Systems of infinitely many hard balls with long range interaction

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Introduction 1

Systems of Brownian balls (BM with the hard core interaction)

$$dX_t^j = dB_t^j + \sum_{k\neq j}(X_t^j - X_t^k)dL_t^{jk}, \quad j \in \Lambda \qquad \text{(SDE-}\Lambda\text{)}$$

where B_t^j, $j \in \Lambda$ are independent Brownian motions,
and L_t^{jk} $j,k \in \Lambda$ are non-decreasing functions satisfying

$$L_t^{jk} = \int_0^t \mathbf{1}(|X_s^j - X_s^k| = r)dL_s^{jk}$$

and $r > 0$ is the diameter of hard balls.

(1) $\Lambda = \{1, 2, \dots, n\}$
Existence and uniqueness of solutions (Saisho-Tanaka [1986])

(2) $\Lambda = \mathbb{N}$, equilibrium case
Existence and uniqueness of solutions (T. [1996])

Introduction 2

Potentials:

$\Phi : \mathbb{R}^d \to (-\infty, \infty]$ self-potential, free potential

$\Psi : \mathbb{R}^d \times \mathbb{R}^d \to (-\infty, \infty)$ pair-interaction potential, $\Psi(x,y) = \Psi(y,x)$

In this talk we consider the case that Φ is smooth, and

$$\Psi = \Psi_{\mathrm{hard}} + \Psi_{\mathrm{sm}}$$

and

$$\Psi_{\mathrm{hard}}(x,y) = \begin{cases} 0 & \text{if} \quad |x-y| \geq r, \\ \infty & \text{if} \quad |x-y| < r, \end{cases}$$ the hard core pair potential

$\Psi_{\mathrm{sm}}(x,y) = \Psi_{\mathrm{sm}}(x-y)$: a translation invariant smooth potential

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Systems of hard balls with interaction

$$dX_t^j = dB_t^j - \frac{1}{2}\nabla\Phi(X_t^j)dt - \frac{1}{2}\sum_{k\in\Lambda, k\neq j}\nabla\Psi_{\mathrm{sm}}(X_t^j - X_t^k)dt$$
$$+ \sum_{k\in\Lambda, k\neq j}(X_t^j - X_t^k)dL_t^{jk}, \quad j\in\Lambda, \qquad \text{(SDE2-}\Lambda\text{)}$$

where B_t^j, $j\in\Lambda$ are independent Brownian motions,
and L_t^{jk} $j,k\in\Lambda$ are non-decreasing functions satisfying

$$L_t^{jk} = \int_0^t \mathbf{1}(|X_s^j - X_s^k| = r)dL_s^{jk}, \quad i,j\in\Lambda.$$

We put

$$b(y,\{x^j\}) = -\frac{1}{2}\nabla\Phi(y) - \frac{1}{2}\sum_j \nabla\Psi_{\mathrm{sm}}(y - X_t^j).$$

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Configuration space of unlabeled balls with diameter $r > 0$ in $\mathbb{R}^d$:

$$\mathfrak{X} = \{\xi = \{x_j\}_{j\in\Lambda} : |x^j - x^k| \geq r \quad j \neq k, \ \Lambda: \text{ countable}\}$$

(Exponential decay)$\Psi_{\rm sm}$ is a potential of short range: For $\{x^j\} \in \mathfrak{X}$

(i) $\sum_{k:k\neq j} |\Psi_{\rm sm}(x^j - x^k)| < \infty$ and $\sum_{k:k\neq j} |\nabla\Psi_{\rm sm}(x^j - x^k)| < \infty$.

(ii) $\sum_{k:k\neq j} |\nabla^2\Psi_{\rm sm}(x^j - x^k)| < \infty$.

(iii) $\exists c_1, c_2, c_3 > 0$ such that for large enough R

$$\sum_{k:|x_j - x_k|>R} |\nabla\Psi_{\rm sm}(x^j - x^k)| \leq c_1 \exp(-c_2 R^{c_3}), \ \{x^j\} \in \mathfrak{X}$$

(3) $\Lambda = \mathbb{N}$, equilibrium case, $\Phi = \text{cons.}$, $\Psi_{\rm sm}$: (i), (ii), (iii)
Existence and uniqueness of solutions (Fradon-Roelly-T. [2000])

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(Polynomial decay) Ψ_{sm} is a potential of long range: : For $\{x^j\} \in \mathfrak{X}$

(i) $\sum_{k:k\neq j} |\Psi_{\rm sm}(x^j - x^k)| < \infty$, and $\sum_{k:k\neq j} |\nabla\Psi_{\rm sm}(x^j - x^k)| < \infty$.

