Weingarten calculus and counting paths on Weingarten graphs

Sho Matsumoto

Kagoshima University, JAPAN

Random matrices and their applications, Kyoto University

Overview

Weingarten calculus

It is a method for computations of mixed moments

$$\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n}] \qquad \text{or} \qquad \mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n} \overline{x_{k_1 l_1} x_{k_2 l_2} \cdots x_{k_m l_m}}]$$

where $X = (x_{ij})$ is a random matrix picked up from a classical compact Lie group.

History:
Don Weingarten (1978), Benoît Collins (2003), B.C. & Piotr Śniady (2006), ...

Today's topics:

- Weingarten calculus on Lie groups $\mathrm{U}(d), \mathrm{O}(d), \mathrm{Sp}(d)$;
- Weingarten calculus on symmetric spaces G/K (COE, chiral unitary matrix);
- Weingarten graphs (joint work with Benoît Collins).

1. Weingarten Calculus for Unitary Groups
2. Weingarten Calculus for Orthogonal Groups
3. Weingarten Calculus for Symplectic Groups
4. Weingarten Calculus for Symmetric Spaces
   - A I – circular orthogonal ensemble (COE)
   - $\mathrm{A\,III}$ – chiral unitary ensemble (chUE)
5. Weingarten Graphs (joint work with Benoît Collins)
   - Unitary group $\mathrm{U}(d)$
   - $\mathrm{A\,III}$ – chiral unitary ensemble (chUE)

Weingarten calculus for unitary groups

$G = \mathrm{U}(d) = \{g \in \mathrm{GL}(d, \mathbb{C}) \mid g\, g^* = I_d\}$. (CUE = circular unitary ensemble)

Any compact Lie group G has the normalized Haar measure $\mu = \mu_G$ such that

$$\int_G f(g_1\, g\, g_2)\, \mu(dg) = \int_G f(g)\, \mu(dg), \qquad \int_G \mu(dg) = 1,$$

where f is any continuous function on G, and g_1, g_2 are any elements in G.

Weingarten calculus for unitary groups

$G = \mathrm{U}(d) = \{g \in \mathrm{GL}(d, \mathbb{C}) \mid g\, g^* = I_d\}$. (CUE = circular unitary ensemble)

Any compact Lie group G has the normalized Haar measure $\mu = \mu_G$ such that

$$\int_G f(g_1\, g\, g_2)\, \mu(dg) = \int_G f(g)\, \mu(dg), \qquad \int_G \mu(dg) = 1,$$

where f is any continuous function on G, and g_1, g_2 are any elements in G.

Let $U = (u_{ij})_{1 \le i,j \le d}$ be a random matrix distributed with respect to $\mu_{\mathrm{U}(d)}$. Consider

$$\mathbb{E}[u_{i_1 j_1}\, u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1}\, u_{i'_2 j'_2} \cdots u_{i'_m j'_m}}]$$

where i_p, j_p, i'_p, j'_p are entries in $\{1, 2, \dots, d\}$. Here $\mathbb{E}$ stands for the expectation (with respect to $\mu_{\mathrm{U}(d)}$). For example, we will compute $\mathbb{E}[u_{11}\, u_{22}\, u_{33} \overline{u_{12}\, u_{23}\, u_{31}}]$.

Fact

The expectation $\mathbb{E}[\cdots]$ vanishes unless $n = m$.

Weingarten calculus for unitary groups

Theorem (Collins, 2003)

Given four sequences

$$\boldsymbol{i} = (i_1, \dots, i_n), \quad \boldsymbol{j} = (j_1, \dots, j_n), \quad \boldsymbol{i}' = (i'_1, \dots, i'_n), \quad \boldsymbol{j}' = (j'_1, \dots, j'_n)$$

in $\{1, 2, \dots, d\}^{\times n}$*, we have*

$$\mathbb{E}\left[u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_n j'_n}} \right] = \sum_{\sigma \in \mathfrak{S}_n} \sum_{\tau \in \mathfrak{S}_n} \delta_\sigma(\boldsymbol{i}, \boldsymbol{i}') \, \delta_\tau(\boldsymbol{j}, \boldsymbol{j}') \, \mathrm{Wg}^{\mathrm{U}}(\sigma^{-1}\tau, d).$$

Here $\mathfrak{S}_n$ *is the symmetric group on* $\{1, 2, \dots, n\}$ *and*

$$\delta_\sigma(\boldsymbol{i}, \boldsymbol{i}') = \begin{cases} 1 & \text{if } (i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(n)}) = (i'_1, i'_2, \dots, i'_n), \\ 0 & \text{otherwise.} \end{cases}$$

The function $\mathrm{Wg}^{\mathrm{U}}(\,\cdot\,, d)$ *on* $\mathfrak{S}_n$ *is given in the next slide.*

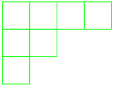
Unitary Weingarten function

Fourier expansion of Wg^{U}

$$\mathrm{Wg}^{\mathrm{U}}(\sigma, d) = \frac{1}{n!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \le d}} \frac{f^{\lambda}}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (d + j - i)} \chi^{\lambda}(\sigma) \qquad (\sigma \in \mathfrak{S}_n).$$

- $\lambda \vdash n$: The sum runs over all partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of n with length $l = \ell(\lambda)$.

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l > 0, \qquad \lambda_i \in \mathbb{Z}_{>0}$$

We identify λ with its Young diagram. Example: $(4, 2, 1) =$ 

- χ^{λ}: the (unnormalized) irreducible character of $\mathfrak{S}_n$ associated with λ.
- f^{λ}: the degree of χ^{λ} i.e. $f^{\lambda} = \chi^{\lambda}(\mathrm{id}_n) \in \mathbb{Z}_{>0}$.
- The product in the denominator runs over all boxes of the Young diagram λ. The quantity $j - i$ is called the content of the box (i, j).

