Dunkl jump processes: relaxation and a phase transition

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Random matrices and their applications
Kyoto University, 2018-05-22

Dunkl jump processes

Dunkl processes are generalizations of multidimensional Brownian motion obtained through the use of differential-difference operators (Dunkl operators) to construct the infinitesimal generator (Dunkl Laplacian). They are associated to root systems, and have discontinuities.

- Continuous part: *radial* Dunkl processes.
  - A_{N-1}: Dyson model ($\beta > 0$)
  - B_N: Wishart-Laguerre processes / interacting Bessel processes ($\beta > 0$, $\nu > -1/2$)
- Discontinuous part: Dunkl Jump processes

Example: process of type A_{N-1}

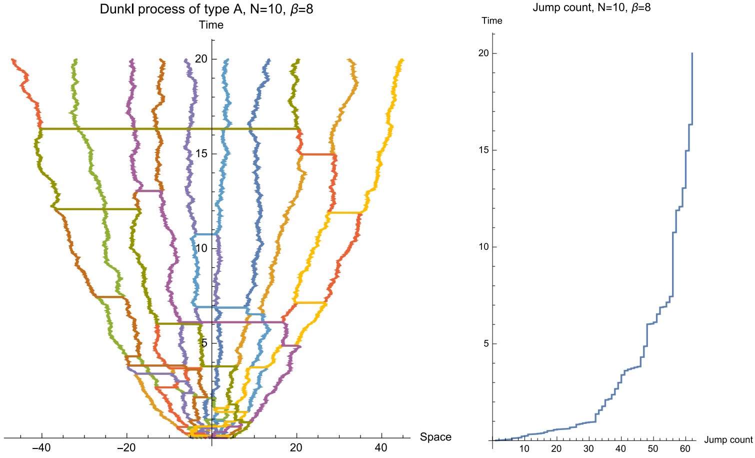

Figure: Sample of the Dunkl process of type A_{N-1} and its jump count for $N = 10$, $\beta = 8$. The horizontal lines represent jumps.

Problems we study

- Dunkl jump process
  - Dynamics → master equation
  - Relaxation → behavior at long times and convergence to equilibrium
- Jump counting process
  - Long-time behavior and jump rate
  - Phase transition in the bulk scaling limit ($t \sim N$) for the processes of type A_{N-1} and B_N at $\beta_c = 1$

For details, please come see the poster!

Eigenvector Distribution and QUE for Deformed Wigner Matrices

Lucas Benigni

LPSM, Université Paris Diderot

Random matrices and their applications
Kyoto University

Description of the Model

- D a diagonal deterministic matrix of size N with some assumptions on its density of states.

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  We will consider the model:

$$D + \sqrt{t}W$$

Different Phases

Good model for phase transitions for eigenvalues and eigenvectors

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	$t \ll N^{-1}$	$N^{-1} \ll t \ll 1$	$t \geqslant 1$
Eigenvalues			
Eigenvectors			

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Good model for phase transitions for eigenvalues and eigenvectors

	$t \ll N^{-1}$	$N^{-1} \ll t \ll 1$	$t \gtrsim 1$
Eigenvalues	$\lambda_i \approx D_i$		
Eigenvectors			

Different Phases

Good model for phase transitions for eigenvalues and eigenvectors

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Good model for phase transitions for eigenvalues and eigenvectors

	$t \ll N^{-1}$	$N^{-1} \ll t \ll 1$	$t \gg 1$
Eigenvalues	$\lambda_i \approx D_i$		RM Universality
Eigenvectors	Supported on $\mathcal{O}(1)$ entries		Completely Delocalized

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Eigenvectors	Supported on $\mathcal{O}(1)$ entries	Non-ergodic Delocalized	Completely Delocalized

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Thank you!

The Stochastic Semigroup Approach to the Edge of β-ensembles

Pierre Yves Gaudreau Lamarre

Princeton University

May 2018

Based on work by V. Gorin and M. Shkolnikov, and joint work with M. Shkolnikov.

Problem: Edge Fluctuations of β-ensembles

Given $\beta > 0$, let $\lambda_1^\beta \geq \lambda_2^\beta \geq \cdots \geq \lambda_N^\beta$ be sampled from

$$\frac{1}{\mathcal{Z}_\beta} \cdot \prod_{i<j} (x_j - x_i)^\beta \cdot \exp\left(-\frac{\beta}{4}\sum_{i=1}^N x_i^2\right), \quad x_1 \geq \cdots \geq x_N.$$

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Problem

Given $k \in \mathbb{N}$, understand the fluctuations of $(\lambda_1^\beta, \ldots, \lambda_k^\beta)$ as $N \to \infty$.

Operator Limit

Define the stochastic Airy opterator (SAO) with parameter $\beta > 0$ as

$$[\mathcal{H}^\beta f](x) := -f''(x) + xf(x) + \frac{2}{\sqrt{\beta}} W_x' f(x), \qquad f : \mathbb{R}_+ \to \mathbb{R}, f(0) = 0,$$

where $(W_x)_{x \geq 0}$ is a Brownian motion.

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Theorem (Dumitriu-Edelman (2002); Edelman-Sutton (2007); Ramírez-Rider-Virág (2011))

Let $\Lambda_1^\beta \leq \Lambda_2^\beta \leq \cdots$ be the eigenvalues of $\mathcal{H}^\beta$. For every $k \in \mathbb{N}$ fixed,

$$N^{1/6}(2\sqrt{N} - \lambda_i^\beta)_{1 \leq i \leq k} \Rightarrow (\Lambda_i^\beta)_{1 \leq i \leq k}.$$

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Advantages of operator limit approach.

1. Unified method (i.e., for all $\beta > 0$) of studying β-ensembles.
2. Study limiting fluctuations through functional analysis, as they arise as the spectrum of a differential operator.

Stochastic Semigroup Approach

Idea. Study the asymptotic extreme value fluctuations of Gaussian β-ensembles through the semigroups generated by the SAOs:

$$\mathcal{U}_T^\beta := \mathrm{e}^{-T\cdot\mathcal{H}^\beta/2}, \qquad T \geq 0.$$

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Theorem

Let $(e_t)_{t\in[0,1]}$ be a Brownian excursion, and let $(\ell^a)_{a\geq 0}$ be its local time process on $[0,1]$.

$$\int_0^1 e_t \,\mathrm{d}t - \frac{1}{2}\int_0^\infty (\ell^a)^2 \,\mathrm{d}a \sim N(0, 1/12)$$

Cauchy noise loss for stochastic optimization of random matrix models via free deterministic equivalents

arXiv:1804.03154, github.com/ThayaFluss/cnl

Tomohiro Hayase

May, 2018

The University of Tokyo

Parameter Estimation of Random Matrix Models

Random Matrix Models

- Compound Wishart Model: $W_{\mathrm{CW}}(B) = Z^* B Z$
- Information-plus-noise Model: $W_{\mathrm{IPN}}(A, \sigma) = (A + \sigma Z)^*(A + \sigma Z)$

where Z is a Gaussian random matrix on a probability space $(\Omega, \mathbb{P})$.

