The Euler characteristic method for the largest eigenvalues of random matrices

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Random matrices and their applications

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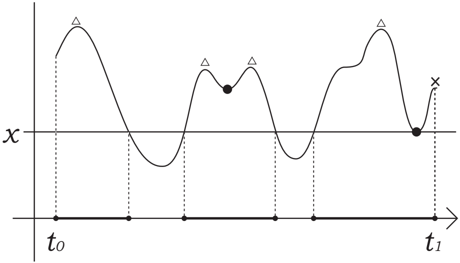
The Euler characteristic method

- $X(t)$, $t \in M$: random field with smooth sample path
- Excursion set

$$M_x = \{t \in M \mid X(t) \geq x\}$$



$$M = [t_0, t_1], \quad \chi(M_x) = 3 \quad \text{(Euler characteristic)}$$

- The Euler characteristic method

$$\Pr\left(\sup_{t \in M} X(t) \geq x\right) \approx E[\chi(M_x)] \quad \text{when } x \text{ is large}$$

- Useful in statistical testing hypothesis, i.e., p-value.

The largest eigenvalue of a Wishart matrix

- The largest eigenvalue is the maximum of a random field:

$$\lambda_1(A) = \max_{U \in M} \mathrm{tr}(UA), \quad M = \begin{cases} \{hh^\top \mid \|h\| = 1\} & \text{(real Wishart)} \\ \{hh^* \mid \|h\| = 1\} & \text{(complex W)} \end{cases}$$

Lemma (Morse's theorem)

The Euler characteristic of the excursion set $M_x = \{U \in M \mid \mathrm{tr}(UA) \geq x\}$ *is*

$$\chi(M_x) = \begin{cases} \sum_{k=1}^n (-1)^{k-1} \mathbb{1}\{\lambda_k(A) \geq x\} & (\textit{real Wishart}) \\ \sum_{k=1}^n \mathbb{1}\{\lambda_k(A) \geq x\} & (\textit{complex Wishart}) \end{cases}$$

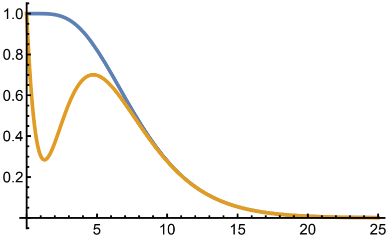
EC method

Theorem

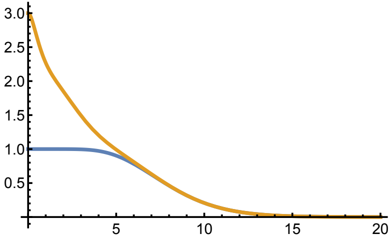
Let $A \sim W_n(N, I_n)$ *or* $CW_n(N, I_n)$. *Let* $\alpha = N - n$.

$$E[\chi(M_x)] = \begin{cases} \frac{\sqrt{\pi}(-1)^{n-1}(n-1)!}{2^{\frac{N+n-1}{2}}\Gamma(\frac{N}{2})\Gamma(\frac{n}{2})} \int_x^\infty \lambda^{\frac{N-n-1}{2}} e^{-\frac{\lambda}{2}} d\lambda \, L_{n-1}^{(\alpha)}(\lambda) & (\textit{real}) \\ \frac{n!}{\Gamma(N)} \int_x^\infty \lambda^{N-n} e^{-\lambda} d\lambda & \\ \times \{ L_{n-1}^{(\alpha)}(\lambda) L_{n-1}^{(\alpha+1)}(\lambda) - L_n^{(\alpha)}(\lambda) L_{n-2}^{(\alpha+1)}(\lambda) \} & (\textit{complex}) \end{cases}$$

- Upper prob of $\lambda_1(A)$ (blue) and the EC method (orange)



real Wishart 3×3, df=4



complex Wishart 3×3, df=4

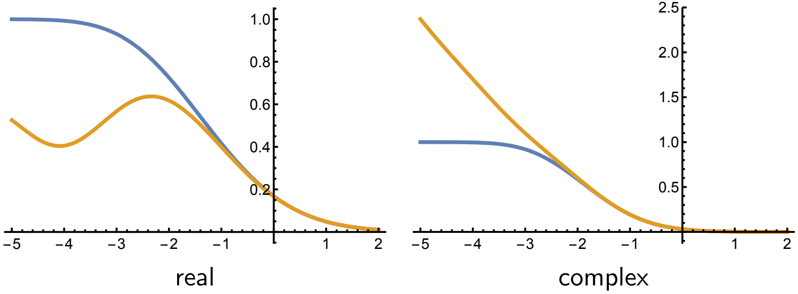
Edge asymptotics

Theorem

Let $A \sim W_n(N, I_n)$ *or* $CW_n(N, I_n)$. *As* $N, n \to \infty$ *s.t.* $N/n \to \gamma$,

$$E[\chi(M_x)]\Big|_{x=\mu_+ + \sigma s} \to \begin{cases} \frac{1}{2}\int_x^\infty \mathrm{Ai}(x)dx & (\textit{real}) \\ \int_x^\infty \{\mathrm{Ai}'(x)^2 - \mathrm{Ai}(x)^2\}dx & (\textit{complex}) \end{cases}$$

- Tracy-Widom (blue) and the EC method (orange)



Other applications

1. Gaussian and beta (MANOVA) matrices can be dealt with in the same way.
2. By changing the index manifold M,

$$\max_{U \in M} \operatorname{tr}(AU)$$

represents various functions of A, e.g.,

- The range of eigenvalues

$$\lambda_1(A) - \lambda_n(A)$$

- Partial sum of the largest eigenvalues

$$\lambda_1(A) + \cdots + \lambda_r(A) \quad (r < n)$$

- The largest singular-value $\sigma_1(A)$
  (when A is not real symmetric/Hermitian).

The EC method works for them.