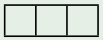
Example: unitary Weingarten functions

$$\mathrm{Wg}^{\mathrm{U}}(\sigma, d) = \frac{1}{n!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \le d}} \frac{f^{\lambda}}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (d+j-i)} \chi^{\lambda}(\sigma) \qquad (\sigma \in \mathfrak{S}_n).$$

Example

Consider $n = 3$ and $\sigma = [3,1,2] = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) = (1\ 3\ 2)$. Suppose $d \ge 3$.

$$\begin{aligned} &\mathrm{Wg}^{\mathrm{U}}([3,1,2], d) \\ =&\frac{1}{3!}\Bigg(\underbrace{\frac{1}{d(d+1)(d+2)} \cdot 1}_{} + \underbrace{\frac{2}{d(d+1)(d-1)} \cdot (-1)}_{} + \underbrace{\frac{1}{d(d-1)(d-2)} \cdot 1}_{} \Bigg) \\ =&\frac{2}{d(d^2-1)(d^2-4)}. \end{aligned}$$



Here we use one-row notation for a permutation. We also use cycle expressions.

Example: Weingarten calculus for $\mathrm{U}(d)$

Example

Let $U = (u_{ij})$ be a Haar-distributed unitary matrix from $\mathrm{U}(d)$. Then

$$\mathbb{E}[u_{12}u_{23}u_{31}\overline{u_{11}u_{22}u_{33}}] = \frac{2}{d(d^2-1)(d^2-4)}.$$

Input $n = 3$, $\boldsymbol{i} = \boldsymbol{i}' = (1, 2, 3)$. $\boldsymbol{j} = (2, 3, 1)$, $\boldsymbol{j}' = (1, 2, 3)$.

$$\mathbb{E}[u_{12}u_{23}u_{31}\overline{u_{11}u_{22}u_{33}}] = \sum_{\sigma \in \mathfrak{S}_3} \sum_{\tau \in \mathfrak{S}_3} \delta_\sigma(\boldsymbol{i}, \boldsymbol{i}')\, \delta_\tau(\boldsymbol{j}, \boldsymbol{j}')\, \mathrm{Wg}^{\mathrm{U}}(\sigma^{-1}\tau, d)$$

Recall

$$\delta_\sigma(\boldsymbol{i}, \boldsymbol{i}') = \begin{cases} 1 & \text{if } (i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(n)}) = (i'_1, i'_2, \dots, i'_n), \\ 0 & \text{otherwise.} \end{cases}$$

Only term for $\sigma = \mathrm{id}_3$ and $\tau = [3, 1, 2]$ contributes (i.e. $\delta_\sigma(\boldsymbol{i}, \boldsymbol{i}')\delta_\tau(\boldsymbol{j}, \boldsymbol{j}') = 1$).

$$= \mathrm{Wg}^{\mathrm{U}}([3, 1, 2], d) = \frac{2}{d(d^2-1)(d^2-4)}.$$

Unitary Weingarten functions

An important invariance for $\mathrm{Wg}^{\mathrm{U}}(\sigma, d)$

The function $\mathfrak{S}_n \ni \sigma \mapsto \mathrm{Wg}^{\mathrm{U}}(\sigma, d) \in \mathbb{Q}$ is central (another name is class function). Namely,

$$\mathrm{Wg}^{\mathrm{U}}(\tau^{-1}\sigma\tau, d) = \mathrm{Wg}^{\mathrm{U}}(\sigma, d) \qquad (\forall\sigma, \forall\tau \in \mathfrak{S}_n)$$

Equivalently,

- It is constant on each conjugacy class of $\mathfrak{S}_n$.
- It depends on only the cycle-type of σ ($\rightarrow$ a partition of n).

We will see that Weingarten functions for other Lie groups $\mathrm{O}(d)$ and $\mathrm{Sp}(d)$ have different invariances.

Preparations: Pair partitions

Definition

Denote by $\mathcal{M}_{2n}$ the set of all pair partitions on $\{1, 2, \dots, 2n\}$.

Example

$\mathcal{M}_4$ consists of three elements

$$\{1,2\}\{3,4\}, \qquad \{1,3\}\{2,4\}, \qquad \{1,4\}\{2,3\}$$

Every element $\mathfrak{p}$ in $\mathcal{M}_{2n}$ is uniquely expressed as

$$\{p_1, p_2\}\{p_3, p_4\} \cdots \{p_{2n-1}, p_{2n}\}$$
$$p_{2j-1} < p_{2j} \ (j = 1, \dots, n), \qquad 1 = p_1 < p_3 < \cdots < p_{2n-1}.$$

We then regard $\mathfrak{p}$ as a permutation in $\mathfrak{S}_{2n}$:

$$\mathcal{M}_{2n} \subset \mathfrak{S}_{2n}, \qquad \mathfrak{p} = [p_1, p_2, \dots, p_{2n-1}, p_{2n}].$$

Preparations: Hyperoctahedral groups

Definition

Denote by $\mathfrak{B}_n \subset \mathfrak{S}_{2n}$ the hyper-octahedral group, which is generated by permutations

$$(2i-1\ 2i)\ (i=1,2,\dots,n), \qquad (2i-1\ 2j-1)(2i\ 2j)\ (1 \le i < j \le n).$$

Example (in cycle notation)

$$\mathfrak{B}_2 = \{\mathrm{id}_4, \quad (1\ 2), \quad (3\ 4), \quad (1\ 2)(3\ 4), \quad (1\ 3)(2\ 4), \quad (1\ 4)(2\ 3), \quad (1\ 3\ 2\ 4), \quad (1\ 4\ 2\ 3)\}$$

The set $\mathcal{M}_{2n}$ forms representatives of left cosets of $\mathfrak{B}_n$ in $\mathfrak{S}_{2n}$:

$$\mathfrak{S}_{2n} = \bigsqcup_{\mathfrak{p} \in \mathcal{M}_{2n}} \mathfrak{p}\mathfrak{B}_n, \qquad \text{i.e.} \qquad \mathcal{M}_{2n} \cong \mathfrak{S}_{2n}/\mathfrak{B}_n$$

(Recall that $\mathfrak{p} \in \mathcal{M}_{2n}$ is regarded as a permutation in $\mathfrak{S}_{2n}$.)

Weingarten calculus for $\mathrm{O}(d)$

Real orthogonal group $\mathrm{O}(d) = \{g \in \mathrm{GL}(d, \mathbb{R}) \mid gg^{\mathrm{T}} = I_d\}$.

