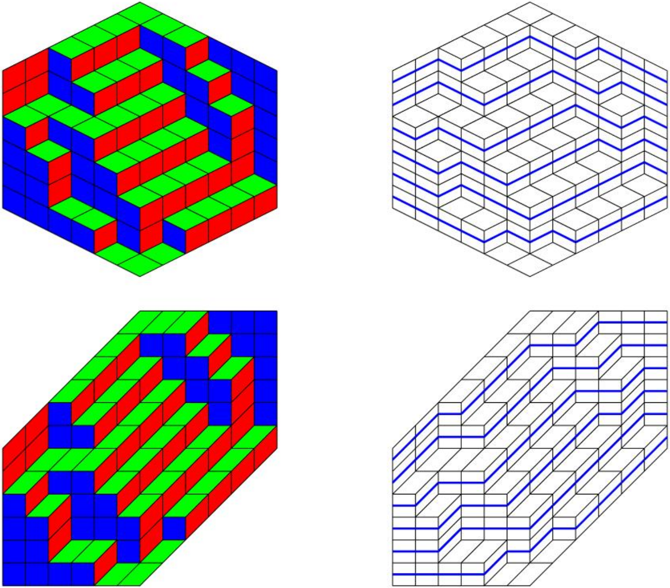
4. Non-intersecting paths

Non-intersecting paths



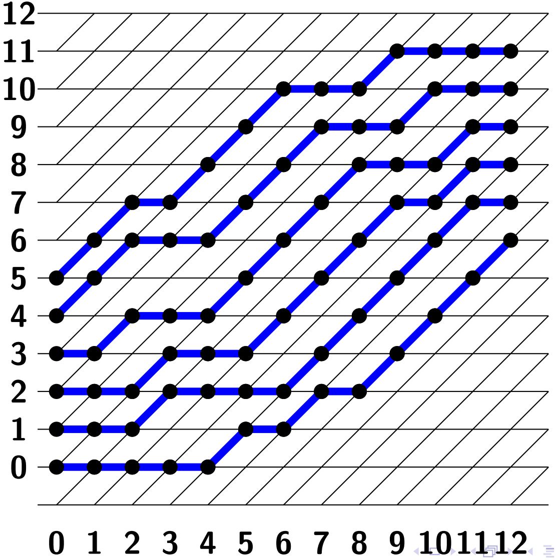


Non-intersecting paths



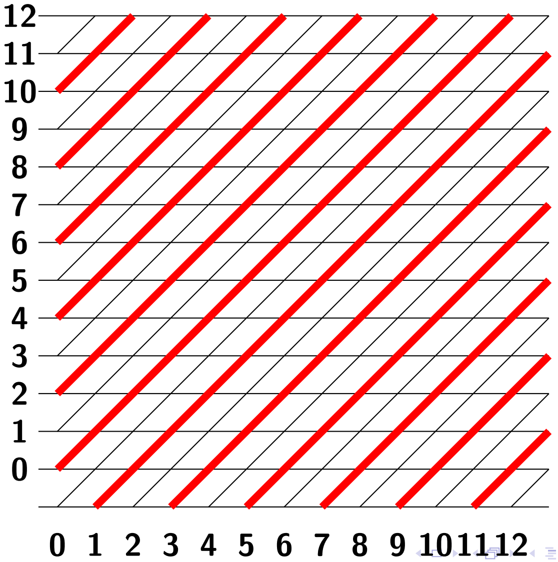
Non-intersecting paths on a graph

Paths fit on a graph

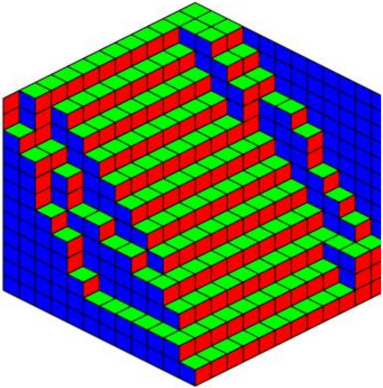


Weights on the graph

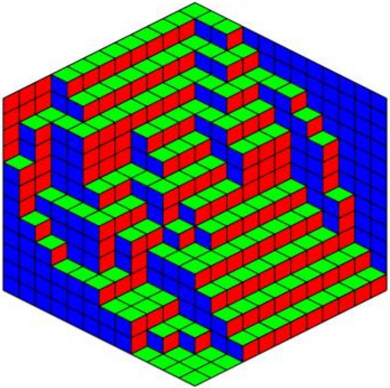
Red edges carry weight α, Other edges have weight 1



