Density of states and level statistics for 1-d Schrödinger operators

Trinh Kahn Duy, Shinnichi Kotani. Fumihiko Nakano

Density of states and level statistics for 1-d Schrödinger operators

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Decaying Coupling

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Known Facts on RSO

 $(\Omega, \mathcal{F}, \textbf{P})$: probability space

$$(H_{\omega}\phi)(x) := \sum_{|y-x|=1} \phi(y) + V_{\omega}(x)\phi(x), \quad \omega \in \Omega, \quad \phi \in \ell^2(\mathbf{Z}^d)$$

 $\{V_{\omega}(x)\}_{x\in\mathbf{Z}^d}$: i.i.d. with "good" distribution μ .

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(1) Spectrum

$$\sigma(H_{\omega}) = \Sigma := [-2d, 2d] + \text{ supp } \mu, \quad a.s.$$

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(1) Spectrum

$$\sigma(H_{\omega}) = \Sigma := [-2d, 2d] + \text{ supp } \mu, \quad a.s.$$

(2) Anderson localization : $\exists I (\subset \Sigma)$ s.t. $\sigma(H_{\omega}) \cap I$ is a.s. pp with exponentially decaying e.f.'s.

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IDS and Level Statistics 1

(1) Integrated Density of States (Macroscopic Limit) : Let $H_L := H|_{[0,L]^d}$ with D-bc.

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IDS and Level Statistics 1

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$$\sharp \{ \text{ e.v.'s of } H_L \leq E \} = N(E) \cdot L^d(1 + o(1)), \quad L \to \infty$$

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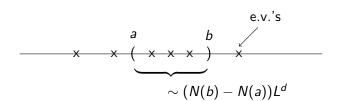
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IDS and Level Statistics 2

(2) Level Statistics (Microscopic Limit, Minami 1996) : Let E_0 : "localized region", $n(E_0):=\frac{d}{dE}N(E_0)$. Then

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(2) Level Statistics (Microscopic Limit, Minami 1996) : Let E_0 : "localized region", $n(E_0):=\frac{d}{dE}N(E_0)$. Then $\sharp \left\{ \text{ e.v.'s of } H_L \text{ in } E_0+\frac{1}{L^d}[a,b] \right\} \overset{d}{\to} \mathsf{Poisson}(n(E_0)(b-a))$

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e.v. s of
$$H_L$$
 iii $E_0 + \frac{1}{L^d}[a,b]$ \rightarrow Poisson($H(E_0)(b-a)$)



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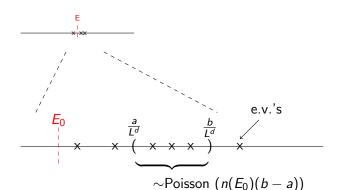
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An Extension (Killip-N, N, 2007)

(1) (Macroscopic Limit) Let

 ϕ_k : e.f. corr. to $E_k(L)$, $x_k := \langle x \rangle_{\phi_k} \in \mathbf{R}^d$: loc. center of ϕ_k .

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$$ar{\xi}_L := rac{1}{L^d} \sum_k \delta_{(E_k(L), \, x_k/L^d)} \stackrel{v}{
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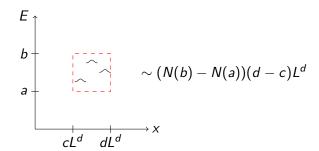
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$$\xi_L := \sum_k \delta_{(L^d(E_k(L) - E_0), \, x_k/L^d)} \stackrel{d}{ o} \mathsf{Poisson}(n(E_0) dE \otimes dx)$$

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$$E_0 + \frac{b}{L^d}$$

$$E_0 + \frac{a}{L^d}$$

$$C_0 + \frac{a}{L^d}$$

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Decaying Potential Model

We consider

$$H:=-rac{d^2}{dt^2}+a(t)F(X_t)$$
 on $L^2(\mathbf{R})$

where a: decaying factor, and F: random potential.