Theorem ((Collins-Śniady, 2006), (Collins-M, 2009))

Let $R = (r_{ij})_{1 \le i,j \le d}$ be a Haar-distributed orthogonal matrix. Given two sequences $\boldsymbol{i} = (i_1, \dots, i_{2n})$, $\boldsymbol{j} = (j_1, \dots, j_{2n})$, we have

$$\mathbb{E}[r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_{2n} j_{2n}}] = \sum_{\mathfrak{p} \in \mathcal{M}_{2n}} \sum_{\mathfrak{q} \in \mathcal{M}_{2n}} \Delta_{\mathfrak{p}}(\boldsymbol{i}) \Delta_{\mathfrak{q}}(\boldsymbol{j}) \mathrm{Wg}^{\mathrm{O}}(\mathfrak{p}^{-1}\mathfrak{q}, d).$$

Here

$$\Delta_{\mathfrak{p}}(\boldsymbol{i}) = \prod_{\{a,b\} \in \mathfrak{p}} \delta_{i_a, i_b}.$$

Moments of odd degree $\mathbb{E}[r_{i_1 j_1} \cdots r_{i_{2n+1} j_{2n+1}}]$ always vanish.

Recall $\mathcal{M}_{2n} \subset \mathfrak{S}_{2n}$ (so $\mathfrak{p}^{-1}\mathfrak{q}$ does make sense as permutations).
The orthogonal Weingarten function $\mathrm{Wg}^{\mathrm{O}}(\cdot, d)$ on $\mathfrak{S}_{2n}$ is described as follows.

Orthogonal Weingarten functions

In order to study $\mathrm{Wg}^{\mathrm{O}}(\cdot, d)$, we review (finite) Gelfand pairs.

Definition

Let $\mathtt{G}$ be a finite group and $\mathtt{H}$ its subgroup. Consider the Hecke algebra

$$\mathcal{H}(\mathtt{G}, \mathtt{H}) = \{f : \mathtt{G} \to \mathbb{C} \mid f(\zeta_1 \sigma \zeta_2) = f(\sigma) \ (\forall \sigma \in \mathtt{G}, \ \forall \zeta_1, \forall \zeta_2 \in \mathtt{H})\}$$

with convolution product $(f_1 * f_2)(\sigma) = \sum_{\tau \in \mathtt{G}} f_1(\sigma\tau^{-1}) f_2(\tau)$. The pair $(\mathtt{G}, \mathtt{H})$ is called a Gelfand pair if $\mathcal{H}(\mathtt{G}, \mathtt{H})$ is *commutative*: $g * f = f * g$.

Fact (well known)

$(\mathfrak{S}_{2n}, \mathfrak{B}_n)$ is a Gelfand pair.

- The unitary Weingarten function $\mathrm{Wg}^{\mathrm{U}}(\cdot, d)$ belongs to the center $\mathcal{Z}\mathbb{C}[\mathfrak{S}_n] = \bigoplus_{\lambda \vdash n} \mathbb{C}\chi^{\lambda}$.
- The orthogonal Weingarten function $\mathrm{Wg}^{\mathrm{O}}(\cdot, d)$ belongs to the Hecke algebra

$$\begin{aligned} \mathcal{H}_n :=& \mathcal{H}(\mathfrak{S}_{2n}, \mathfrak{B}_n) \\ =& \{f : \mathfrak{S}_{2n} \to \mathbb{C} \mid f(\zeta_1 \sigma \zeta_2) = f(\sigma) \ (\sigma \in \mathfrak{S}_{2n}, \ \zeta_1, \zeta_2 \in \mathfrak{B}_n)\}. \end{aligned}$$

Orthogonal Weingarten functions

- Zonal spherical functions ω^λ ($\lambda \vdash n$) form a linear basis of $\mathcal{H}_n$.

$$\omega^\lambda(\sigma) = \frac{1}{2^n n!} \sum_{\zeta \in \mathfrak{B}_n} \chi^{2\lambda}(\sigma\zeta) \qquad (\sigma \in \mathfrak{S}_{2n}),$$

where $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$. $\mathcal{H}_n = \bigoplus_{\lambda \vdash n} \mathbb{C}\omega^\lambda$.

- They are constant on each double cosets $\mathfrak{B}_n \sigma \mathfrak{B}_n$.

Theorem (Collins-M, 2009)

$$\mathrm{Wg}^{\mathrm{O}}(\sigma, d) = \frac{2^n n!}{(2n)!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \le d}} \frac{f^{2\lambda}}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (d + 2j - i - 1)} \omega^\lambda(\sigma) \quad (\sigma \in \mathfrak{S}_{2n}).$$

$$\mathrm{Wg}^{\mathrm{U}}(\sigma, d) = \frac{1}{n!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \le d}} \frac{f^{\lambda}}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (d + j - i)} \chi^\lambda(\sigma) \qquad (\sigma \in \mathfrak{S}_{n}).$$

Example. Weingarten calculus for $\mathrm{O}(d)$

Example

Let $R = (r_{ij})_{1 \le i,j \le d}$ be a Haar-distributed orthogonal matrix in $\mathrm{O}(d)$.
Let us compute

$$\mathbb{E}[r_{11} r_{12} r_{21} r_{22} r_{32} r_{32}].$$

Input $n = 3$, $\boldsymbol{i} = (1,1,2,2,3,3)$. $\boldsymbol{j} = (1,2,1,2,2,2)$.

Contributions: $\mathfrak{p}_1 = \{1,2\}\{3,4\}\{5,6\}$ and

$$\mathfrak{q}_1 = \{1,3\}\{2,4\}\{5,6\}, \ \mathfrak{q}_2 = \{1,3\}\{2,5\}\{4,6\}, \ \mathfrak{q}_3 = \{1,3\}\{2,6\}\{4,5\}$$

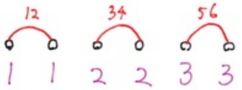
$$\begin{aligned}\mathbb{E}[r_{11} r_{12} r_{21} r_{22} r_{32} r_{32}] &= \sum_{\mathfrak{p} \in \mathcal{M}_6} \sum_{\mathfrak{q} \in \mathcal{M}_6} \Delta_{\mathfrak{p}}(\boldsymbol{i}) \Delta_{\mathfrak{q}}(\boldsymbol{j}) \mathrm{Wg}^{\mathrm{O}}(\mathfrak{p}^{-1}\mathfrak{q}, d) \\ &= \mathrm{Wg}^{\mathrm{O}}({\mathfrak{p}_1}^{-1}\mathfrak{q}_1, d) + \mathrm{Wg}^{\mathrm{O}}({\mathfrak{p}_1}^{-1}\mathfrak{q}_2, d) + \mathrm{Wg}^{\mathrm{O}}({\mathfrak{p}_1}^{-1}\mathfrak{q}_3, d) \\ &= \frac{-1}{d(d+4)(d-1)(d-2)} + \frac{2}{d(d+2)(d+4)(d-1)(d-2)} \times 2 \\ &= -\frac{1}{d(d+2)(d+4)(d-1)}.\end{aligned}$$