Question

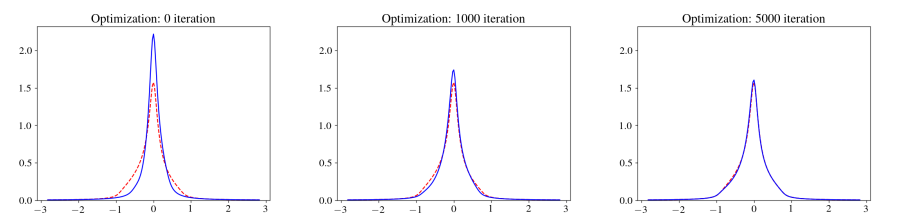
Estimate a parameter ϑ_0 from a single-shot observation $W(\vartheta_0)(\omega), \omega \in \Omega$.

Our method is based on

- Free Probability Theory (FDE, Subordination, Linearization, etc.)
- Stochastic Optimization (Stochastic (online) Gradient Descent)

Example

(CW) A "mollified" spectral distribution of a model $W_{\mathrm{CW}}(B)$ gets close to that of a true model $W_{\mathrm{CW}}(B_0)$ as the iteration progresses;



(IPN) Rank reduction: our algorithm estimated the true rank of the signal part (i.e. rank A) even if the true rank is not low.

More general random matrix models are in the scope of our method.

The Euler characteristic method for the largest eigenvalues of random matrices

Satoshi Kuriki (Inst. Statist. Math., Tokyo)

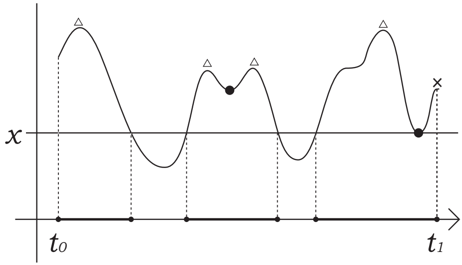
Random matrices and their applications

Kyoto U, Tue 22 May 2018

The Euler characteristic method

- $X(t)$, $t \in M$: random field with smooth sample path
- Excursion set

$$M_x = \{t \in M \mid X(t) \geq x\}$$



$$M = [t_0, t_1], \quad \chi(M_x) = 3 \quad \text{(Euler characteristic)}$$

- The Euler characteristic method

$$\Pr\left(\sup_{t \in M} X(t) \geq x\right) \approx E[\chi(M_x)] \quad \text{when } x \text{ is large}$$

- Useful in statistical testing hypothesis, i.e., p-value.

The largest eigenvalue of a Wishart matrix

- The largest eigenvalue is the maximum of a random field:

$$\lambda_1(A) = \max_{U \in M} \mathrm{tr}(UA), \quad M = \begin{cases} \{hh^\top \mid \|h\| = 1\} & \text{(real Wishart)} \\ \{hh^* \mid \|h\| = 1\} & \text{(complex W)} \end{cases}$$

Lemma (Morse's theorem)

The Euler characteristic of the excursion set $M_x = \{U \in M \mid \mathrm{tr}(UA) \ge x\}$ *is*

$$\chi(M_x) = \begin{cases} \sum_{k=1}^{n} (-1)^{k-1} \mathbb{1}\{\lambda_k(A) \ge x\} & \textit{(real Wishart)} \\ \sum_{k=1}^{n} \mathbb{1}\{\lambda_k(A) \ge x\} & \textit{(complex Wishart)} \end{cases}$$

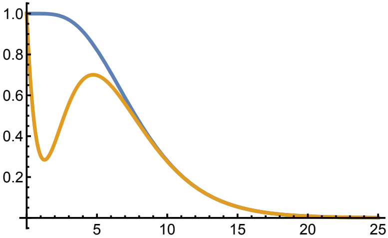
EC method

Theorem

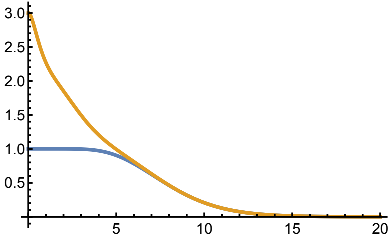
Let $A \sim W_n(N, I_n)$ or $CW_n(N, I_n)$. Let $\alpha = N - n$.

$$E[\chi(M_x)] = \begin{cases} \frac{\sqrt{\pi}(-1)^{n-1}(n-1)!}{2^{\frac{N+n-1}{2}}\Gamma(\frac{N}{2})\Gamma(\frac{n}{2})} \int_x^\infty \lambda^{\frac{N-n-1}{2}} e^{-\frac{\lambda}{2}} d\lambda \, L_{n-1}^{(\alpha)}(\lambda) & (\textit{real}) \\ \frac{n!}{\Gamma(N)} \int_x^\infty \lambda^{N-n} e^{-\lambda} d\lambda & \\ \times \left\{ L_{n-1}^{(\alpha)}(\lambda) L_{n-1}^{(\alpha+1)}(\lambda) - L_n^{(\alpha)}(\lambda) L_{n-2}^{(\alpha+1)}(\lambda) \right\} & (\textit{complex}) \end{cases}$$

- Upper prob of $\lambda_1(A)$ (blue) and the EC method (orange)



real Wishart 3×3, df=4



complex Wishart 3×3, df=4

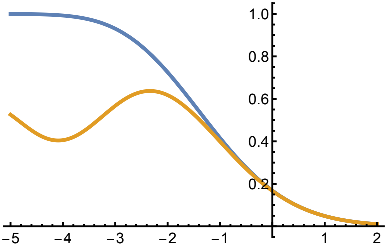
Edge asymptotics

Theorem

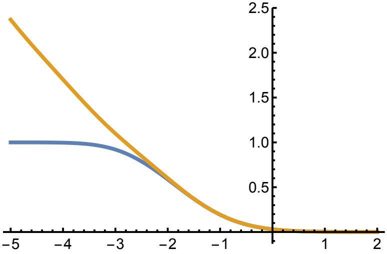
Let $A \sim W_n(N, I_n)$ *or* $CW_n(N, I_n)$. *As* $N, n \to \infty$ *s.t.* $N/n \to \gamma$,

$$E[\chi(M_x)]\Big|_{x=\mu_+ + \sigma s} \to \begin{cases} \frac{1}{2}\int_x^\infty \mathrm{Ai}(x)dx & (\textit{real}) \\ \int_x^\infty \{\mathrm{Ai}'(x)^2 - \mathrm{Ai}(x)^2\}dx & (\textit{complex}) \end{cases}$$

- Tracy-Widom (blue) and the EC method (orange)



real



complex

Other applications

1. Gaussian and beta (MANOVA) matrices can be dealt with in the same way.
2. By changing the index manifold M,

$$\max_{U \in M} \mathrm{tr}(AU)$$

represents various functions of A, e.g.,

- The range of eigenvalues

$$\lambda_1(A) - \lambda_n(A)$$

- Partial sum of the largest eigenvalues

$$\lambda_1(A) + \cdots + \lambda_r(A) \quad (r < n)$$

- The largest singular-value $\sigma_1(A)$
(when A is not real symmetric/Hermitian).