(ii) $\sum_{k:k\neq j} |\nabla^2\Psi_{\rm sm}(x^j - x^k)| < \infty$.

Examples

(2.1) Lennard-Jones 6-12 potential ($d = 3$, $\Psi_{6,12}(x) = \{|x|^{-12} - |x|^{-6}\}$.)

$$b(y, \{x^k\}) = \frac{\beta}{2} \sum_k \{ \frac{12(y - x^j)}{|y - x^k|^{14}} - \frac{6(y - x^k)}{|y - x^k|^8} \}$$

(2.2) Riesz potentials ($d < a \in \mathbb{N}$ and $\Psi_a(x) = (\beta/a)|x|^{-a}$.)

$$b(y, \{x^k\}) = \frac{\beta}{2} \sum_k \frac{y - x^k}{|y - x^k|^{a+2}}$$

Skorohod equation

D : a domain in $\mathbb{R}^m$, $\quad W(\mathbb{R}^m) = C([0,\infty) \to \mathbb{R}^m)$

For $x \in \overline{D}$ and $w \in W_0(\mathbb{R}^m) = \{w \in W(\mathbb{R}^m) : w(0) = 0\}$ we consider the following equation called Skorohod equation

$$\zeta(t) = x + w(t) + \varphi(t), \quad t \geq 0 \tag{Sk}$$

A solution is a pair (ζ, φ) satisfying (Sk) with the following two conditions

(1) $\zeta \in W(\overline{D})$

(2) φ is an $\mathbb{R}^m$-valued continuous function with bounded variation on each finite time interval satisfying $\varphi(0) = 0$ and

$$\varphi(t) = \int_0^t \mathbf{n}(s) d\|\varphi\|_s, \quad \|\varphi\|_t = \int_0^t \mathbf{1}_{\partial D}(\zeta(s)) d\|\varphi\|_s$$

where $\mathbf{n}(s) \in \mathcal{N}_{\zeta(s)}$ if $\zeta(s) \in \partial D$, $\|\varphi\|_t$ denotes the total variation of φ on $[0,t]$.

Conditions(A) and (B)

$\mathcal{N}_x = \mathcal{N}_x(D)$ is the set of inward normal unit vectors at $x \in \partial D$,

$$\mathcal{N}_x = \bigcup_{\ell > 0} \mathcal{N}_{x,\ell} \quad \mathcal{N}_{x,\ell} = \{\mathbf{n} \in \mathbb{R}^{\mathbf{m}} : |\mathbf{n}| = 1, U_\ell(x - \ell\mathbf{n}) \cap D = \emptyset\}$$

(A) (Uniform exterior sphere condition) There exists a constant $\alpha_0 > 0$ such that

$$\forall x \in \partial D, \quad \mathcal{N}_x = \mathcal{N}_{x,\alpha_0} \neq \emptyset$$

(B) There exists constants $\delta_0 > 0$ and $\beta_0 \in [1, \infty)$ such that for any $x \in \partial D$ there exists a unit vector $\mathbf{l}_x$ verifying

$$\forall \mathbf{n} \in \bigcup_{y \in U_{\delta_0}(x) \cap \partial D} \mathcal{N}_y, \quad \langle \mathbf{l}_x, \mathbf{n} \rangle \geq \frac{1}{\beta_0}$$

Under (A) and (B), the unique solution of (SK) exists (Saisho[1987]). (Ref. Tanaka[1979], Lions-Sznitman[1984])

(i) The configuration space of n balls with diameter $r > 0$:

$$D_n = \{\mathbf{x} = (x^1, x^2, \dots, x^n) \in (\mathbb{R}^d)^n : |x^j - x^k| > r, \quad j \neq k\}$$

satisfies conditions (A) and (B) (Saisho-Tanaka [1986])