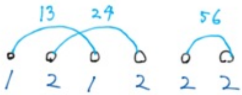
Diagrams

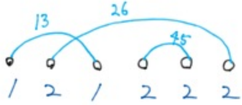
Observation

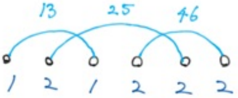
$\boldsymbol{i} = (1,1,2,2,3,3)$, $\boldsymbol{j} = (1,2,1,2,2,2)$.

If $\Delta_{\mathfrak{p}}(\boldsymbol{i}) = \prod_{\{a,b\}\in\mathfrak{p}} \delta_{i_a,i_b} = 1$ and $\Delta_{\mathfrak{q}}(\boldsymbol{j}) = \prod_{\{a,b\}\in\mathfrak{q}} \delta_{j_a,j_b} = 1$ then we can choose
$\mathfrak{p}_1 = \{1,2\}\{3,4\}\{5,6\}$
$\mathfrak{q}_1 = \{1,3\}\{2,4\}\{5,6\}, \ \mathfrak{q}_2 = \{1,3\}\{2,5\}\{4,6\}, \ \mathfrak{q}_3 = \{1,3\}\{2,6\}\{4,5\}$









Symplectic groups

Consider a skew-symmetric bi-linear form on $\mathbb{C}^{2d}$ given by

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_J = \boldsymbol{v}^{\mathrm{T}} J \boldsymbol{w}, \qquad J = J_d = \begin{pmatrix} O_d & I_d \\ -I_d & O_d \end{pmatrix}$$

Definition ((unitary) symplectic group)

$$\mathrm{Sp}(d) = \{ g \in \mathrm{U}(2d) \mid \langle g\boldsymbol{v}, g\boldsymbol{w} \rangle_J = \langle \boldsymbol{v}, \boldsymbol{w} \rangle_J \ (\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^{2d}) \}.$$

Recall

$$\mathrm{O}(d) = \{ g \in \mathrm{GL}(d, \mathbb{R}) \mid \langle g\boldsymbol{v}, g\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle \ (\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^d) \}$$

with the standard inner product $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^{\mathrm{T}} \boldsymbol{w}$.

Weingarten calculus for $\mathrm{Sp}(d)$

Theorem ((Collins-Stolz, 2008), (M, 2013))

Let $S = (s_{ij})_{1 \le i,j \le 2d}$ be a Haar-distributed symplectic matrix. Given two sequences $\boldsymbol{i} = (i_1, \dots, i_{2n})$, $\boldsymbol{j} = (j_1, \dots, j_{2n})$,

$$\mathbb{E}[s_{i_1 j_1} s_{i_2 j_2} \cdots s_{i_{2n} j_{2n}}] = \sum_{\mathfrak{p} \in \mathcal{M}_{2n}} \sum_{\mathfrak{q} \in \mathcal{M}_{2n}} \Delta'_{\mathfrak{p}}(\boldsymbol{i}) \Delta'_{\mathfrak{q}}(\boldsymbol{j}) \mathrm{Wg}^{\mathrm{Sp}}(\mathfrak{p}^{-1}\mathfrak{q}, d).$$

Here

$$\Delta'_{\mathfrak{p}}(\boldsymbol{i}) = \prod_{\{a,b\} \in \mathfrak{p}} \langle \boldsymbol{e}_{i_a}, \boldsymbol{e}_{i_b} \rangle_J \qquad \in \{0, +1, -1\},$$

and $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{2d}\}$ is a standard basis of $\mathbb{C}^{2d}$.
Moments of odd degree $\mathbb{E}[s_{i_1 j_1} \cdots s_{i_{2n+1} j_{2n+1}}]$ always vanish.

The symplectic Weingarten function $\mathrm{Wg}^{\mathrm{Sp}}(\,\cdot\,, d)$ on $\mathfrak{S}_{2n}$ and $\mathfrak{B}_n$-twisted:

$$\mathrm{Wg}^{\mathrm{Sp}}(\zeta_1 \sigma \zeta_2, d) = \mathrm{sgn}(\zeta_1)\,\mathrm{sgn}(\zeta_2) \mathrm{Wg}^{\mathrm{Sp}}(\sigma, d) \quad (\sigma \in \mathfrak{S}_{2n},\ \zeta_1, \zeta_2 \in \mathfrak{B}_n).$$

It is described by using the theory of a twisted Gelfand pair.

Comparison of three Weingarten functions

Unitary	**Orthogonal**	**Symplectic**
$\mathfrak{S}_n$	$\mathcal{M}_{2n}$, $(\mathfrak{S}_{2n}, \mathfrak{B}_n)$	$\mathcal{M}_{2n}$, $(\mathfrak{S}_{2n}, \mathfrak{B}_n, \mathrm{sgn}\,\vert_{\mathfrak{B}_n})$
center $\mathcal{Z}(\mathbb{C}[\mathfrak{S}_n])$	Hecke algebra $\mathcal{H}_n$	twisted Hecke algebra $\mathcal{H}_n^{\epsilon_n}$
irr. char. χ^λ	zonal spherical ω^λ	twisted spherical π^λ
central	$\mathfrak{B}_n$-invariant	$\mathfrak{B}_n$-twisted

$\mathcal{M}_{2n}$: pair partitions, $\mathfrak{B}_n$: hyperoctahedral subgroup.