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$$a(t) \in C^{\infty}(\mathbf{R}), \quad a(-t) = a(t), \quad \searrow \text{ for } t > 0$$

 $a(t) = t^{-\alpha}(1 + o(1)), \quad t \to \infty, \quad \alpha > 0$

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 $a(t) = t^{-\alpha}(1 + o(1)), \quad t \to \infty, \quad \alpha > 0$
 $F \in C^{\infty}(M), \quad M : torus, \quad \langle F \rangle := \int_{M} F(x) dx = 0,$
 $\{X_{t}\}_{t \in \mathbf{R}} : \text{BM. on } M.$

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Spectrum of H

Kotani-Ushiroya(1988) : $\sigma(H) \cap [0, \infty)$ is

(1)(Rapid decay)

$$\alpha > \frac{1}{2} \implies \frac{ac}{0}$$

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(2)(Slow decay)

$$0 \le \alpha < \frac{1}{2} \implies \frac{pp}{0}$$

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Spectrum of H

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(1)(Rapid decay)

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$$0 \le \alpha < \frac{1}{2} \implies \frac{pp}{0}$$

(3)(Critical decay)

$$\alpha = \frac{1}{2} \implies \frac{pp}{E_c} \xrightarrow{sc} \cdots$$

Schrödinger operators

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We have

$$IDS(E) = IDS_{free}(E) = \frac{1}{\pi}\sqrt{E}.$$

as far as $\alpha > 0$.

operators

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$$N_L(\kappa_1, \kappa_2) := \sharp \left\{ \text{ e.v.'s of } H_L \text{ in } (\kappa_1^2, \kappa_2^2) \right\}, \quad 0 < \kappa_1 < \kappa_2.$$

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Then

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi} (\kappa_2 - \kappa_1) (1 + o(1)), \quad L \to \infty.$$

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Q: 2nd order ("CLT")?

References

Fluctuation of IDS (Notation)

Let

(1) $\{G(x)\}_{x>0}$: the Gaussian field with

$$\langle G(x), G(y) \rangle = \delta_{xy} C(x),$$

(2) G_0 : a Gaussian independent of $\{G(\cdot)\}$.

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Fluctuation of IDS (Notation)

Let

(1) $\{G(x)\}_{x>0}$: the Gaussian field with

$$\langle G(x), G(y) \rangle = \delta_{xy} C(x),$$

(2) G_0 : a Gaussian independent of $\{G(\cdot)\}$.

Further, let

$$G(\kappa_1, \kappa_2) := G(\kappa_2) - G(\kappa_1) + \left(\frac{1}{\kappa_1} - \frac{1}{\kappa_2}\right) G_0$$

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Fluctuation of IDS (Results 1)

Theorem 0 [N2017]

(1) (AC case)
$$\alpha > \frac{1}{2}$$
 :

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi} (\kappa_2 - \kappa_1) + \text{bounded}$$

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$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi} (\kappa_2 - \kappa_1) + C(\kappa_1, \kappa_2) \log L + G(\kappa_1, \kappa_2) \sqrt{\log L} + \cdots$$

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Fluctuation of IDS (Results 1)

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(3) (PP Case) $\alpha < \frac{1}{2}$:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi} (\kappa_2 - \kappa_1) + C_2(\kappa_1, \kappa_2) L^{1-2\alpha} + C_3(\kappa_1, \kappa_2) L^{1-3\alpha} + \cdots$$

 $1 - m\alpha$

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Fluctuation of IDS (Results 2)

Theorem 0 (continued)

(2) (Critical Case)
$$\alpha = \frac{1}{2}$$
:

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi} (\kappa_2 - \kappa_1) + C(\kappa_1, \kappa_2) \log L + G(\kappa_1, \kappa_2) \sqrt{\log L} + \cdots$$

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Fluctuation of IDS (Results 2)

Theorem 0 (continued)

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(3) (PP Case)
$$\frac{1}{2m} \le \alpha < \frac{1}{2(m-1)}$$
, $m = 2, 3, \cdots$, :

$$N_L(\kappa_1, \kappa_2) = \frac{L}{\pi} (\kappa_2 - \kappa_1)$$

$$+ C_2(\kappa_1, \kappa_2) L^{1-2\alpha} + C_3(\kappa_1, \kappa_2) L^{1-3\alpha} + \cdots$$

$$+ C_m(\kappa_1, \kappa_2) L^{1-m\alpha} + L^{\frac{1}{2}-\alpha} G(\kappa_1, \kappa_2) + \cdots$$

 $C_i(\kappa_1, \kappa_2)$: non-random const.'s

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Level statistics

$$\label{eq:HL} \textit{H}_{\textit{L}} := \textit{H}|_{[0,\textit{L}]}, \quad \text{ Dirichlet b.c. } \; ,$$