(ii) Suppose that D satisfies conditions (A) and (B).
Let $\zeta^{(i)}$ is the solution of (Sk) for $x^{(i)}$ and $w^{(i)}$, $i = 1, 2$. Then there exists a constant $C = C(\alpha_0, \beta_0, \delta_0)$ such that

$$|\zeta^{(1)}(t) - \zeta^{(2)}(t)| \leq (\|w^{(1)} - w^{(2)}\|_t + |x^{(1)} - x^{(2)}|)e^{C(\|\varphi^{(1)}\|_t + \|\varphi^{(2)}\|_t)}$$

and for each $T > 0$

$$\|\varphi^{(i)}\|_t \leq f(\Delta_{0,T,\cdot}(w), \|w^{(i)}\|_t), \quad 0 \leq t \leq T, \ i = 1, 2,$$

where f is a function on $W_0(\mathbb{R}^+) \times \mathbb{R}^+$ depending on $\alpha_0, \beta_0, \delta_0$ and $\Delta_{0,T,\delta}(w)$ denote the modulus of continuity of w in $[0, T]$.

Prelimiary 1

Configuration space of unlabeled particle in $\mathbb{R}^d$:

$$\mathfrak{M} = \mathfrak{M}(\mathbb{R}^d) = \Big\{ \xi(\cdot) = \sum_{j \in \Lambda} \delta_{x^j}(\cdot) : \xi(K) < \infty, \ \forall K \subset \mathbb{R}^d \text{ compact} \Big\}$$

The index set Λ is countable.
$\mathfrak{M}$ is a Polish space with the vague topology.

$$\mathfrak{N} \subset \mathfrak{M} \text{ is relative compact} \Leftrightarrow \sup_{\xi \in \mathfrak{N}} \xi(K) < \infty, \ \forall K \subset \mathbb{R}^d \text{ compact}$$

Configuration space of unlabeled balls with radius $r > 0$ in$\mathbb{R}^d$:

$$\mathfrak{X} = \{ \xi = \{x_j\}_{j \in \Lambda} : |x^j - x^k| \geq r \quad j \neq k, \ \Lambda : \text{ countable} \}$$

$\mathfrak{X}$ is compact with the vague topology, and the space of probability measures on $\mathfrak{X}$ is also compact with the weak topology.

Polynomial functions

A function f on $\mathfrak{M}$ (or $\mathfrak{X}$) is called a polynomial function if it is represented as

$$f(\xi) = Q\left(\langle\varphi_1, \xi\rangle, \langle\varphi_2, \xi\rangle, \dots, \langle\varphi_\ell, \xi\rangle\right)$$

with a polynomial function Q on $\mathbb{R}^\ell$, and smooth functions ϕ_j, $1 \leq j \leq \ell$, with compact supports, where

$$\langle\varphi, \xi\rangle = \int_{\mathbb{R}^d} \varphi(x)\xi(dx).$$

We denote by $\mathcal{P}$ the set of all polynomial functions on $\mathfrak{M}$. A polynomial function is local and smooth: $\exists K$ compact s.t.

$$f(\xi) = f(\pi_K(\xi)) \quad \text{and} \quad f(\xi) = f(x_1, \dots, x_n) \text{ is smooth}$$

where $n = \xi(K)$ and $\pi_K(\xi)$ is the restriction of ξ on K.

Square fields

For $f \in \mathcal{P}$ we introduce the square field on $\mathfrak{M}$ defined by

$$\mathbf{D}(f,g)(\xi) = \frac{1}{2}\int_{\mathbb{R}^d} \xi(dx)\nabla_x f(\xi)\cdot\nabla_x g(\xi).$$

For a RPF (a probability measure μ on $\mathfrak{M}$), we introduce the bilinear form on $L^2(\mu)$ defined by

$$\mathcal{E}^\mu(f,g) = \int_{\mathfrak{M}} \mathbf{D}(f,g)(\xi)\mu(d\xi), \quad f,g \in \mathcal{D}_\circ^\mu$$
$$\mathcal{D}_\circ^\mu = \{f \in \mathcal{P} : \| f \|_1 < \infty\}.$$

where

$$\| f \|_1^2 = \| f \|_{L^2(\mu)}^2 + \mathcal{E}^\mu(f,f).$$

Quasi Gibbs state

Hamiltonian for Φ, Ψ on $S_\ell = \{x \in \mathbb{R}^d : |x| \le \ell\}$

$$H_\ell(\zeta) = \sum_{x \in \mathrm{supp}\zeta \cap S_\ell} \Phi(x) + \sum_{x,y \in \mathrm{supp}\zeta \cap S_\ell, x \neq y} \Psi(x,y),$$