$$\mathrm{Wg}^{\mathrm{U}}(\sigma, d) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{f^\lambda}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (d+j-i)} \chi^\lambda(\sigma) \quad (\sigma \in \mathfrak{S}_n).$$

$$\mathrm{Wg}^{\mathrm{O}}(\sigma, d) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (d+2j-i-1)} \omega^\lambda(\sigma) \quad (\sigma \in \mathfrak{S}_{2n}).$$

$$\mathrm{Wg}^{\mathrm{Sp}}(\sigma, d) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{f^{\lambda \cup \lambda}}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (2d-2i+j+1)} \pi^\lambda(\sigma) \quad (\sigma \in \mathfrak{S}_{2n}).$$

1. Weingarten Calculus for Unitary Groups
2. Weingarten Calculus for Orthogonal Groups
3. Weingarten Calculus for Symplectic Groups
4. Weingarten Calculus for Symmetric Spaces
   - A I – circular orthogonal ensemble (COE)
   - A III – chiral unitary ensemble (chUE)
5. Weingarten Graphs (joint work with Benoît Collins)
   - Unitary group $\mathrm{U}(d)$
   - A III – chiral unitary ensemble (chUE)

COE matrix

Let U be a $d \times d$ Haar-distributed unitary matrix picked up from $\mathrm{U}(d)$. Then we call

$$V = UU^{\mathrm{T}}$$

a COE matrix. (Note: U itself is also called a CUE matrix (circular unitary emseble).)

An ensemble of such V is well known as the circular orthogonal ensemble (COE). The random matrix V is symmetric and unitary, and has invariance

$$U_0 V U_0{}^{\mathrm{T}} \stackrel{dist}{=} V \qquad \text{for any } d \times d \text{ unitary matrix } U_0.$$

The distribution of V is invariant under the conjugacy action of $\mathrm{O}(d)$.

COE matrix

Let U be a $d \times d$ Haar-distributed unitary matrix picked up from $\mathrm{U}(d)$. Then we call

$$V = UU^{\mathrm{T}}$$

a COE matrix. (Note: U itself is also called a CUE matrix (circular unitary emseble).)

An ensemble of such V is well known as the circular orthogonal ensemble (COE). The random matrix V is symmetric and unitary, and has invariance

$$U_0 V U_0{}^{\mathrm{T}} \stackrel{dist}{=} V \qquad \text{for any } d \times d \text{ unitary matrix } U_0.$$

The distribution of V is invariant under the conjugacy action of $\mathrm{O}(d)$.

Aim

1. We establish Weingarten calculus for a COE matrix.
2. We explain how the COE matrix arises from a framework of the compact symmetric space (CSS) $\mathrm{U}(d)/\mathrm{O}(d)$.
3. We consider random matrices associated with other CSS.

Weingarten calculus for COE

Theorem (M, 2012)

Let $V = (v_{ij})_{1 \le i,j \le d}$ *be a COE matrix. For two sequences* $\boldsymbol{i} = (i_1, i_2, \dots, i_{2n})$ *and* $\boldsymbol{j} = (j_1, j_2, \dots, j_{2n})$, *we have*

$$\mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2n-1} i_{2n}} \overline{v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2n-1} j_{2n}}}] = \sum_{\sigma \in \mathfrak{S}_{2n}} \delta_\sigma(\boldsymbol{i}, \boldsymbol{j}) \mathrm{Wg}^{\mathrm{COE}}(\sigma, d).$$

The Weingarten function $\mathrm{Wg}^{\mathrm{COE}}(\sigma, d)$ *coincides with the orthogonal Weingarten function with a parameter shift:*

$$\mathrm{Wg}^{\mathrm{COE}}(\sigma, d) = \mathrm{Wg}^{\mathrm{O}}(\sigma, d+1) \qquad (\sigma \in \mathfrak{S}_{2n}).$$

Moments of the form $\mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2n-1} i_{2n}} \overline{v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2m-1} j_{2m}}}]$ *with* $n \ne m$ *always vanish.*

Note:

- Different from Lie group cases, the formula includes a single summation.

Compact symmetric spaces

G: a compact linear Lie group (We deal with only either $\mathrm{U}(d)$, $\mathrm{O}(d)$, or $\mathrm{Sp}(d)$).
$\Omega : G \to G$: an involutive automorphism (called a Cartan involution),
$K = \{k \in G \mid \Omega(k) = k\}$.

$$G/K \quad \cong \quad \mathcal{S} := \{g\,\Omega(g)^{-1} \mid g \in G\} \quad \subset \quad G.$$

We take a Haar-distributed random matrix Z from G, and then consider an $\mathcal{S}$-valued random matrix

$$V := Z\,\Omega(Z)^{-1}$$

associated with the compact symmetric space G/K.

Example (COE)

$G = \mathrm{U}(d)$, $K = \mathrm{O}(d)$, $\Omega(g) = \overline{g}$.

$$\mathrm{U}(d)/\mathrm{O}(d) \cong \mathcal{S} = \{gg^{\mathrm{T}} \mid g \in \mathrm{U}(d)\} = \{d \times d \text{ symmetric unitary matrices}\}.$$

The random matrix $V = U\Omega(U)^{-1} = UU^{\mathrm{T}}$ is a COE matrix.

Classification for CSS

Classical CSS are classified by E. Cartan (1927) as follows.

Class $\mathcal{C}$	CSS	random matrix
A I	$\mathrm{U}(d)/\mathrm{O}(d)$	circular orthogonal ensemble (COE)
A II	$\mathrm{U}(2d)/\mathrm{Sp}(d)$	circular symplectic ensemble (CSE)
A III	$\mathrm{U}(d)/(\mathrm{U}(a) \times \mathrm{U}(b))$	chiral unitary ensemble (chUE)
BD I	$\mathrm{O}(d)/(\mathrm{O}(a) \times \mathrm{O}(b))$	chiral orthogonal ensemble (chOE)
C II	$\mathrm{Sp}(d)/(\mathrm{Sp}(a) \times \mathrm{Sp}(b))$ $(d = a + b)$	chiral symplectic ensemble (chSE)
D III	$\mathrm{O}(2d)/\mathrm{U}(d)$	Bogoliubov-de Gennes (BdG) ensemble
C I	$\mathrm{Sp}(d)/\mathrm{U}(d)$	

For each CSS, we have a matrix ensemble.

Theorem (M, 2013)

We have established Weingarten calculus for all of them, with an explicit Fourier expansion for each Weingarten function.

