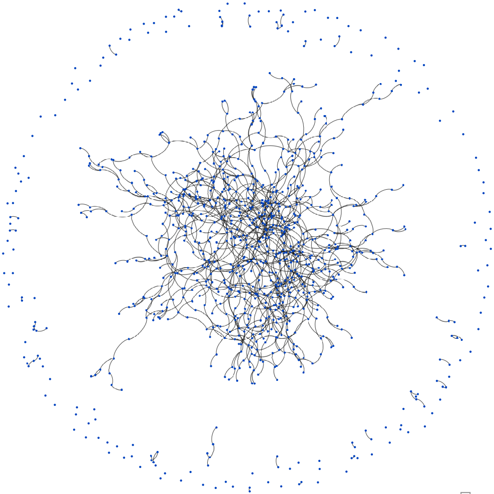
Spectral analysis of sparse random graphs

Justin Salez (LPSM)



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Question: how does μ_G typically look when G is large?

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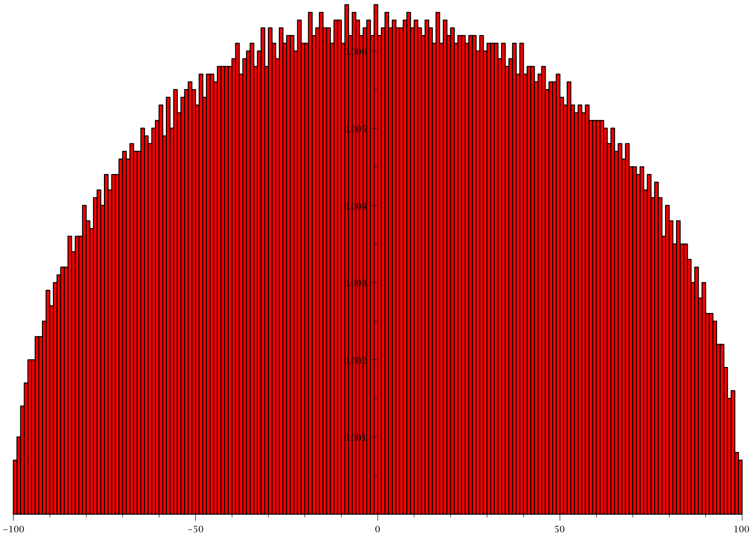
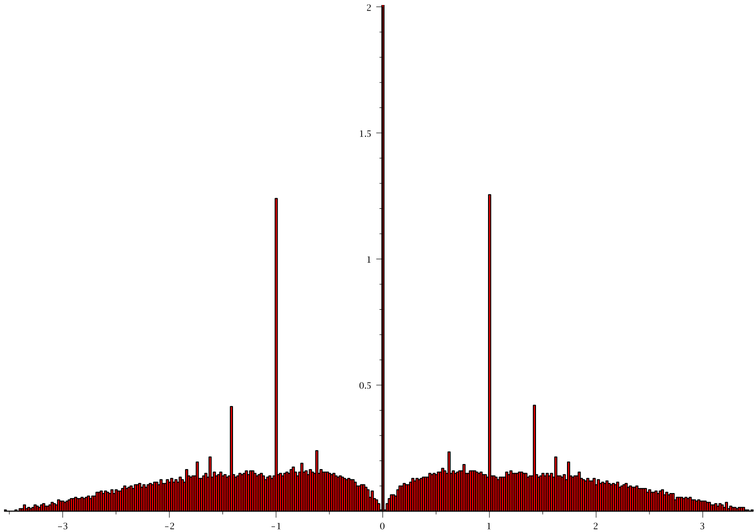
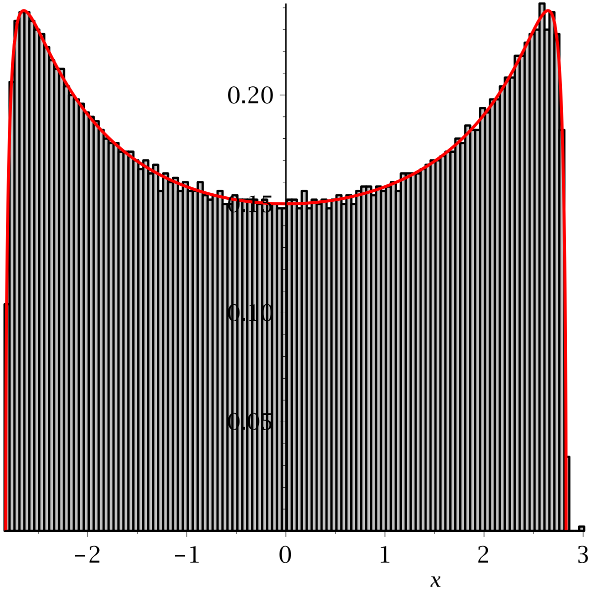


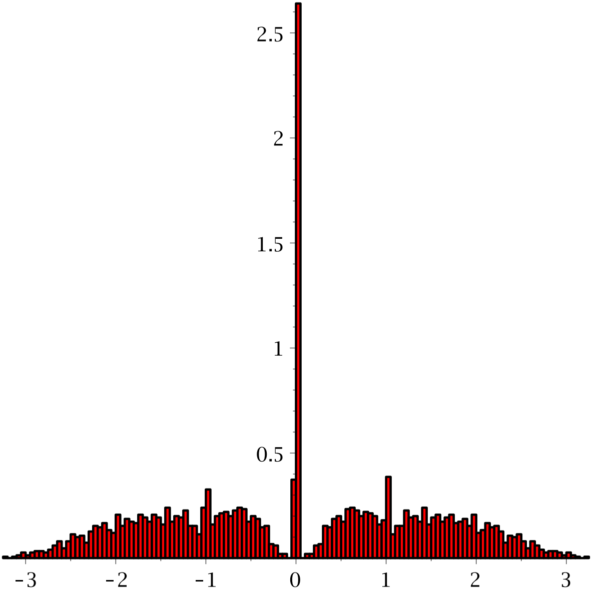
Erdős-Rényi model with average degree 3



Uniform random 3-regular graph



Uniform random tree on 3000 nodes



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Conjectures were proposed by Bordenave, Sen, Virág (JEMS 2017).