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$$H_L:=H|_{[0,L]}, \quad ext{Dirichlet b.c.} \ , \ 0<\kappa^2_{n_0}(L)<\kappa^2_{n_0+1}(L)<\cdots, \quad ext{positive e.v.'s of } H_L$$

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$$\begin{split} &H_L:=H|_{[0,L]}, \quad \text{Dirichlet b.c.} \ , \\ &0<\kappa_{n_0}^2(L)<\kappa_{n_0+1}^2(L)<\cdots, \quad \text{positive e.v.'s of } H_L \\ &E_0=\kappa_0^2>0 \ : \quad \text{reference energy (fixed)} \end{split}$$

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Level statistics

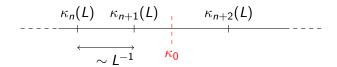
$$H_L := H|_{[0,L]},$$
 Dirichlet b.c. , $0 < \kappa_{n_0}^2(L) < \kappa_{n_0+1}^2(L) < \cdots$, positive e.v.'s of H_L $E_0 = \kappa_0^2 > 0$: reference energy (fixed)

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To study the local statistics of e.v.'s near E_0 , we consider

$$\underline{\xi_L} := \sum_{n > n_0} \delta_{L(\kappa_n(L) - \kappa_0)}.$$

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$$\begin{split} &H_L:=H|_{[0,L]}, \quad \text{Dirichlet b.c.} \ , \\ &0<\kappa_{n_0}^2(L)<\kappa_{n_0+1}^2(L)<\cdots, \quad \text{positive e.v.'s of } H_L \\ &E_0=\kappa_0^2>0 \ : \quad \text{reference energy (fixed)} \end{split}$$

To study the local statistics of e.v.'s near E_0 , we consider

$$\xi_{L} := \sum_{n \geq n_0} \delta_{L(\kappa_n(L) - \kappa_0)}.$$

Problem : $\xi_L \rightarrow ?$ as $L \rightarrow \infty$.

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Known Results

(1) Killip-Stoiciu (2009): For CMV matrices,

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Known Results

(1) Killip-Stoiciu (2009): For CMV matrices,

$$\xi_L \to \left\{ \begin{array}{l} \text{ (i) } \alpha > \frac{1}{2} : \text{ clock process} \\ \text{ (ii) } \alpha < \frac{1}{2} : \text{ Poisson process} \\ \text{ (iii) } \alpha = \frac{1}{2} : \text{ limit of circular } \beta\text{-ensembles} \end{array} \right.$$

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(2) Krichevski-Valko-Virag (2012): For 1-dim discrete Sch. op., $\alpha = \frac{1}{2}$, Nakano

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$$\xi_L \to \alpha = \frac{1}{2}$$
: Sine_{\beta}-process (limit of Gaussian \beta-ensembles)

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Fast Decay $(\alpha > \frac{1}{2})$: Assumption

For free Hamiltonian ($V\equiv 0$), $\kappa_n=n\pi/L$, so that the atoms of ξ_L are

$$L(\kappa_n - \kappa_0) = n\pi - \kappa_0 L.$$

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 $\kappa_0 L$: must converge modulo π .

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$$L(\kappa_n - \kappa_0) = n\pi - \kappa_0 L.$$

 $\kappa_0 L$: must converge modulo π .

Assumption (A)

Subsequence $\{L_j\}$ satisfies $L_j \overset{j \to \infty}{\to} \infty$ and

$$\kappa_0 L_j = m_j \pi + \boldsymbol{\beta} + o(1), \quad j \to \infty, \quad m_j \in \mathbf{N}, \quad \boldsymbol{\beta} \in [0, \pi).$$

References

Fast Decay $(\alpha > \frac{1}{2})$: Results

Theorem 1 ([KN2014] AC-case \Longrightarrow clock process)

Assume (A). Then we have

$$\lim_{j\to\infty} \mathbf{E}\left[e^{-\xi_{L_j}(f)}\right] = \int_0^\pi d\mu_\beta(\phi) \exp\left(-\sum_{n\in\mathbf{Z}} f(n\pi - \phi)\right)$$

for some probability measure μ_{β} on $[0, \pi]$.

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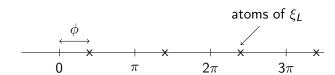
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Definition

(1) The circular β -ensemble with n-points is given by

$$\mathbf{P} \left(\begin{array}{c} e^{i\theta_n} \\ \bullet \\ \cdot \end{array} \right) \propto |\triangle(e^{i\theta_1}, \cdots, e^{i\theta_n})|^{\boldsymbol{\beta}}$$

 \triangle : Vandermonde determinant, $\beta > 0$.