The EC method works for them.

Large permutation invariant matrices are asymptotically free over the diagonal

Camille Male

Institut de Mathématiques de Bordeaux & CNRS

16 mai 2018

Free probability probability :

1. Generalizes classical probability : Free independence and associated CLT, cumulants, entropy, harmonic analysis...
2. Robust for the spectral analysis of large random multi-matrix models : e.g. unitarily invariant random matrices and Wigner matrices.

Traffic probability : to accommode models beyond this scope.

1. Generalizes non-commutative probability : a single independence which unifies the three non-commutative notions.
2. Permutation invariance is the canonical model of traffic independence in the large N limit.

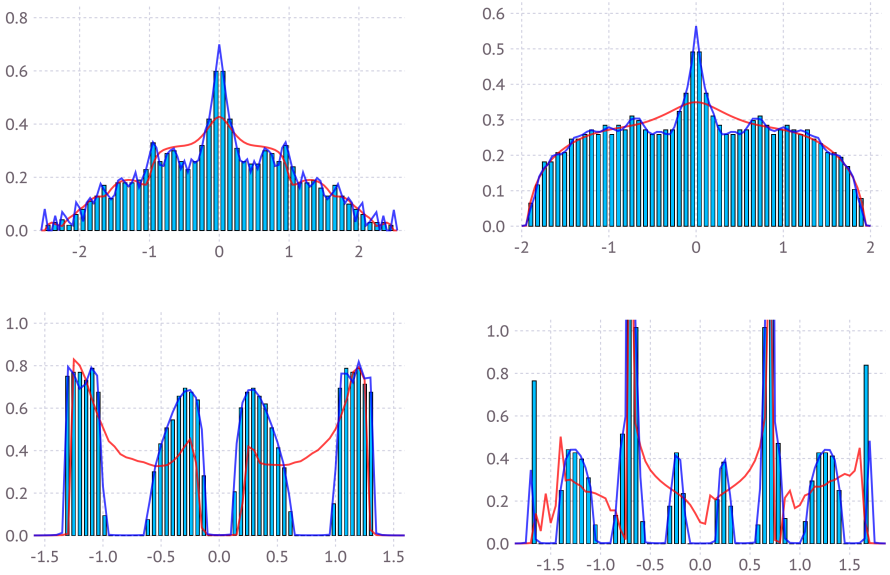
We show in the context of large random matrices that Voiculescu's notion of **conditional expectation provides an analytic tool for traffic independence**.

We write $\mathcal{M}_N$ for the set of N by N matrices, $\mathcal{D}_N \subset \mathcal{M}_N$ for the subset of diagonal matrices, $\Delta : \mathcal{M}_N \to \mathcal{D}_N$ for the diagonal map.

Theorem (Au, Cébron, Dahlqvist, Gabriel, M.)

Let $\mathbf{A}_{N,1} = (A_{N,1}^{(k)})_{k \in K}, \dots, \mathbf{A}_{N,L} = (A_{N,L}^{(k)})_{k \in K}$ be independent families of random matrices which are uniformly bounded in operator norm and permutation invariant. Then $\mathbf{A}_{N,1}, \dots, \mathbf{A}_{N,L}$ are asymptotically free over the diagonal in the operator valued non-commutative probability space $(\mathcal{M}_N, \mathcal{D}_N, \Delta)$.

Diagonal version of the usual fixed point equations remains valid $\Rightarrow$ numerical method


0.8
0.6
0.4
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0.0
-2
-1
0
1
2
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-2
-1
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1
2
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-1.0
-0.5
0.0
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1.0
1.5

KU LEUVEN

A Riemann-Hilbert approach to the Muttalib-Borodin ensemble

Joint work with prof. Arno Kuijlaars

L.D. Molag

KU Leuven

May 22, 2018, Kyoto

– The Muttalib-Borodin ensemble

The **Muttalib-Borodin ensemble** is the following probability density function for n particles on the half line $[0, \infty)$

$$P(x_1, \dots, x_n) = \frac{1}{Z_n} \prod_{j<k} (x_k - x_j)(x_k^\theta - x_j^\theta) \prod_{j=1}^{n} w(x_j), \qquad x_j \geq 0,$$

where $\theta > 0$ and $w(x) = x^\alpha e^{-nV(x)}$ is a weight function having enough decay at infinity. It forms a **determinantal point process**:

$$P(x_1, \dots, x_n) = \frac{1}{n!} \det \left[K_{V,n}^{\alpha,\theta}(x_j, x_k) \right]_{j,k=1}^{n}$$

Borodin proved a **hard edge scaling limit** for specific weights $w(x)$ in 1999

$$\lim_{n\to\infty} \frac{1}{n^{1+1/\theta}} K_{V,n}^{\alpha,\theta}\left(\frac{x}{n^{1+1/\theta}}, \frac{y}{n^{1+1/\theta}} \right) = \mathbb{K}^{(\alpha,\theta)}(x, y), \qquad x, y > 0$$

– Orthogonal polynomials and Riemann-Hilbert problem

By **universality** we expect Borodin's result to hold for a much larger class of weights $w(x)$. We prove this for $\theta = \frac{1}{2}$.

Our approach: we identify the ensemble with a **type II MOP ensemble**.

$$K_{V,n}^{\alpha,\frac{1}{2}}(x,y) = w(y) \sum_{j=0}^{n-1} p_j(x) q_j(\sqrt{y}),$$

where p_j and q_j are polynomials, and

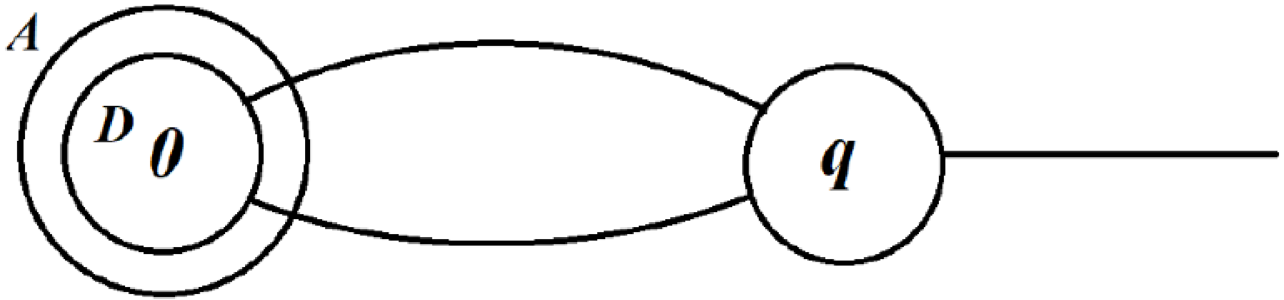
$$\int_0^\infty p_n(x) x^k w(x) dx = \int_0^\infty p_n(x) x^k \sqrt{x} w(x) dx = 0, \qquad k = 0, 1, \ldots, \frac{n}{2} - 1.$$

In turn such a MOP ensemble can be identified with a **Riemann-Hilbert problem** of size 3×3, solved with the Deift-Zhou steepest descent method. A vector equilibrium problem is needed to normalize the RHP.