Def.(Quasi Gibbs state)
A RPF μ is called a (Φ, Ψ)-quasi Gibbs state, if

$$\mu_{\ell,\xi}^m(d\zeta) = \mu(d\zeta | \pi_{S_\ell^c}(\xi) = \pi_{S_r^c}(\zeta), \zeta(S_\ell) = m),$$

satisfies that for $\ell, m, k \in \mathbb{N}$, μ-a.s. ξ

$$c^{-1} e^{-H_\ell(\zeta)} \Lambda_\ell^m(d\zeta) \le \mu_{\ell,\xi}^m(\pi_{S_\ell} \in d\zeta) \le c e^{-H_\ell(\zeta)} \Lambda_\ell^m(d\zeta)$$

where $c = c(\ell, m, \xi) > 0$ is a constant depending on ℓ, m, ξ,
Λ_ℓ^m is the rest. of PRF with int. meas. dx on $\mathfrak{M}_\ell^m = \{\xi(S_\ell) = m\}$.

Systems of unlabeled particles

We make assumptions on RPF μ

(A1) μ is a (Φ, Ψ)-quasi Gibbs state, and $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ satisfy

$$c^{-1}\Phi_0(x) \le \Phi(x) \le c\ \Phi_0(x)$$
$$c^{-1}\Psi_0(x-y) \le \Psi(x,y) \le c\ \Psi_0(x-y)$$

for some $c > 1$ and locally bounded from below and lower semi-continuous function Φ_0, Ψ_0 with $\{x \in \mathbb{R}^d : \Psi_0(x) = \infty\}$ being compact.

Let $k \in \mathbb{N}$.

(A2.k) $\displaystyle\sum_{k=1}^{\infty} k^n \mu(\mathfrak{M}_r^k) = \int_{\mathfrak{M}} \xi(S_r)\mu(d\xi) < \infty$, $\forall r, n \in \mathbb{N}$.

Systems of unlabeled particles 2

Th. (Osada 13)
Suppose that **(A1)-(A2.1)**. Then $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}_{\circ})$ is closable on $L^2(\mathfrak{M}, \mu)$, and the closure $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$ is a quasi-regular Dirichlet form. Moreover the associated diffusion process $(\Xi_t, \mathbb{P}_{\xi})$ can be constructed.

Remark

1) For a RPF μ on $\mathfrak{X}$, **(A2.k)** is always satisfied. If $\Psi = \Psi_{\rm hard} + \Psi_{\rm sm}$ and $\Psi_{\rm sm}$ is continuous, we can construct diffusion process on $\mathfrak{X}$ if μ is a quasi-Gibbs state.
2) Gibbs states of Ruelle class are quasi-Gibbs state. Then for Lennard-Jones 6-12 potential : $\Psi_{\rm sm} = \Psi_{6,12}(x) = \{|x|^{-12} - |x|^{-6}\}$. and Riesz potential : $\Psi_{\rm sm} = \Psi_a(x) = (\beta/a)|x|^{-a}$., $d < a \in \mathbb{N}$ we can apply the above theorem.

Systems of labeled particles

Suppose that $\mu(\mathfrak{X}_\infty) = 1$, where

$$\mathfrak{X}_\infty = \{\xi = \{x^j\}_{j\in\mathbb{N}} : |x^j - x^k| \geq r \quad j \neq k\}.$$

We call a function $\mathsf{l} : \mathfrak{X}_\infty \to (\mathbb{R}^d)^{\mathbb{N}}$ is called a label function, if

$$\mathsf{l}(\xi) = (x^j)_{j\in\mathbb{N}} \equiv \mathbf{x}, \qquad \text{if } \xi = \textstyle\sum_{j\in\mathsf{N}} \delta_{x^j} \equiv \mathsf{u}(\mathbf{x}).$$

We make the following assumption.

