A Riemann-Hilbert approach to the Muttalib-Borodin ensemble

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The **Muttalib-Borodin ensemble** is the following probability density function for n particles on the half line $[0,\infty)$

$$P(x_1,\ldots,x_n) = \frac{1}{Z_n}\prod_{j<k}(x_k-x_j)(x_k^\theta-x_j^\theta)\prod_{j=1}^n w(x_j), \qquad x_j\geq 0,$$

where $\theta>0$ and $w(x)=x^\alpha e^{-nV(x)}$ is a weight function having enough decay at infinity. It forms a **determinantal point process**:

$$P(x_1,\ldots,x_n) = \frac{1}{n!}\det\left[K_{V,n}^{\alpha,\theta}(x_j,x_k)\right]_{j,k=1}^n$$

Borodin proved a **hard edge scaling limit** for specific weights $w(x)$ in 1999

$$\lim_{n\to\infty}\frac{1}{n^{1+1/\theta}}K_{V,n}^{\alpha,\theta}\left(\frac{x}{n^{1+1/\theta}},\frac{y}{n^{1+1/\theta}}\right) = \mathbb{K}^{(\alpha,\theta)}(x,y), \qquad x,y>0$$

– Orthogonal polynomials and Riemann-Hilbert problem

By **universality** we expect Borodin's result to hold for a much larger class of weights $w(x)$. We prove this for $\theta = \frac{1}{2}$.

Our approach: we identify the ensemble with a **type II MOP ensemble**.

$$K_{V,n}^{\alpha,\frac{1}{2}}(x,y) = w(y)\sum_{j=0}^{n-1} p_j(x) q_j(\sqrt{y}),$$

where p_j and q_j are polynomials, and

$$\int_0^\infty p_n(x) x^k w(x) dx = \int_0^\infty p_n(x) x^k \sqrt{x} w(x) dx = 0, \qquad k = 0, 1, \ldots, \tfrac{n}{2} - 1.$$

In turn such a MOP ensemble can be identified with a **Riemann-Hilbert problem** of size 3×3, solved with the Deift-Zhou steepest descent method. A vector equilibrium problem is needed to normalize the RHP.

– The local parametrix and the matching condition

Based on recent articles we expected that **Meijer G-functions** should turn up in our analysis. Indeed, we construct the local parametrix with these.

Matching the local parametrix with the global parametrix is often complicated in higher dimensional RHPs. We devised an **iterative method** to obtain the **matching condition**, for this we need an extra annulus A around the domain D of the local parametrix.