$\mathrm{A\,III}$ case – chiral unitary ensembles (chUE)

$G = \mathrm{U}(d)$, $K = \mathrm{U}(a) \times \mathrm{U}(b)$, $d = a + b$.

$$\Omega(g) = I'_{ab} g I'_{ab}, \qquad I'_{ab} = \mathrm{diag}(\underbrace{1, \dots, 1}_{a}, \underbrace{-1, \dots, -1}_{b}) = \begin{pmatrix} I_a & O \\ O & -I_b \end{pmatrix}.$$

For a Haar-distributed unitary matrix U from $G = \mathrm{U}(d)$, we consider a Hermitian and unitary random matrix

$$X = X^{\mathrm{A\,III}} = U I'_{ab} U^*$$

rather than $V = U\Omega(U)^{-1} = U I'_{ab} U^* I'_{ab}$. The matrix X is called a chiral unitary matrix, or a random matrix of class $\mathrm{A\,III}$.

Recall the Schur symmetric polynomial

$$s_\lambda(x_1, \dots, x_d) = \frac{\det(x_j^{\lambda_i + d - i})_{1 \le i,j \le d}}{\det(x_j^{d-i})_{1 \le i,j \le d}}$$

for partitions λ. This is a character for an irreducible representation of $\mathrm{U}(d)$.

Weingarten calculus for chiral unitary matrix

Theorem (M, 2013)

Let $X = (x_{ij})_{1 \le i,j \le d}$ be a chiral unitary matrix from $\mathrm{U}(a+b)/(\mathrm{U}(a) \times \mathrm{U}(b))$. For two sequences $\boldsymbol{i} = (i_1, i_2, \dots, i_n)$ and $\boldsymbol{j} = (j_1, j_2, \dots, j_n)$, we have

$$\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n}] = \sum_{\sigma \in \mathfrak{S}_n} \delta_\sigma(\boldsymbol{i}, \boldsymbol{j}) \mathrm{Wg}^{\mathrm{A\,III}}(\sigma, a, b).$$

The Weingarten function $\mathrm{Wg}^{\mathrm{A\,III}}(\sigma, a, b)$ $(\sigma \in \mathfrak{S}_n)$ has the Fourier expansion

$$\mathrm{Wg}^{\mathrm{A\,III}}(\sigma, a, b) = \frac{1}{n!} \sum_{\lambda \vdash n} f^\lambda \frac{s_\lambda(\overbrace{1, \dots, 1}^{a}, \overbrace{-1, \dots, -1}^{b})}{s_\lambda(\underbrace{1, \dots, 1}_{d=a+b})} \chi^\lambda(\sigma) \quad (\sigma \in \mathfrak{S}_n).$$

- Different from Weingarten function appeared so far, this Weingarten function $\mathrm{Wg}^{\mathrm{A\,III}}$ has two parameters a, b.

Weingarten graph for $\mathrm{U}(d)$

Joint work with Benoît Collins (2017)

Our goal is to reformulate various Weingarten functions via a *Weingarten graph*.

Definition (Weingarten graph for unitary groups)

We define an infinite directed graph $\mathcal{G}^{\mathrm{U}} = (V, E^{\mathrm{red}} \sqcup E^{\mathrm{blue}})$ as follows.

- Vertex set V. $V = \bigsqcup_{n=0}^{\infty} \mathfrak{S}_n$ with $\mathfrak{S}_0 = \{\emptyset\}$. We call the vertex $\emptyset$ the root.
- Red edges. (keep level)

$$\mathfrak{S}_n \ni \sigma \longleftrightarrow \tau \in \mathfrak{S}_n : \quad \exists \text{ tranposition } (i \;\, n) \text{ such that } \tau = (i \;\, n)\sigma.$$

- Blue edges. (lower level)

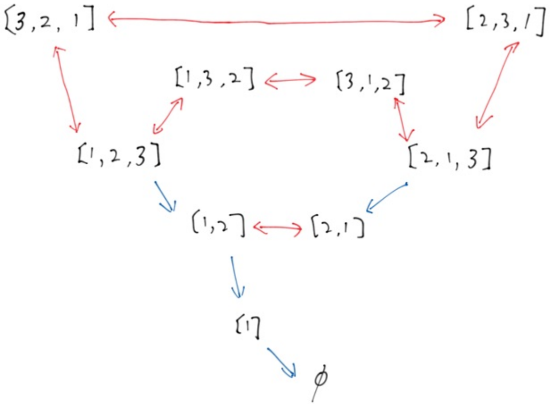
$$\mathfrak{S}_n \ni \sigma \longrightarrow \sigma' \in \mathfrak{S}_{n-1}$$

  if

  - the letter n is fixed by σ;
  - $\sigma' \in \mathfrak{S}_{n-1}$ is obtained from σ by removing the trivial cycle (n).

Weingarten graph for $\mathrm{U}(d)$

A part of Weingarten graph.



- $[3,2,1]$ $\longleftrightarrow$ $[2,3,1]$: they are switched by the transposition (2 3).
- $[2,1,3]$ $\longrightarrow$ $[2,1]$: the letter 3 is fixed in $[2,1,3]$, and erasing 3 in it we obtain $[2,1]$.

Asymptotics for $\mathrm{U}(d)$

Theorem (Collins-M, 2017)

Let $\sigma \in \mathfrak{S}_n$ and let $p(\sigma, k)$ be the number of paths from σ to $\emptyset$ going through exactly k red edges on the Weingarten graph $\mathcal{G}^{\mathrm{U}}$. Assume $d \geq n$. Then

$$\mathrm{Wg}^{\mathrm{U}}(\sigma, d) = (-1)^{|\sigma|} \sum_{j \geq 0} p(\sigma, |\sigma| + 2j) d^{-(n+|\sigma|+2j)}.$$

Here $|\sigma| = n - \ell(\mu)$, where $\ell(\mu)$ is the length of the cycle-type μ of σ.