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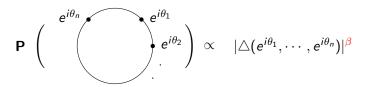
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Circular β -ensemble

Definition

(1) The circular β -ensemble with n-points is given by



- \triangle : Vandermonde determinant, $\beta > 0$.
- (2) The scaling limit ζ_{β}^{C} of the circular β -ensemble is defined by

$$\zeta_{\beta}^{\mathbf{C}} := \lim_{n \to \infty} \sum_{j=1}^{n} \delta_{n\theta_{j}}.$$

Characterization of ζ_{β}^{C}

Theorem (Killip-Stoiciu (2009))

$$\mathbf{E}[e^{-\zeta_{\beta}^{\mathcal{C}}(f)}] = \mathbf{E}\left[\int_{0}^{2\pi} \frac{d\theta}{2\pi} \exp\left(-\sum_{n \in \mathbf{Z}} f\left(\Psi_{1}^{-1}(2n\pi + \theta)\right)\right)\right]$$

Reference

Characterization of $\zeta_{\beta}^{\mathcal{C}}$

Theorem (Killip-Stoiciu (2009))

$$\mathbf{E}[e^{-\zeta_{\beta}^{\mathcal{C}}(f)}] = \mathbf{E}\left[\int_{0}^{2\pi} \frac{d\theta}{2\pi} \exp\left(-\sum_{n \in \mathbf{Z}} f\left(\Psi_{1}^{-1}(2n\pi + \theta)\right)\right)\right]$$

where $\{\Psi_t(\cdot)\}_{t\geq 0}$ is the strictly-increasing function valued process s.t. $\{\Psi_t(\lambda)\}_{t>0}$ is the solution to :

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$$d\Psi_t(\lambda) = \lambda dt + rac{2}{\sqrt{\beta t}} Re \left\{ (e^{i\Psi_t(\lambda)} - 1) dZ_t \right\},$$
 $\Psi_0(\lambda) = 0$

 Z_t : complex B.M.

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Gaussian β -ensemble

Definition

(1) The Gaussian β -ensemble with n-points is given by

$$\mathbf{P}(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n) \propto \exp\left(-\frac{\beta}{4} \sum_{k=1}^n \lambda_k^2\right) |\triangle(\{\lambda_j\})|^{\beta}$$

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(2) The scaling limit $\zeta_{\beta}^{\mathcal{G}}$ of the Gaussian β -ensemble is defined by

$$\zeta_{\beta}^{G} := \lim_{n \to \infty} \sum_{i=1}^{n} \delta_{\sqrt{4n}\lambda_{i}}$$

which is called the Sine β -process.

Characterization of ζ_{β}^{G}

Theorem (Valko-Virag 2009)

Let $\alpha_t(\lambda)$ be the solution to the following SDE.

$$dlpha_t(\lambda) = \lambda \cdot rac{eta}{4} e^{-rac{eta}{4}t} dt + Re\left[\left(e^{ilpha_t} - 1
ight) dZ_t
ight],$$
 $lpha_0(\lambda) = 0.$

Then for $\lambda > 0$, $t \mapsto |\alpha_t(\lambda)/2\pi|$ is non-decreasing and $\alpha_{\infty}(\lambda) := \exists \lim_{t \to \infty} \alpha_t(\lambda) \in 2\pi \mathbf{Z}$, a.s. Then Sine_{\beta}-process on each interval is given by

$$\zeta_{\beta}^{G}[\lambda_{1},\lambda_{2}] \stackrel{d}{=} \frac{\alpha_{\infty}(\lambda_{2}) - \alpha_{\infty}(\lambda_{1})}{2\pi}.$$

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(1)(Valko-Virag (2009) Universality in the bulk) Let μ_n (reference energy) is away from the Tracy-Widom region : $n^{\frac{1}{6}}(2\sqrt{n}-|\mu_n|)\to\infty$.