– The local parametrix and the matching condition

Based on recent articles we expected that **Meijer G-functions** should turn up in our analysis. Indeed, we construct the local parametrix with these.

Matching the local parametrix with the global parametrix is often complicated in higher dimensional RHPs. We devised an **iterative method** to obtain the **matching condition**, for this we need an extra annulus A around the domain D of the local parametrix.



Stationary KPZ Fluctuations For the Stochastic Higher Spin Six Vertex Model

MUCCICONI MATTEO
based on a collaboration with T. IMAMURA and T. SASAMOTO

Random matrices and their applications
Kyoto university

May 21-25 2018

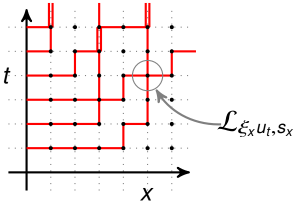
Tokyo Tech

Stochastic Higher Spin Six Vertex Model [Corwin-Petrov '15]

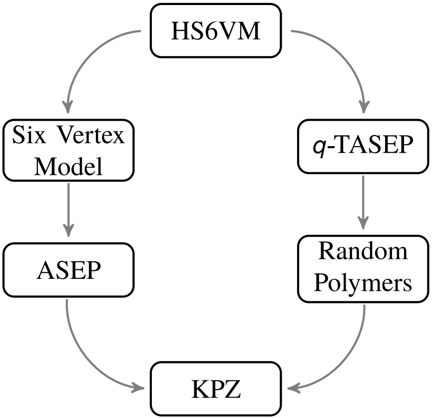
Boltzmann vertex weights

	g / g	$g-1$ / g	g / g	$g+1$ / g
$\mathcal{L}_{\xi u,s}$	$\frac{1-s\xi u q^g}{1-s\xi u}$	$\frac{-s\xi u+s\xi u q^g}{1-s\xi u}$	$\frac{-s\xi u+s^2 q^g}{1-s\xi u}$	$\frac{1-s^2 q^g}{1-s\xi u}$

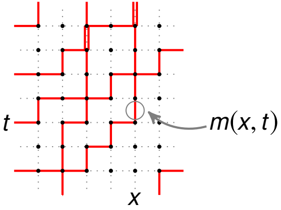
Construct a measure on the set of directed path on $\mathbb{Z}^2_{\geq 1}$



Map of principal degenerations of the HS6VM



In its stationary state the HS6VM can be defined on the full lattice $\mathbb{Z}^2$



Stationary product measure

$$\mathbb{P}(m(x,t)=M) \propto \left(\frac{\rho}{s_x \xi_x}\right)^M \frac{(s_x^2;q)_M}{(q;q)_M}$$

An important observable is the stationary height $\mathcal{H}$

$$\mathcal{H}(x,t) - \mathcal{H}(x+\Delta x, t) = \#\text{ of paths in } [x, x+\Delta x] \text{ at time } t,$$

$$\mathcal{H}(x, t+\Delta t) - \mathcal{H}(x,t) = \#\text{ of paths crossing } x$$

$$\text{during the time interval } [t, t+\Delta t].$$

Exact formulas for the statistics of $\mathcal{H}$ are a consequence of

- Yang Baxter equation

$$L_{\frac{u_1}{u_2\sqrt{q}}, \frac{1}{\sqrt{q}}} * L_{u_1,s} * L_{u_2,s} = L_{u_1,s} * L_{u_2,s} * L_{\frac{u_1}{u_2\sqrt{q}}, \frac{1}{\sqrt{q}}}$$

- Elliptic determinants

$$\frac{\overline{\Theta}(\zeta A/Z)}{\overline{\Theta}(\zeta)} \frac{\prod_{1\le i<j\le n} \overline{\Theta}(a_i/a_j)\overline{\Theta}(z_j/z_i)}{\prod_{i,j=1}^n \overline{\Theta}(a_i/z_j)} = \det_{i,j=1}^n \left(\frac{\overline{\Theta}(\zeta a_i/z_j)}{\overline{\Theta}(\zeta)\overline{\Theta}(a_i/z_j)}\right)$$

We obtain

$$\left\langle \frac{1}{(\zeta q^{\mathcal{H}(x,t)};q)_\infty} \right\rangle$$

$$= \frac{1}{(q;q)_\infty} \sum_{k\geq 0} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} \det\left(1 - f_{\zeta q^{-k}} A\right) G(\zeta q^{-k}),$$

with

$$f(n) = \frac{1}{1-q^n/\zeta},$$

$$A(n,m) = \frac{1}{(2\pi i)^2} \int_D \frac{dw}{w} \int_C dz \frac{z^m}{w^n} \frac{\exp\{xg(z)\}}{\exp\{xg(w)\}} \frac{(q\frac{v}{w};q)_\infty}{(q\frac{v}{z};q)_\infty} \frac{1}{z-w},$$

and G has a more complicated expression.

Our formulas are good for asymptotic analysis!

Theorem (IMS)

$$\frac{\mathcal{H}(x,\kappa x)-\eta x}{\gamma x^{1/3}} \xrightarrow[x\to\infty]{\mathcal{D}} F_{BR}.$$

Here F_{BR} is the Baik-Rains distribution [Baik-Rains'00].

Topological Recursion

Anas A. Rahman
Supervised by:
Peter J. Forrester and Paul Norbury

The University of Melbourne

Let $\rho(\lambda)$ be the eigenvalue density or empirical spectral measure.

$$W_1(x) := \sum_{k=0}^{\infty} \frac{m_k}{x^{k+1}}, \quad m_k := \int_{\mathbb{R}} \lambda^k \rho(\lambda) \, \mathrm{d}\lambda$$

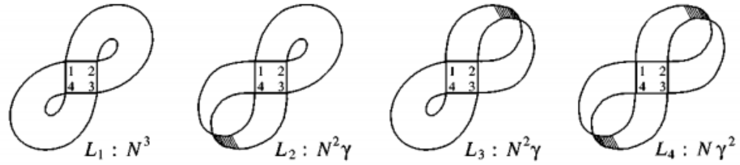
is the resolvent. Along with analogues $W_n(x_1, \ldots, x_n)$, satisfies recursion:

$$\kappa W_{n+2}(x, x, I) + \kappa \sum_{J \subseteq I} W_{|J|+1}(x, J) W_{|I-J|+1}(x, I - J)$$
$$+ \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{W_n(x, I - \{x_i\}) - W_n(I)}{x - x_i} + (\kappa - 1) \frac{\partial}{\partial x} W_{n+1}(x, I)$$
$$= \kappa N \left(V'(x) W_{n+1}(x, I) - P_n(x; I) \right), \quad I = (x_1, \ldots, x_n).$$

- Inverse Stieltjes Transform $\to$ Smoothed density

$$\tilde{\rho}(\lambda) = \frac{1}{2\pi \mathrm{i}} \lim_{\epsilon \to 0} \left[W_1(\lambda - \mathrm{i}\epsilon) - W_1(\lambda + \mathrm{i}\epsilon) \right]$$

- What makes it *topological*?