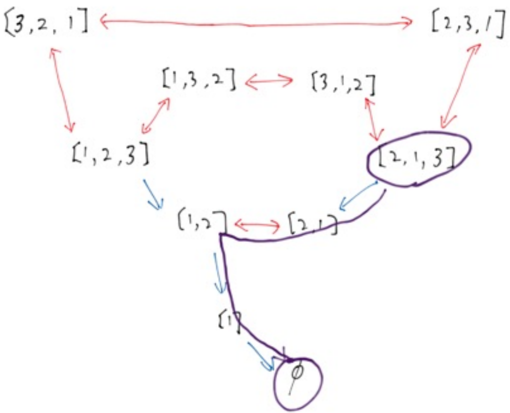
Example

Consider $\sigma = [2, 1, 3] \in \mathfrak{S}_3$. Then $|\sigma| = 1$.

$$\mathrm{Wg}^{\mathrm{U}}([2, 1, 3], d) = -\left(p([2, 1, 3], 1)d^{-4} + p([2, 1, 3], 3)d^{-6} + \cdots\right).$$

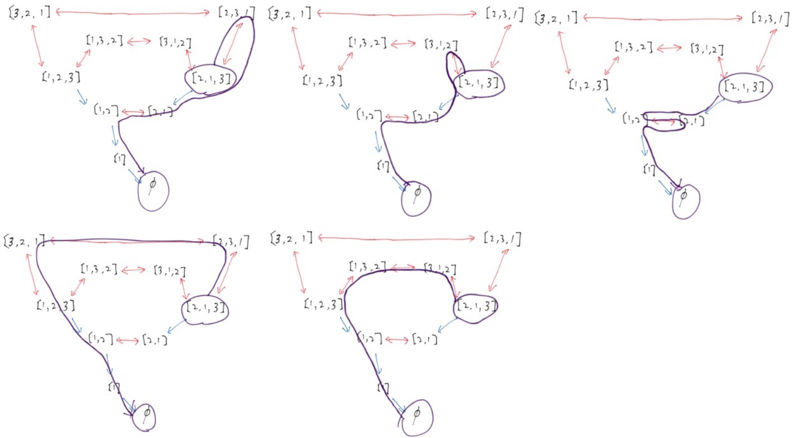
Let us compute the coefficients $p([2, 1, 3], 1)$ and $p([2, 1, 3], 3)$.

$p([2,1,3],1) = 1$



$p([2,1,3],1)$ is the number of path(s) from $[2,1,3]$ to $\emptyset$ going through red edge(s) exactly 1 time on the Weingarten graph.

$p([2,1,3],3) = 5$



$p([2,1,3],3)$ is the number of paths from $[2,1,3]$ to $\emptyset$ going through red edges exactly 3 times on the Weingarten graph.

$$\mathrm{Wg}^{\mathrm{U}}([2,1,3],d) = -(1d^{-4} + 5d^{-6} + \cdots).$$

Uniform bound for Wg^{U}

It is well known that

$$p(\sigma, |\sigma|) = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}(\mu_j - 1)$$

with Catalan numbers $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$.

Theorem (Collins-M, 2017)

For any $\sigma \in \mathfrak{S}_n$ and nonnegative integer j, we have

$$(n-1)^j p(\sigma, |\sigma|) \ \le \ p(\sigma, |\sigma| + 2j) \ \le \ (6n^{7/2})^j p(\sigma, |\sigma|).$$

Corollary

For any $\sigma \in \mathfrak{S}_n$ and $d > \sqrt{6} n^{7/4}$,

$$\frac{1}{1 - \frac{n-1}{d^2}} \ \le \ \frac{(-1)^{|\sigma|} d^{n+|\sigma|} \mathrm{Wg}^{\mathrm{U}}(\sigma, d)}{p(\sigma, |\sigma|)} \ \le \ \frac{1}{1 - \frac{6n^{7/2}}{d^2}}.$$

Connection to monotone factorizations

The expansion

$$\mathrm{Wg}^{\mathrm{U}}(\sigma, d) = (-1)^{|\sigma|} \sum_{j \geq 0} p(\sigma, |\sigma| + 2j) d^{-(n+|\sigma|+2j)}.$$

is equivalent to the result in [M-Novak (2013)].

Definition (Monotone factorizations)

Let σ be a permutation in $\mathfrak{S}_n$. A sequence $f = (\tau_1, \dots, \tau_k)$ of k transpositions is called a monotone factorization of length k for σ if:

- $\tau_i = (s_i, t_i)$ with $1 \leq s_i < t_i \leq n$;
- $\sigma = \tau_1 \tau_2 \cdots \tau_k$;
- $n \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 1$ (monotonicity).

Example

$$f = ((3, 5), (2, 5), (2, 4), (1, 2))$$

is a monotone factorization of length 4 of $\sigma = [4, 1, 5, 3, 2]$.

Connection to monotone factorizations

Proposition

The number of monotone factorizations of length k for σ is equal to $p(\sigma, k)$. Specifically, we have one-to-one correspondence between the following two objects:

- *monotone factorizations of length k for σ;*
- *paths from σ to $\emptyset$ going through k red edges on the Weingarten graph $\mathcal{G}^{\mathrm{U}}$.*

Example

- $$[4,1,5,3,2] \xrightarrow{(3,5)} [4,1,3,5,2] \xrightarrow{(2,5)} [4,1,3,2,5] \longrightarrow [4,1,3,2] \xrightarrow{(2,4)} [2,1,3,4] \longrightarrow [2,1,3] \longrightarrow [2,1] \xrightarrow{(1,2)} [1,2] \longrightarrow [1] \longrightarrow \emptyset.$$
- $$f = ((3,5),(2,5),(2,4),(1,2))$$

Chiral unitary matrix – class $\mathrm{A\,III}$

Let us consider the compact symmetric space $\mathrm{U}(d)/(\mathrm{U}(a) \times \mathrm{U}(b))$, with $d = a + b$, of class $\mathrm{A\,III}$ again. The corresponding random matrix is, the chiral unitary matrix, which is a unitary and Hermitian matrix given by

$$X = (x_{ij})_{1 \le i,j \le d} := U \cdot \begin{pmatrix} I_a & O \\ O & -I_b \end{pmatrix} \cdot U^*,$$

where U is a $d \times d$ Haar-distributed unitary matrix.

(Recall the Fourier expansion for $\mathrm{Wg}^{\mathrm{A\,III}}$.)