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$$\sum_{j=1}^n \delta_{\Lambda_j} \to \zeta_\beta^G, \quad \text{ where } \Lambda_j := \sqrt{4n - \mu_n^2} (\lambda_j - \mu_n).$$

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- (i) Killip-Stoiciu : SDE has singularity at t=0, but Ψ_t^{KS} is continuous for any t>0
- (ii) Valko-Virag : SDE has no singularity, but $\Psi_{t-}^{VV} \in 2\pi \mathbf{Z}$

Critical Case

Go back to our model and let $\alpha = \frac{1}{2}$: $a(t) = t^{-\frac{1}{2}}(1 + o(1))$.

Theorem 3

(1) [KN2014]
$$\xi_L \stackrel{L \to \infty}{\to} \zeta_{\beta}^{\mathbf{C}}$$
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 $\gamma(E)$: "Lyapunov exponent" in the sense that the solution ψ to $H\psi=E\psi$ satisfies $\psi(x)\sim |x|^{-\gamma(E)}$.

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 $\gamma(E)$: "Lyapunov exponent" in the sense that the solution ψ to $H\psi=E\psi$ satisfies $\psi(x)\sim |x|^{-\gamma(E)}$.

"Non-Universality"

$$\begin{array}{cccc}
0 & \text{p.p.} & E_c & \text{s.c.} \\
& & & & \\
0 \leftarrow & \beta < 2 & \beta = 2 & \beta > 2 & \rightarrow \infty
\end{array}$$

All β 's are realized.

Coincidence of two β -ensembles

Corollary 4

The limits of C_{β} -ensemble and G_{β} -ensemble are equal :

$$\zeta_{\beta}^{C} \stackrel{d}{=} \zeta_{\beta}^{G}$$
.

for all $\beta > 0$.

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Coincidence of two β -ensembles

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(1) This fact had previously been known for specific β 's, e.g., $\beta = 1, 2, 4$.

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Coincidence of two β -ensembles

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for all $\beta > 0$.

Remark

- (1) This fact had previously been known for specific β 's, e.g., $\beta = 1, 2, 4$.
- (2) Valko-Virag (2016) have "direct" proof of this fact.

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0 p.p.
$$E_c$$
 s.c.
$$\beta < 2 \qquad \beta = 2 \qquad \beta > 2$$

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 $\mathsf{Sine}_{\beta}\text{-process}$ has a "phase transition" between at $\beta=2$.

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- (1)(Valko-Virag (2009))
- (i) $\beta < 2$: $\Psi_t(\lambda)$ approaches to $2\pi \mathbf{Z}$ from above a.s.
- (ii) $\beta > 2$: $\Psi_t(\lambda)$ approaches to $2\pi \mathbf{Z}$ from below with pos. prob.

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- (ii) $\beta > 2$: $\Psi_t(\lambda)$ approaches to $2\pi \mathbf{Z}$ from below with pos. prob.
- (2)(Valko, private communication)

$$\exists H_{Dirac}$$
 on s.t. $\sigma(H_{Dirac}) \stackrel{d}{=} Sine_{\beta}$.

$$\beta \leq 2 \Longrightarrow H_{Dirac}$$
: limit point

$$\beta > 2 \Longrightarrow H_{Dirac}$$
: limit circle

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0 p.p.
$$E_c$$
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0 $\leftarrow \beta < 2$ $\beta = 2$ $\beta > 2$ $\rightarrow \infty$

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(1) [N2015] As
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- (1) [N2015] As $\beta \uparrow \infty$, Sine $\beta \stackrel{d}{\rightarrow}$ Clock process
- (2) (Allez Dumaz (2014)) As $\beta \downarrow 0$, Sine $_{\beta} \stackrel{d}{\to}$ Poisson process with intensity $(2\pi)^{-1}d\lambda$.

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- (1) [N2015] As $\beta \uparrow \infty$, Sine $\beta \xrightarrow{d}$ Clock process
- (2) (Allez Dumaz (2014)) As $\beta \downarrow 0$, Sine $_{\beta} \stackrel{d}{\to}$ Poisson process with intensity $(2\pi)^{-1} d\lambda$.
- (3) (Benaych-Georges Péché (2015), Duy-N (2016)) For $G\beta E$, $\xi_n \to {\sf Poisson}(\rho(E_0))$, if $n\beta = {\sf const.}$

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PP case
$$(\alpha < \frac{1}{2})$$