- Accessible derivation for Gaussian, Laguerre, Jacobi (*Aomoto's method: integration by parts, etc.*)
- Spectral curves and enumerative geometry (*The Eynard-Orantin generalisation, Seiberg-Witten representation, pairs-of-pants decomposition, etc.*)

Quantized Vershik–Kerov Theory and q-deformed Gelfand–Tsetlin Graph

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Abstract

We propose natural quantized character theory for inductive systems of compact quantum groups. Also, we provided a q-deformation of the approximation theorem for ordinary characters of group due to Vershik–Kerov. This relate to Gorin's analysis on q-Gelfand–Tsetlin graph explicitly when the given quantum groups are quantum unitary groups.

Character theory of inductive limit groups G_∞

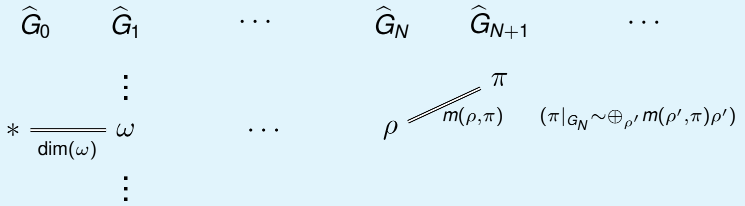
A continuous function $\chi : G \to \mathbb{C}$ is a **character** if it is

- **positive-type** $([\chi(g_i g_j^{-1})]_{i,j} \geq 0, \quad \forall g_1, \dots, g_n \in G)$,
- **central** $(\chi(gh) = \chi(hg), \quad \forall g, h \in G)$,
- **normalized** $(\chi(e) = 1)$.

$G_\infty := \varinjlim_N G_N, \quad G_N$: compact group, $\quad G_0 = \{e\}$

Examples: $S(\infty), \quad U(\infty), \quad O(\infty), \quad SO(\infty), \quad \dots$

Branching Graph:



character theory of inductive groups

probability theory on branching graphs

$\chi : G_\infty \to \mathbb{C}$; character $\longrightarrow$ $(\mathbb{P}_N)_N$; coherent system

($\mathbb{P}_N$; probability on $\widehat{G}_N$

with a certain relation)

$$\chi|_{G_N} = \sum_\rho \mathbb{P}_N(\rho)\chi_\rho$$

$$\mathbb{P}(C_t) = \frac{\mathbb{P}_N(\rho)}{\dim(\rho)}$$

$\bigstar C_t$ is the cylinder set of a finite path t from $*$ to $\rho \in \widehat{G}_N$

$\mathbb{P}$; central measure

(probability on the path space

with a certain invariance)

A character χ is extremal if and only if the corresponding central probability measure $\mathbb{P}$ is ergodic with respect to the group of finite permutations of paths.

For every extremal character χ on G_∞ there exists a sequence $\pi_1 \prec \pi_2 \prec \cdots$ such that $\pi_N \in \widehat{G_N}, \quad \pi_N \subset \pi_{N+1}|_{G_N}$ and

$$\chi|_{G_N} = \lim_{\substack{L\to\infty \\ L\geq N}} \chi_{\pi_L}|_{G_N},$$

where χ_{π_L} is the irreducible character of the representation π_L.

This is called the *ergodic method*.

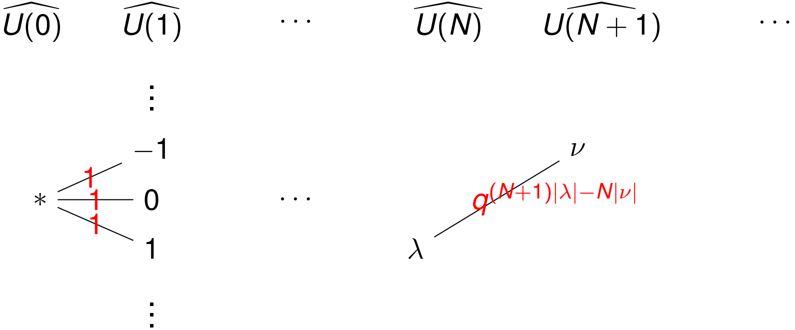
The set of extremal characters (and ergodic central measures) of:

- $S(\infty)$ (resp. the Young graph) are parametrized by
$$\left\{ (\alpha, \beta) \middle| \begin{matrix} \alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0), \\ \beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0), \end{matrix} \sum_{i\geq 1} (\alpha_i + \beta_i) \leq 1 \right\}$$
- $U(\infty)$ (resp. the Gelfand–Tsetlin graph) are parametrized by
$$\left\{ (\alpha^+, \beta^+, \alpha^-, \beta^-, \delta^+ . \delta^-) \middle| \begin{matrix} \alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0), \\ \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0), \\ \sum_{i\geq 1} (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \\ \beta_1^+ + \beta_1^- \leq 1 \end{matrix} \right\}$$

This is called the *boundary theorem*.

q-Deformed Gelfand–Tsetlin Graph (Gorin, 2012)

the q-Gelfand–Tsetlin graph = the Gelfand–Tsetlin graph + the weights on edges



$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$ joins $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{N+1})$ by an edge if $\nu_1 \geq \lambda_1 \geq \nu_2 \cdots \geq \nu_N \geq \lambda_N \geq \nu_{N+1}$.

There exists q-coherent systems and q-central measures which are q-deformations for coherent systems and central measures on the Gelfand–Tsetlin graph.

The q-Deformation v.s. Character theory of CQGs

Compact quantum group(=CQG) is a q-deformation of the ring $C(G)$ of continuous functions on a compact group G.

★ Quantum unitary group $U_q(N)$ is a q-deformation of the ring generated by the continuous functions

$$u_{ij} : U(N) \ni U \mapsto U_{ij} \in \mathbb{C}, \quad i, j = 1, \ldots, N,$$

where U_{ij} is the (i, j)-entry of $U \in U(N)$.