Two periodic hexagon (size 30)



$\alpha = 0.1$



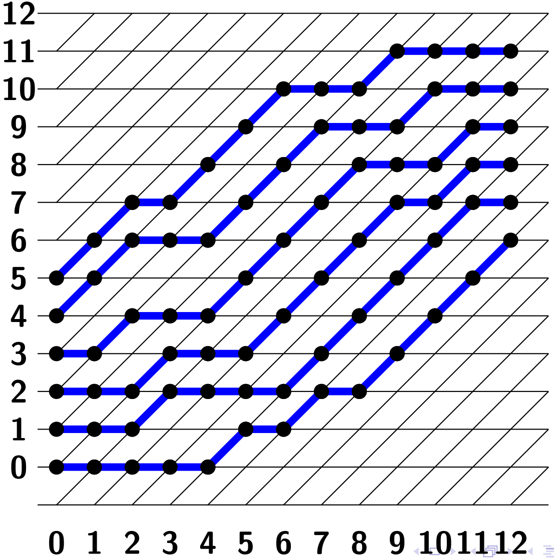
$\alpha = 0.18$

- **For $0 < \alpha < 1$: punishment to cover the red edges.**
- **Appearance of the staircase region in the middle.**

5. Determinantal point process : known results

Particle configuration

Focus on positions of particles along the paths.



Transitions and LGV theorem

Particles at level m: $x_j^{(m)}$, $j = 0, \ldots, N-1$.

Proposition

$$\textbf{Prob}\left((x_j^{(m)})_{j=0,m=1}^{N-1,2N-1} \right) = \frac{1}{Z_n} \prod_{m=0}^{2N-1} \det \left[T_m(x_j^{(m)}, x_k^{(m+1)}) \right]_{j,k=0}^{N-1}$$

with $x_j^{(0)} = j$, $x_j^{(2N)} = N + j$ and transition matrices

$$T_m(x,x) = 1$$

$$T_m(x, x+1) = \begin{cases} \alpha, & \textbf{if } m + x \textbf{ is even}, \\ 1, & \textbf{if } m + x \textbf{ is odd}, \end{cases}$$

$$T_m(x,y) = 0 \quad \textbf{otherwise}, \qquad x, y \in \mathbb{Z}$$

This follows from Lindström Gessel Viennot lemma.

Lindström (1973) **Gessel-Viennot (1985)**

Determinantal point process

Such a product of determinants defines a determinantal point process on $\mathcal{X} = \{0, \ldots, 2N\} \times \mathbb{Z}$:

Corollary

There is a correlation kernel $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that for every finite $\mathcal{A} \subset \mathcal{X}$

$$\textbf{Prob}\left[\exists \textbf{ particle at each } (m, x) \in \mathcal{A}\right] = \det\left[K((m,x),(m',x'))\right]_{(m,x),(m',x') \in \mathcal{A}}$$

Eynard Mehta formula

Notation for $m < m'$

$$T_{m,m'} = T_{m'-1} \cdot \cdots \cdot T_{m+1} \cdot T_m$$

is transition matrix from level m **to level** m'**, and**

$$G = [T_{0,2N}(i,j)]_{i,j=0}^{2N-1}$$

is finite section of $T_{0,2N}$**.**

Eynard-Mehta (1998) formula for correlation kernel

$$K((m,x),(m',x')) = -\chi_{m>m'} T_{m',m}(x',x) + \sum_{i,j=0}^{2N-1} T_{0,m}(i,x) \left[G^{-1}\right]_{j,i} T_{m',2N}(x',j)$$

- **How to invert the matrix** G**?**

6. Determinantal point process: new result for periodic T_m

Periodic transition matrices

T_m **is** 2**-periodic:** $T_m(x+2, y+2) = T_m(x, y)$ **for** $x, y \in \mathbb{Z}$