-Theorem 5 ([KN2017] PP case \Longrightarrow Poisson process)

$$\xi_L(dx) \stackrel{d}{\to} \mathsf{Poisson}\left(\frac{1}{\pi}dx\right)$$

PP case $(\alpha < \frac{1}{2})$

Theorem 5 ([KN2017] PP case \Longrightarrow Poisson process) -

$$\xi_L(dx) \stackrel{d}{\to} \text{Poisson}\left(\frac{1}{\pi}dx\right)$$

Summary

- (1) $\alpha > \frac{1}{2} : \xi_L(dx) \xrightarrow{d} \text{Clock process}$
- (2) $\alpha = \frac{1}{2} : \xi_L(dx) \xrightarrow{d} \mathsf{Sine}_{\beta}$
- (3) $\alpha < \frac{1}{2} : \xi_L(dx) \stackrel{d}{\to} \text{Poisson}(\frac{1}{\pi}dx)$

References

Outline of proof 1

Let x_t be the solution to $H_L x_t = \kappa^2 x_t$ which we write in the Prüfer coordinate.

$$\begin{pmatrix} x_t \\ x'_t/\kappa \end{pmatrix} = r_t \begin{pmatrix} \sin \theta_t \\ \cos \theta_t \end{pmatrix}, \quad \theta_0 = 0.$$

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Let

$$\Psi_L(\lambda) := \theta_L(\kappa_0 + \frac{\lambda}{L}) - \theta_L(\kappa_0), \quad \kappa_0 := \sqrt{E_0}$$

be the relative Prüfer phase.

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$$\Psi_L(\lambda) := \theta_L(\kappa_0 + \frac{\lambda}{L}) - \theta_L(\kappa_0), \quad \kappa_0 := \sqrt{E_0}$$

be the relative Prüfer phase. Then we have

$$\mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E}\left[\exp\left(-\sum_{n \geq n(L) - m(\kappa_0, L)} f\left(\Psi_L^{-1}(n\pi - \phi(\kappa_0, L))\right)\right)\right]$$

where
$$\theta_L(\kappa_0, L) = m(\kappa_0, L)\pi + \phi(\kappa_0, L)$$
, $m(\kappa_0, L) \in \mathbf{Z}$, $\phi(\kappa_0, L) \in [0, \pi)$.

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Outline of Proof 2

We replace L by n, and consider

$$\begin{split} & \Psi_t^{(n)}(\lambda) := \theta_{nt}(\kappa_{\lambda}) - \theta_{nt}(\kappa_{0}), \\ & \sim \lambda t + \frac{1}{2\kappa_{0}} Re \int_{0}^{nt} a(s) \left(e^{2i\theta_{s}(\kappa_{\lambda})} - e^{2i\theta_{s}(\kappa_{0})} \right) F(X_{s}) ds \\ & \kappa_{\lambda} := \kappa_{0} + \frac{\lambda}{n} \quad n > 0, \quad t \in [0, 1]. \end{split}$$

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By using "Ito's formula",

$$e^{2i\kappa s}F(X_s)ds = d(e^{2i\kappa s}g_{\kappa}(X_s)) - e^{2i\kappa s}\nabla g_{\kappa}(X_s)dX_s$$

 $g_{\kappa} := (L+2i\kappa)^{-1}F, \quad L : \text{generator of } X_s,$

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 $g_{\kappa} := (L+2i\kappa)^{-1}F, \quad L: \text{generator of } X_s,$

we have

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{\frac{1}{2}-\alpha} \frac{1}{2\kappa_0} Re \int_0^t s^{-\alpha} (e^{2i\Psi_s^{(n)}(\lambda)} - 1) \nabla g_\kappa dX_s$$

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Outline of Proof 3

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{\frac{1}{2}-\alpha} \frac{1}{2\kappa_0} Re \int_0^t s^{-\alpha} (e^{2i\Psi_s^{(n)}(\lambda)} - 1) \nabla g_\kappa dX_s$$

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Outline of Proof 3

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{rac{1}{2}-lpha} rac{1}{2\kappa_0} ext{Re} \int_0^t s^{-lpha} (e^{2i\Psi_s^{(n)}(\lambda)} - 1)
abla g_\kappa dX_s$$

(1) AC case
$$(\alpha>\frac{1}{2})$$
 : $\Psi_t^{(n)}(\lambda) o \lambda t$, a.s.

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Outline of Proof 3

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{rac{1}{2}-lpha} rac{1}{2\kappa_0} ext{Re} \int_0^t s^{-lpha} (e^{2i\Psi_s^{(n)}(\lambda)} - 1)
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(1) AC case
$$(\alpha>\frac{1}{2}): \Psi_t^{(n)}(\lambda) \to \lambda t$$
, a.s.