The $U_q(N)$ and the $U(N)$ have the same representation theory. In particular, the inductive system of $U_q(N)$ has the same branching graph : the Gelfand–Tsetlin graph.

At first glance, it looks like “character theory of CQGs” does not provide the q-deformation of its branching graph. However, we can obtain representation-theoretic interpretation of the q-deformation of the Gelfand–Tsetlin graph from $U_q(N)$.

Let G be a CQG.

In general, for unitary irreducible representation ρ of G ρ and ρ^{cc} are not unitary equivalent, but there exists a unique positive invertible intertwiner F_ρ from ρ to ρ^{cc} such that $\mathrm{Tr}(F_\rho) = \mathrm{Tr}(F_\rho^{-1})$.

The trace $\mathrm{Tr}(F_\rho)$ is called the *quantum dimension* of ρ. When a given quantum group is the quantum unitary group $U_q(N)$, we have

$$\mathrm{Tr}(F_\rho) = \sum_{\substack{(e_1,\dots,e_N):\\ \text{finite path from } * \text{ to } \rho \\ \text{on the Gelfand–Tsetlin graph}}} w(e_1)\cdots e(e_N),$$

where $w(\cdot)$ is the weight of an edge of the Gelfand–Tsetlin graph.

$\to$ The irreducible quantized character is defined by $\mathrm{Tr}(F_\rho\ \cdot)/\mathrm{Tr}(F_\rho)$.
$\to$ A general quantized characters are defined by states which are invariant under the action given by $\prod_{\rho\in\widehat{G}}\mathrm{Ad}(F_\rho^{it})$ on $\bigoplus_{\rho\in\widehat{G}} B(\mathcal{H}_\rho)$, that is, KMS states.

This definition is compatible with an inductive system of CQGs.

$\therefore$ We can find the correspondence quantized characters of inductive system of $U_q(N)$ and q-central measures on the Gelfand–Tsetlin graph.

Approximation theorem (S. 2018):
For every extremal quantized character χ of an inductive system of compact quantum group there exists a sequence $\pi_1 \prec \pi_2 \prec \cdots$ such that $\pi_N \in \widehat{G_N}, \quad \pi_N \subset \pi_{N+1}|_{G_N}$ and

$$\chi|_{G_N} = \lim_{\substack{L\to\infty \\ L\geq N}} \chi_{\pi_L}|_{G_N},$$

where χ_{π_L} is the irreducible quantized character of the representation π_L.

Boundary theorem (Gorin 2012, S. 2018):
The set of extremal quantized characters of the inductive system of quantum unitary groups $U_q(N)$ (and ergodic q-central measures on q-Gelfand–Tsetlin graph) are parametrized by

$$\{\theta = (\theta_i)_{i=1}^{\infty} \in \mathbb{Z}^{\infty} \mid \theta_1 \leq \theta_2 \leq \cdots\}.$$

Reference:
V. Gorin, *The q-Gelfand–Tsetlin graph, Gibbs measures and q-Toeplitz matrices*, Adv. Math **229** (2012), no. 1, 201–266

R. Sato, *Quantized Vershik–Kerov theory and quantized central measures on branching graphs*, arXiv:1804.02644

Masaki Tezuka
(Department of Physics, Kyoto University)

Random-matrix behavior in the energy spectrum of the Sachdev-Ye-Kitaev model and in the Lyapunov spectra of classical chaos systems

J. S. Cotler, …, MT, JHEP **1705**, 118 (2017)
(arXiv:1611.04650)

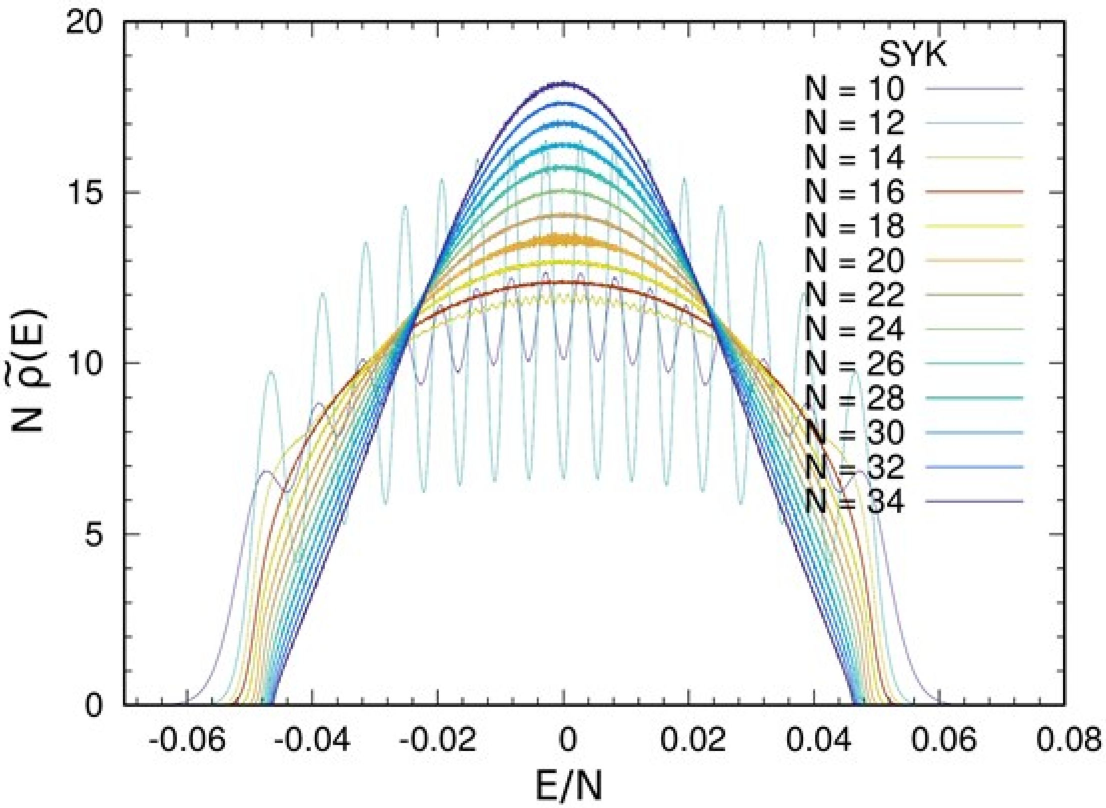
SYK model

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

[A. Kitaev: talks at KITP (Apr 7 and May 27, 2015)]

1. Solvable at large-N (strong coupling when βJ>>1), finite entropy / N at $T\to 0$
2. Holographically corresponds to 1+1D black holes
3. Satisfies the chaos bound

"Fast quantum information scrambler"
(Conjectured upper bound of the Lyapunov exponent $\lambda_{\mathrm{L}} = 2\pi k_{\mathrm{B}}T/\hbar$ realized, as in black holes)



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Random-matrix behavior in the energy spectrum of the Sachdev-Ye-Kitaev model