Block Toeplitz matrix $T_m = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & B_0 & B_1 & \ddots & \\ \ddots & B_{-1} & B_0 & B_1 & \ddots \\ & \ddots & B_{-1} & B_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$

with **block symbol**

$$A_m(z) = \sum_{j=-\infty}^{\infty} B_j z^j = B_0 + B_1 z = \begin{cases} \begin{pmatrix} 1 & \alpha \\ z & 1 \end{pmatrix} & \text{if } m \text{ is even}, \\ \begin{pmatrix} 1 & 1 \\ \alpha z & 1 \end{pmatrix} & \text{if } m \text{ is odd}. \end{cases}$$

- **Notation** $A(z) = A_1(z) A_0(z)$

Double contour integral formula

Theorem (Duits + K for this special case)

Suppose hexagon of size $2N$. Then

$$\begin{pmatrix} K(2m,2x;2m',2y) & K(2m,2x+1;2m',2y) \\ K(2m,2x;2m',2y+1) & K(2m,2x+1,2m',2y+1) \end{pmatrix}$$

$$= -\frac{\chi_{m>m'}}{2\pi i} \oint_\gamma A^{m-m'}(z) z^{y-x} \frac{dz}{z}$$

$$+ \frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma A^{2N-m'}(w) R_N(w,z) A^m(z) \frac{w^y}{z^{x+1} w^{2N}} dz dw$$

where $R_N(w,z)$ is a reproducing kernel for matrix valued polynomials with respect to weight matrix

$$W_N(z) = \frac{A^{2N}(z)}{z^{2N}} = \frac{1}{z^{2N}} \begin{pmatrix} 1+z & 1+\alpha \\ (1+\alpha)z & 1+\alpha^2 z \end{pmatrix}^{2N}$$

7. Matrix Valued Orthogonal Polynomials (MVOP)

MVOP

- **Matrix valued polynomial** $P_j(z) = \sum_{i=0}^{j} C_i z^i$
- **Orthogonality**

$$\frac{1}{2\pi i} \oint_{\gamma} P_j(z) W_N(z) P_k^t(z)\, dz = H_j \delta_{j,k}$$

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Definition

Reproducing kernel for matrix polynomials

$$R_N(w,z) = \sum_{j=0}^{N-1} P_j^t(w) H_j^{-1} P_j(z)$$

- **If** Q **has degree** $\leq N-1$**, then**

$$\frac{1}{2\pi i}\oint_\gamma Q(w) W_N(w) R_N(w,z) dw = Q(z)$$

Riemann-Hilbert problem

- **There is a Christoffel-Darboux formula for R_N and a Riemann Hilbert problem for MVOP**

$Y : \mathbb{C} \setminus \gamma \to \mathbb{C}^{4\times 4}$ **satisfies**

- Y **is analytic,**
- $Y_+ = Y_- \begin{pmatrix} I_2 & W_N \\ 0_2 & I_2 \end{pmatrix}$ **on** γ,
- $Y(z) = (I_4 + O(z^{-1})) \begin{pmatrix} z^N I_2 & 0_2 \\ 0_2 & z^{-N} I_2 \end{pmatrix}$ **as** $z \to \infty$.

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Christoffel Darboux formula

$$R_N(w,z) = \frac{1}{z-w} \begin{pmatrix} 0_2 & I_2 \end{pmatrix} Y^{-1}(w) Y(z) \begin{pmatrix} I_2 \\ 0_2 \end{pmatrix}$$

Delvaux (2010)

Matrix weights and genus

Lozenge tiling of hexagon

- $A(z) = \begin{pmatrix} 1+z & 1+\alpha \\ (1+\alpha)z & 1+\alpha^2 z \end{pmatrix}$ **has eigenvalues**

$$1 + \tfrac{1+\alpha^2}{2} z \pm \tfrac{1-\alpha^2}{2} \sqrt{z\left(z + \tfrac{4}{(1-\alpha)^2}\right)}$$

that "live" on $y^2 = z\left(z + \frac{4}{(1-\alpha)^2}\right)$ $\rightarrow$ **genus zero**