(2) Critical Case
$$(\alpha = \frac{1}{2}) : \Psi_t^{(n)}(\lambda) \stackrel{d}{\to} \Psi_t(\lambda) : \text{sol. to SDE,}$$

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(3) PP case $(\alpha < \frac{1}{2}) : \Psi_t^{(n)}(\lambda) \xrightarrow{d}$ Poisson jump process. (Using the idea of Allez - Dumaz(2014))

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abla g_\kappa dX_s$$

- (1) AC case $(\alpha > \frac{1}{2}): \Psi_t^{(n)}(\lambda) \to \lambda t$, a.s.
- (2) Critical Case $(\alpha = \frac{1}{2}) : \Psi_t^{(n)}(\lambda) \xrightarrow{d} \Psi_t(\lambda) : \text{sol. to SDE,}$
- (3) PP case $(\alpha < \frac{1}{2}) : \Psi_t^{(n)}(\lambda) \xrightarrow{d}$ Poisson jump process. (Using the idea of Allez Dumaz(2014))

Moreover in (3), $\Psi_t^{(n)}(\lambda) \stackrel{d}{\to} \pi \int_{[0,t]\times[0,\lambda]} \hat{P}(dsd\lambda')$, where $\hat{P} := Poisson(\pi^{-1}1_{[0,1]}(s)dsd\lambda')$.

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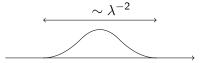
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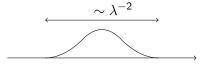
In 1-dim, $H=-\triangle+\lambda V$ generically has localization length $\sim \lambda^{-2}$.



References

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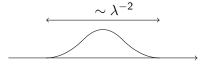


So, for
$$H_L := H|_{[0,L]}$$
,

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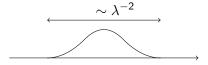


So, for $H_L := H|_{[0,L]}$, we expect

(1)
$$L \ll \frac{1}{\lambda^2} (\Leftrightarrow \lambda \ll \frac{1}{\sqrt{L}}) \Longrightarrow$$
 "extended" $\Longrightarrow \xi_L \to \mathsf{clock}$

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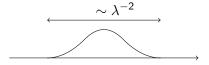
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Decaying Coupling Model

In 1-dim, $H=-\triangle+\lambda V$ generically has localization length $\sim \lambda^{-2}$.



So, for $H_L := H|_{[0,L]}$, we expect

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$$L \gg \frac{1}{\lambda^2} (\Leftrightarrow \lambda \gg \frac{1}{\sqrt{L}}) \Longrightarrow$$
 "localized" $\Longrightarrow \xi_L \to \mathsf{Poisson}$

(3)
$$L \sim \frac{1}{\lambda^2} (\Leftrightarrow \lambda \sim \frac{1}{\sqrt{L}}) \Longrightarrow$$
 "critical" $\Longrightarrow \xi_L \to \beta$ -ensemble ?

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Hamiltonian

$$H_{\lambda} := -\frac{d^2}{dt^2} + \lambda F(X_t)$$

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Hamiltonian

$$egin{aligned} H_{\lambda} &:= -rac{d^2}{dt^2} + \lambda F(X_t) \ H_L &:= H_{\lambda_L}|_{[0,L]}, \quad \lambda_L = L^{-lpha} \end{aligned}$$

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Hamiltonian

$$H_{\lambda} := -\frac{d^2}{dt^2} + \lambda F(X_t)$$
 $H_L := H_{\lambda_L}|_{[0,L]}, \quad \lambda_L = L^{-\alpha}$

In this section, we always assume :

Assumption Subseq. $\{L_j\}$ satisfies $L_j \overset{j \to \infty}{\to} \infty$ and

$$\kappa_0 L_j = m_j \pi + \beta + o(1), \quad j \to \infty.$$

for some $m_i \in \mathbf{N}$, $\beta \in [0, \pi)$.

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Results

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$$\alpha > \frac{1}{2}$$
) $N_n(\kappa_1, \kappa_2) = \lfloor \frac{\kappa_2 n}{\pi} \rfloor - \lfloor \frac{\kappa_1 n}{\pi} \rfloor$

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Theorem 3.2 [N2014], [KN2017]

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Density of states and level statistics for 1-d Schrödinger operators

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