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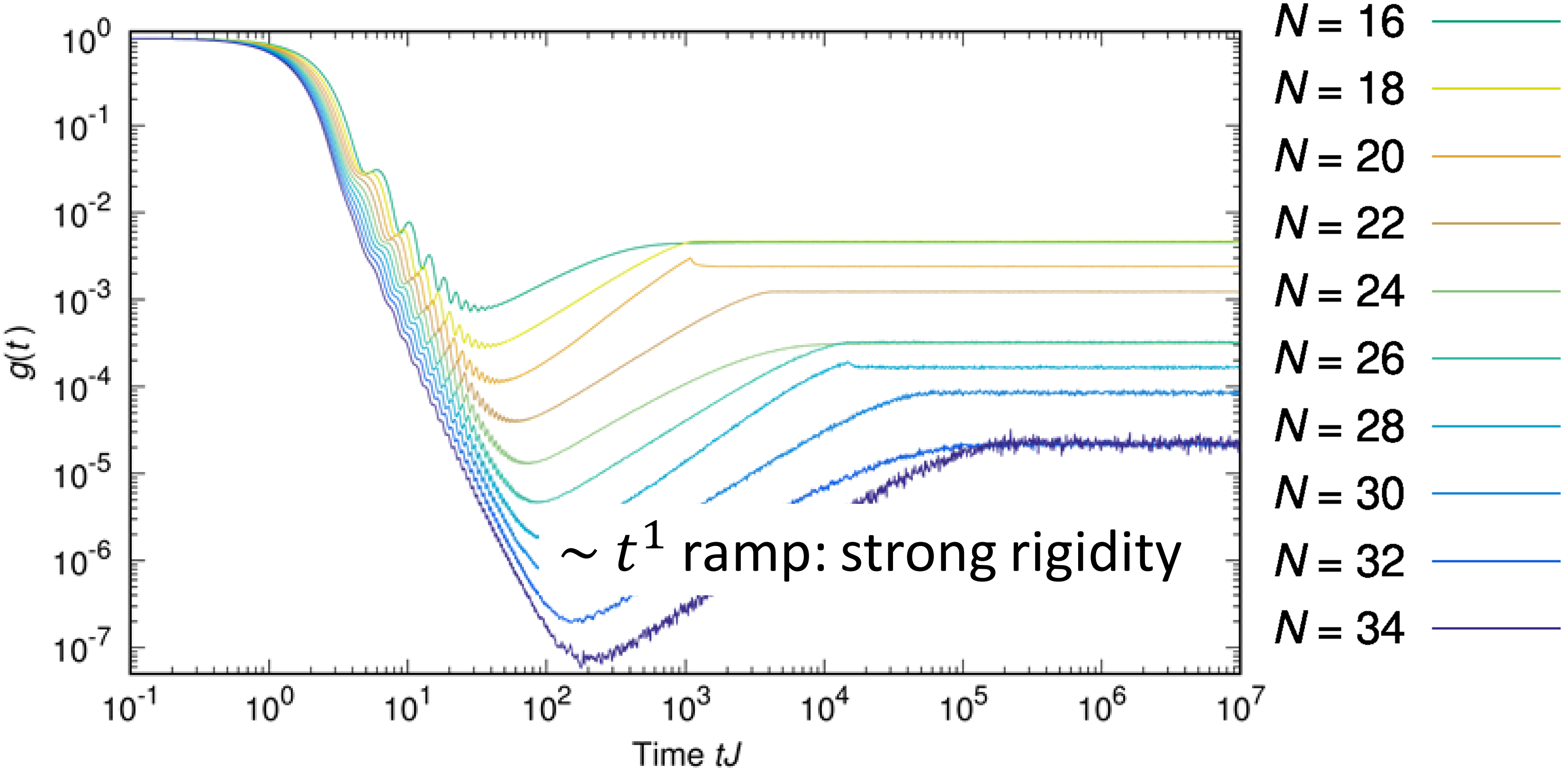
Spectral form factor

$$g(\beta,t)=\frac{\langle |Z(\beta,t)|^2\rangle_J}{\langle Z(\beta)\rangle_J^{\ 2}} \qquad Z(\beta,t)=\mathrm{Tr}\left(\mathrm{e}^{-\beta\hat{H}-\mathrm{i}\hat{H}t}\right)$$

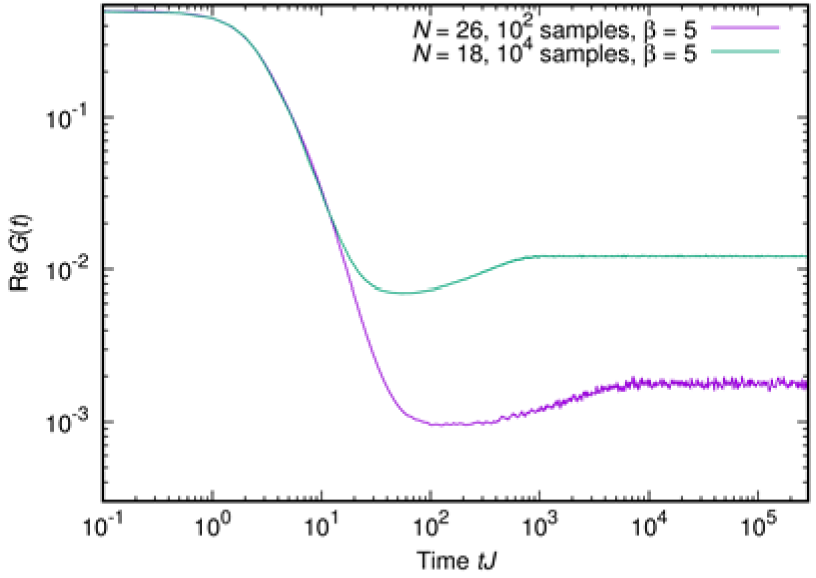


[You, Ludwig, Xu 2017]

$N_\chi \pmod 8$	0	1	2	3	4	5	6	7
qdim	1	$\sqrt{2}$	2	$2\sqrt{2}$	2	$2\sqrt{2}$	2	$\sqrt{2}$
lev. stat.	GOE	GOE	GUE	GSE	GSE	GSE	GUE	GOE

$$G(t)=\langle \chi_a(t)\chi_a(0)\rangle$$

Dip-ramp-plateau structure similar to $g(\beta,t)$ for $N\equiv 2 \pmod 8$



Random-matrix behavior
in the energy spectrum of the Sachdev-Ye-Kitaev model
and in the Lyapunov spectra of classical chaos systems

Masaki Tezuka
(Department of Physics, Kyoto University)