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Two periodic Aztec diamond

- **Similar analysis leads to** $\begin{pmatrix} 2\alpha z & \alpha(z+1) \\ \alpha^{-1} z(z+1) & 2\alpha^{-1} z \end{pmatrix}$ **with eigenvalues**

$$(\alpha + \alpha^{-1}) z \pm \sqrt{z(z+\alpha^2)(z+\alpha^{-2})}$$

$\rightarrow$ **genus one and this leads to gas phase**

8. Results for Aztec diamond

Explicit formulas

- **MVOP of degree N is explicit for N even**

$$P_N(z) = (z-1)^N z^{N/2} A^{-N}(z)$$

- **Explicit formula for correlation kernel (double contour part only)**

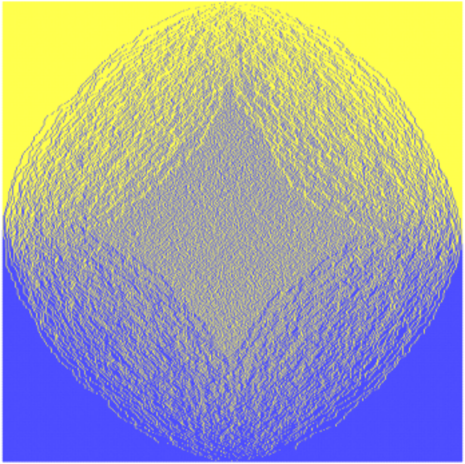
$$\frac{1}{(2\pi i)^2} \oint_{\gamma_{0,1}} \frac{dz}{z} \oint_{\gamma_1} \frac{dw}{z-w} A^{N-m'}(w) F(w) A^{-N+m}(z) \times \frac{z^{N/2}(z-1)^N}{w^{N/2}(w-1)^N} \frac{w^{(m'+n')/2}}{z^{(m+n)/2}}$$

with $F(w) = \frac{1}{2} I_2$

$$+ \frac{1}{2\sqrt{w(w+\alpha^2)(w+\alpha^{-2})}} \begin{pmatrix} (\alpha-\alpha^{-1})w & \alpha(w+1) \\ \alpha^{-1}w(w+1) & -(\alpha-\alpha^{-1})w \end{pmatrix}$$

Steepest descent

- **Classical steepest descent** for integrals on the Riemann surface explains the phases and transitions between phases



9. Results for hexagon

Scalar orthogonality

MVOP for two periodic hexagon are expressed in terms of scalar OP of degree $2N$

$$\frac{1}{2\pi i}\oint_{\gamma_1} P_{2N}(\zeta)\left(\frac{(\zeta-\alpha)^2}{\zeta(\zeta-1)^2}\right)^{2N}\zeta^k d\zeta = 0, \qquad k = 0, 1, \ldots, 2N-1.$$

- **Non-hermitian orthogonality with respect to varying weight**

Scalar orthogonality

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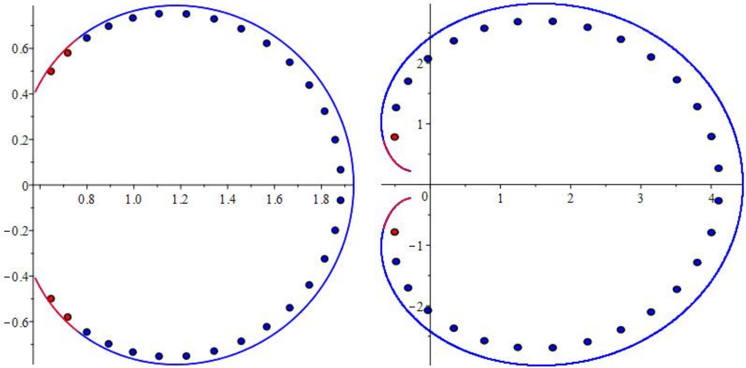
$$k = 0, 1, \ldots, 2N-1.$$

- **Non-hermitian orthogonality with respect to varying weight**
- **We can see the phase transition at $\alpha = 1/9$ in the behavior of the zeros of P_{2N} as $N \to \infty$.**

Zeros



$\alpha = 1/2$ $\alpha = 1/8$

- **Curve closes for $\alpha = 1/9$.**
- **Analysis uses logarithmic potential theory, *S*-curves in external field, and the Riemann-Hilbert problem**

Thank you for your attention