We now change the parameters

$$d = a + b, \qquad e = a - b \qquad \in \quad \mathbb{Z}.$$

The corresponding Weingarten function

$$\mathrm{Wg}^{\mathrm{A\,III}}(\sigma, d, e) = \frac{1}{n!} \sum_{\lambda \vdash n} f^\lambda \frac{s_\lambda(\overbrace{1, \dots, 1}^{(d+e)/2}, \overbrace{-1, \dots, -1}^{(d-e)/2})}{s_\lambda(\underbrace{1, \dots, 1}_{d})} \chi^\lambda(\sigma) \quad (\sigma \in \mathfrak{S}_n).$$

is a class function on a symmetric space $\mathfrak{S}_n$. Suppose $d \ge n$.

Weingarten graph for $\mathrm{A\,III}$

Definition (Weingarten graph for $\mathrm{A\,III}$)

We define an infinite directed graph $\mathcal{G}^{\mathrm{A\,III}} = (V, E^{\mathrm{red}} \sqcup E^{\mathrm{blue}} \sqcup E^{\mathrm{green}})$ as follows.

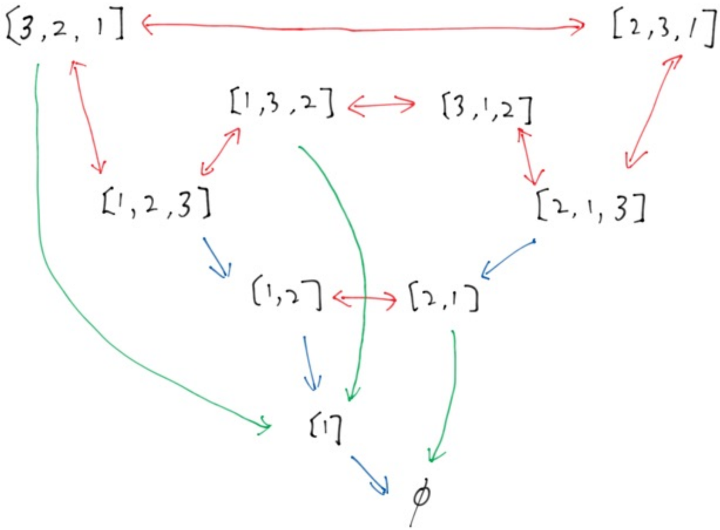
- Vertex set V. $V = \bigsqcup_{n=0}^{\infty} \mathfrak{S}_n$.
- Red edges and Blue edges are the same with the unitary case.
- green edges. (lower the level by two)

$$\mathfrak{S}_n \ni \sigma \longrightarrow \sigma^\flat \in \mathfrak{S}_{n-2}$$

if

- the letter n belongs to a 2-cycle $(j\ n)$ of σ;
- $\sigma^\flat \in \mathfrak{S}_{n-2}$ is obtained by removing the 2-cycle $(j\ n)$ from σ and by shifting letters $1, 2, \dots, \widehat{j}, \dots, n-1$ to $1, 2, \dots, n-2$ while keeping order.

Weingarten graph for $\mathrm{A\,III}$



Asymptotics for $\mathrm{Wg}^{\mathrm{A\,III}}$

Theorem (Collins-M, 2017)

Let $\sigma \in \mathfrak{S}_n$. Assume $d \geq n$. Then

$$\mathrm{Wg}^{\mathrm{A\,III}}(\sigma, d, e) = \sum_{p:\sigma \to \emptyset} (-1)^{\mathrm{red}(p)} e^{\mathrm{blue}(p)} d^{-(\mathrm{red}(p)+\mathrm{blue}(p)+\mathrm{green}(p))}$$

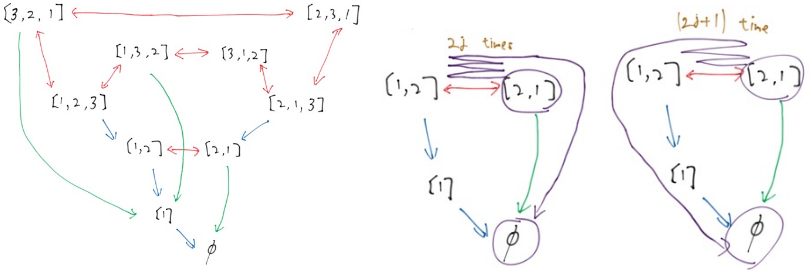
summed over all paths from σ to $\emptyset$ on $\mathcal{G}^{\mathrm{A\,III}}$. Here $\mathrm{red}/\mathrm{blue}/\mathrm{green}(p)$ stands for the number of the edges in p with the color.

Example

Let $X = (x_{ij})_{1 \leq i,j \leq d} = UI_{ab}U^*$ be a chiral unitary matrix. Put $e = a - b$.
Take $\sigma = [2,1] \in \mathfrak{S}_2$.

$$\begin{aligned}\mathrm{Wg}^{\mathrm{A\,III}}([2,1], d, e) =& \mathbb{E}[x_{12}x_{21}] = \mathbb{E}[|x_{12}|^2] \\ =& \sum_{p:[2,1]\to\emptyset} (-1)^{\mathrm{red}(p)} e^{\mathrm{blue}(p)} d^{-(\mathrm{red}(p)+\mathrm{blue}(p)+\mathrm{green}(p))}\end{aligned}$$

Paths from $\sigma = [2, 1]$



Example

$$\mathrm{Wg}^{\mathrm{A\,III}}([2,1], d, e) = \sum_{p:\sigma\to\emptyset} (-1)^{\mathrm{red}(p)} e^{\mathrm{blue}(p)} d^{-(\mathrm{red}(p)+\mathrm{blue}(p)+\mathrm{green}(p))}$$

$$= \sum_{j\geq 0} (-1)^{2j} e^{0} d^{-(2j+0+1)} + \sum_{j\geq 0} (-1)^{2j+1} e^{2} d^{-(2j+1+2+0)}$$

$$= \frac{d^2 - e^2}{d(d^2-1)}.$$

Thank you! Let's enjoy Weingarten calculus together.

1. Weingarten Calculus for Unitary Groups
2. Weingarten Calculus for Orthogonal Groups
3. Weingarten Calculus for Symplectic Groups
4. Weingarten Calculus for Symmetric Spaces
   - A I – circular orthogonal ensemble (COE)
   - $\mathrm{A\,III}$ – chiral unitary ensemble (chUE)
5. Weingarten Graphs (joint work with Benoît Collins)
   - Unitary group $\mathrm{U}(d)$
   - $\mathrm{A\,III}$ – chiral unitary ensemble (chUE)