(A3) Ξ_t is an $\mathfrak{X}$-valued diffusion process in which each tagged particle never explodes

Under **(A3)** we can label particles, $\mathsf{l}(\Xi)_t = (X_t^i)_{i\in\mathbb{N}} \equiv \mathbf{X}_t$, such that

$$\Xi_t = \sum_{j\in\mathbb{N}} \delta_{X_t^j}, \quad \mathsf{l}(\Xi_0) = (X_0^j)_{j\in\mathbb{N}}.$$

ISDE representation

Let μ_x, $x \in \mathbb{R}^d$ be the Palm measure and $\mu^{[1]}$ be teh Campbell measure of PPF μ. Then $\mu^{[1]}(dxd\eta) = \mu_x(d\eta)\rho^1(x)dx$, where $\rho^1(x)$ is the correlation function of the first order. We make the following assumption:

(A4) A PPF μ on $\mathfrak{X}$ has the log derivarive $\mathsf{d}_\mu(x,\eta) \in L^1_{loc}(\mathbb{R}^d \times \mathfrak{X}, \mu^{[1]})$ **:** for any $f \in C_0^\infty(\mathbb{R}^d) \times \mathcal{P}$

$$- \int_{\mathbb{R}^d \times \mathfrak{M}} \nabla_x f(x,\eta)\mu^{[1]}(dxd\eta) = \int_{\mathbb{R}^d \times \mathfrak{M}} \mathsf{d}_\mu(x,\eta) f(x,\eta)\mu^{[1]}(dxd\eta)$$
$$+ \int_{\{(x,\eta):\eta\in\mathfrak{X}, x\in S_\eta\}} \mathcal{S}_\eta(dx)\mu_x(d\eta)\mathbf{n}_\eta(x)f(x,\eta),$$

where $\mathcal{S}_\eta$ is the surface measure of

$$S_\eta = \{x \in \mathbb{R}^d : |x-y| = r \text{ for some } y \in \eta\},$$

and $\mathbf{n}_\eta(x)$ is the normal vector of S_η at x.

We can extend the notion of log derivative in the distribution and write

$$\overline{\mathrm{d}}_\mu(x,\eta) = \mathbf{1}_{S_\eta^c}(x)\mathrm{d}_\mu(x,\eta) + \mathbf{1}_{S_\eta}(x)\mathbf{n}_\eta(x)\delta_x.$$

If the log derivative exists, we put $b(x,\eta) = \frac{1}{2}\mathrm{d}_\mu(x,\eta)$.

Th.(A modification of the result in Osada 12)
Assume the conditions **(A1)**, **(A3)** and **(A4)**. Then $(\mathbf{X}_t, \mathbf{P}_\mathbf{x})$ satisfies the following ISDE:

$$dX_t^j = dB_t^j + b\left(X_t^j, \sum_{k\neq j}\delta_{X_t^k}\right)dt + \sum_{k\neq j}\left(X_t^j - X_t^k\right)dL_t^{jk}$$

$$L_t^{jk} = \int_0^t \mathbf{1}(|X_s^j - X_s^k| = r)dL_s^{jk} \quad j \in \mathbb{N}. \qquad \text{(ISDE)}$$

where L_t^{jk} is a non-decreasing function.

ISDE representation

Ex. 1.

Let $\Psi_{\rm sm}$ be a potential such that

(i) $\sum_{k\neq j} |\Psi_{\rm sm}(x^j - x^k)| < \infty$, and $\sum_{kk\neq j} |\nabla\Psi_{\rm sm}(x^j - x^k)| < \infty$,

and μ is a canonical Gibbs state with potential (Φ, Ψ). Then the log derivative of μ exists and

$$\mathsf{d}_\mu(x, \eta) = -\nabla\Phi(x) - \sum_{y\in\eta} \nabla\Psi_{\rm sm}(x - y).$$

If $|\nabla\Phi|$ is of linear growth, **(A3)** holds and $(\mathbf{X}_t, \mathbf{P}_\mathbf{x})$ has the following ISDE representation

$$dX_t^j = dB_t^j - \frac{1}{2}\nabla\Phi(X_t^j)dt - \frac{1}{2}\sum_{k\in\mathbb{N}, k\neq j} \nabla\Psi_{\rm sm}(X_t^j - X_t^k)dt$$
$$+ \sum_{k\in\Lambda, k\neq j} (X_t^j - X_t^k)dL_t^{jk}, \quad j\in\mathbb{N}. \qquad \text{(ISDE2)}$$