M. Hanada, H. Shimada, and MT,
Phys. Rev. E **97**, 022224 (2018) (arXiv:1702.06935)

Deviation at t initial infinitesimal deviation

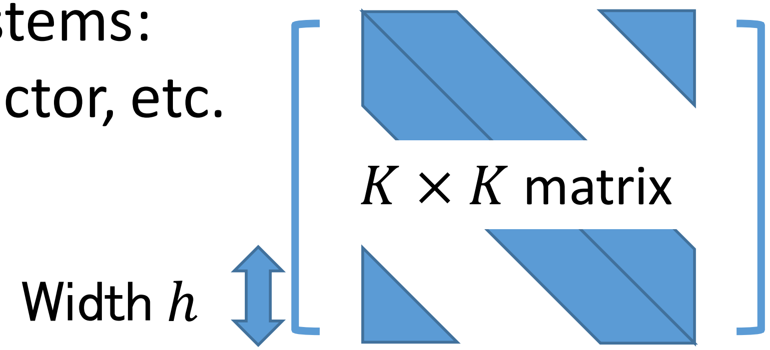
$$\delta\phi_i(t) = T_{ij}\delta\phi_j(0)$$

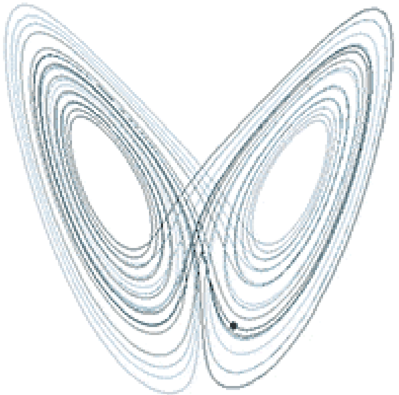
Singular values of T_{ij}: $\{a_k(t)\}_{k=1}^{K}$
Time-dependent Lyapunov spectrum

$$\left\{\lambda_k(t) = \frac{\log a_k(t)}{t}\right\}_{k=1,2,\dots,K}$$

Spectral correlation in $\lambda_k(t)$ observed for various classical chaos systems:
Logistic map, Lorenz attractor, etc.

Random matrix product





__Ongoing work__

Quantum chaos systems e.g. the SYK model:
Definition of Lyapunov spectra and study of its behavior

Unbounded largest eigenvalues of sample covariance matrices: Asymptotics, fluctuations and applications to long memory stationary processes

Peng TIAN

Paris East University

based on a joint work with F. Merlevède and J. Najim

Kyoto University - 22 May 2018

- We consider
$$S_N := \frac{1}{n} T_N^{\frac{1}{2}} Z_N Z_N^* T_N^{\frac{1}{2}}$$
where Z_N is a $N \times n$ matrix with i.i.d centered, reduced entries, and T_N is a nonnegative definite hermitian matrix.

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- As $N, n \to \infty, N/n \to r \in (0, \infty)$, what are the asymptotics and fluctuations of the top eigenvalues of S_N, if

$$\mu^{T_N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(T_N)} \xrightarrow{\mathcal{D}} \mu \quad \text{with} \quad \sup \operatorname{supp} \mu = \infty\,?$$

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- This question was raised in the study of *long memory stationary process*. If a process $(\mathcal{X}_t)_{t \in \mathbb{Z}}$ satisfies
$$\mathbb{E}\mathcal{X}_t = 0, \quad \operatorname{Cov}(\mathcal{X}_{t+h}, \mathcal{X}_t) = \gamma(h), \quad \forall t, h \in \mathbb{Z}$$
with the autocovariance function γ satisfying
$$\sum_{h \in \mathbb{Z}} |\gamma(h)| = \infty.$$
Then $(\mathcal{X}_t)_{t \in \mathbb{Z}}$ is a (centered) long memory stationary process.

- People are initially interested in the asymptotics and fluctuations of top eigenvalues of

$$Q_N := \frac{1}{n}\sum_{j=1}^{n} \vec{X}_j \vec{X}_j^*,$$

where $\vec{X}_i$ are i.i.d obersations of $\begin{pmatrix}\mathcal{X}_1 & \cdots & \mathcal{X}_N\end{pmatrix}^\top$ drawn from a centered long memory stationary process $(\mathcal{X}_t)_{t\in\mathbb{Z}}$.

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- These questions are tightly related to the asymptotic properties of the population covariance matrix, which is the following Toeplitz matrix:

$$T_N := \mathrm{Cov}\begin{pmatrix}\mathcal{X}_1 \\ \vdots \\ \mathcal{X}_N\end{pmatrix} = (\gamma(i-j))_{i,j=1}^{N},$$

with $\mu^{T_N} \xrightarrow{\mathcal{D}} \mu$ and $\sup \operatorname{supp} \mu = \infty$ as natural properties due to the long memory of the process.

- To answer the above questions, we study the following additional properties of Toeplitz matrices:

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- After these, the asymptotics and joint fluctuations of any p (a fixed integer) top eigenvalues of

$$S_N = \frac{1}{n} T_N^{\frac{1}{2}} Z_N Z_N^* T_N^{\frac{1}{2}}$$

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$$S_N = \frac{1}{n} T_N^{\frac{1}{2}} Z_N Z_N^* T_N^{\frac{1}{2}}$$

  are studied.
- In the general model S_N, the fluctuations depend not only on the entry distribution but also on the eigenvectors of T_N. But for some Toeplitz T_N, the universality holds.

Recent Developments for the Singular Values of Skew-Symmetric Gaussian Random Matrices

Donald Richards

Penn State University and the Institute of Statistical Mathematics

$\mathcal{A}$: The space of $p \times p$, real, skew-symmetric matrices.

$A = (a_{ij}) \in \mathcal{A}$: A noncentral Gaussian random matrix with p.d.f.

$$f(A) = (2\pi)^{-p(p+1)/4} \exp\left[-\tfrac{1}{4} \operatorname{tr} (A-M)(A-M)'\right],$$

where $M = E(A)$.

The singular values of A: $\sigma_1 > \cdots > \sigma_q > 0$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$D_\sigma = \begin{cases} \sigma_1 J \oplus \cdots \oplus \sigma_q J, & \text{if } p \text{ is even, } p = 2q \\ \sigma_1 J \oplus \cdots \oplus \sigma_q J \oplus 0, & \text{if } p \text{ is odd, } p = 2q+1 \end{cases}$$

Kuriki (2010) considered the singular value decomposition:

$$A = H D_\sigma H', \quad \text{where} \quad H \in SO(p).$$

The motivation: Problems in mathematical statistics, and a statistical analysis of a Japanese league's baseball scores.

Kuriki was led to Harish-Chandra's integral for $SO(p)$:

$$I_p(\sigma, \nu) = \int_{SO(p)} \exp\left(\tfrac{1}{2} \operatorname{tr} \, H D_\sigma H' D_\nu'\right) \, \mathrm{d}H$$

Note the remarkable connection:

Baseball scores $\longleftrightarrow$ Harish-Chandra's integral!

My poster will raise open problems concerning the *total positivity* properties of $I_p(\sigma, \nu)$.