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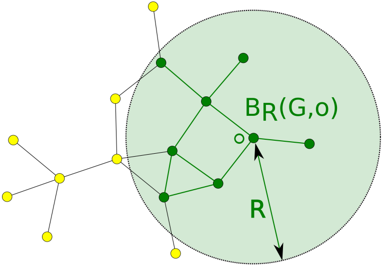

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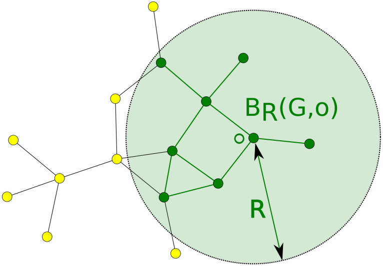

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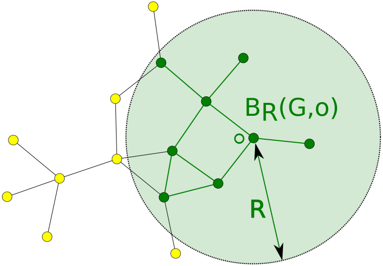

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$\mathcal{G}_\star := \{\text{locally finite, connected rooted graphs}\}$ is a Polish space.

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Intuition: $\mathcal{L}$ describes the local geometry around a typical vertex.

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Fact: uniform rooting confers to every local weak limit $\mathcal{L}$ a powerful form of stationarity known as **unimodularity**.

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Note: in general, the existence of $(A_G - z)^{-1}$ is a delicate issue...

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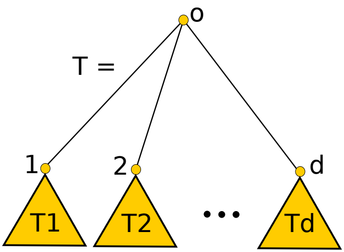

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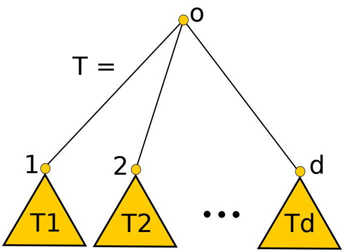

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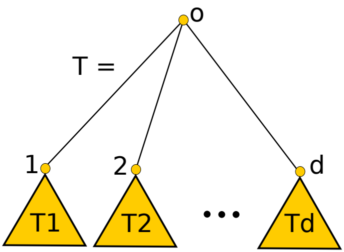

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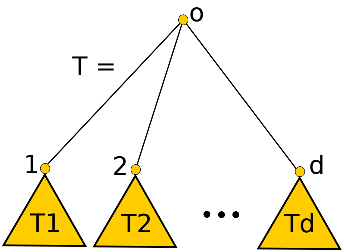

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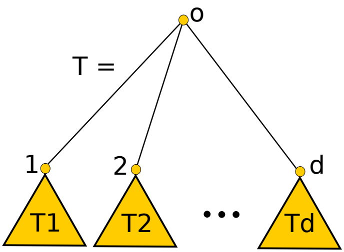

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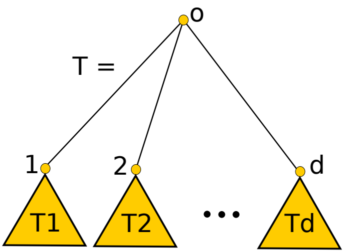

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- See the wonderful survey by Bordenave for details.

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Theorem (Main result). On a unimodular random tree, the connected components of $\mathcal{S}$ are almost-surely finite. Moreover,

$$\mu_{\mathcal{L}}(\{\lambda\}) \quad = \quad \mathbb{P}\left(o \in \mathcal{S}\right) - \frac{1}{2}\mathbb{E}\left[\deg_{\mathcal{S}}(o)\mathbf{1}_{(o \in \mathcal{S})}\right] - \mathbb{P}\left(o \in \partial\mathcal{S}\right).$$

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Remark. $\mathbb{A}$ is dense in $\mathbb{R}$, so these graphs have "rough" spectrum.

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Corollary. If $\mathcal{L}$ is supported on trees with degrees in $\{\delta, \ldots, \Delta\}$,

$$\Sigma_{pp}(\mathcal{L}) \quad \subseteq \quad \left\{\lambda \colon \tau(\lambda) < \frac{\Delta - 2}{\delta - 2}\right\}.$$

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Corollary. If $\mathcal{L}$ is supported on trees with degrees in $\{\delta, \ldots, \Delta\}$,

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- If $\frac{\Delta-2}{\delta-2} \leq 4$ then $\Sigma_{pp}(\mathcal{L}) \subseteq \{-\sqrt{2}, -1, 0, +1, \sqrt{2}\}$.

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Anchored isoperimetric constant:

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Remark. The anchored isoperimetric constant of a GWT conditioned on non-extinction is positive (Chen & Peres, 2004).

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This was conjectured by Bordenave, Sen & Virág (JEMS 2017).

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- When $\pi = \mathit{Poisson}(c)$, does $\mu_{ac}(\mathbb{R}) > 0$ as soon as $c > 1$?

Thank you !