Uniqueness of solutions

Let $\mathbf{X}$ be a solution of (ISDE). We consider the following SDE for $m \in \mathbb{N}$:

$$dY_t^{m,j} = dB_t^j + b_{\mathsf{X}}^{m,j}(Y_t^{m,j}, \mathbf{Y}_t^m)dt + \sum_{\ell=m+1}^{\infty} (Y_t^{m,j} - X_t^\ell) dL_t^{m,j\ell}$$

$$+ \sum_{k \neq j} (Y_t^{m,j} - Y_t^{m,k}) dL_t^{m,jk}, \qquad \text{(SDE-m)}$$

$$L_t^{m,jk} = \int_0^t \mathbf{1}(|Y_s^{m,j} - Y_s^{m,k}| = r) dL_s^{jk}, \quad j,k = 1,2,\ldots,m,$$

$$L_t^{m,j\ell} = \int_0^t \mathbf{1}(|Y_s^{m,j} - X_s^\ell| = r) dL_s^{j\ell}, \quad \ell = m+1, \ldots,$$

$$Y_0^{m,j} = X_0^j, \quad j = 1, \ldots, m,$$

where $b_{\mathbf{X}}^{m,j}(t, \mathbf{y}) = b\Big(y^j, \sum_{k \neq j}^{m} \delta_{y^k} + \sum_{k=m+1}^{\infty} \delta_{X_t^k}\Big).$

Uniqueness of solutions 2

We make the following assumption.

(IFC) $\forall m \in \mathbb{N}$, a solution of (SDE-m) exists and is pathwise unique.

We also introduce the conditions on the process $(\mathbf{X}, \mathbf{P_s})$.

$$\left(\Xi_t = \mathsf{u}(\mathbf{X}_t), \quad \mathbb{P}_{\mathsf{u}(\mathbf{x})} = \mathbf{P}_{\mathbf{x}} \circ \mathsf{u}^{-1}, \quad \mathbb{P}_\mu = \int_{\mathfrak{X}} \mathbb{P}_\xi \, \mu(d\xi) \right)$$

(μ-AC) (μ-absolutely continuity condition): $\forall t > 0$

$$\mathbb{P}_\mu \circ \Xi_t^{-1} \prec \mu \quad \forall t > 0$$

(NBJ) (No big jump condition): $\forall r, \forall T \in \mathbb{N}$

$$\mathbb{P}_\mu \circ \mathsf{l}^{-1}(\mathsf{m}_{r,T}(\mathbf{X}) < \infty) = 1,$$

where

$$\mathsf{m}_{r,T}(\mathbf{X}) = \inf\{m \in \mathbb{N} \, ; \, |X^n(t)| > r, \forall n > m, \forall t \in [0, T]\}.$$

Uniqueness of solutions 3

The tail σ-field on $\mathfrak{X}$ is defined as

$$\mathcal{T}(\mathfrak{X}) = \bigcap_{r=1}^{\infty} \sigma(\pi_{S_r^c})$$

We introduce the following condition on a RPF μ.

(TT) (tail trivial) $\quad \mu(\mathsf{A}) \in \{0, 1\} \quad$ for any $\mathsf{A} \in \mathcal{T}(\mathfrak{X})$,.

Th. (A modification of the result in Osada-T. arXiv:1412.8674v8)
(i) Suppose that **(A1)**, **(A3)**, **(A4)** and **(TT)**. Then there exists a strong solution $(\mathbf{X}, \mathbf{B})$ of (ISDE) satisfying **(IFC)**, **(μ-AC)** and **(NBJ)**.

(ii) Solutions of (ISDE) satisfying **(IFC)**, **(μ-AC)** and **(NBJ)** are pathwise unique.

Non tail trivial case

Remark In case condition **(TT)** is not satisfied, we can discuss the uniqueness by using the decomposition

$$\mu = \int_{\mathfrak{X}} \mu(d\eta)\mu^{\eta}_{\mathrm{Tail}}$$

where $\mu^{\eta}_{\mathrm{Tail}} = \mu(\cdot|\mathcal{T}(\mathfrak{X}))(\eta)$: the regular conditional distribution with respect to the tail σ-field. In this case the uniqueness is derived if
(μ_{Tail}-AC) for μ-a.s. η

$$\mu^{\eta}_{\mathrm{Tail}} \circ \Xi_t^{-1} \prec \mu^{\eta}_{\mathrm{Tail}} \quad \forall t > 0$$

is satisfied instead of **(μ-AC)**. This means that there is no $A \in \mathcal{T}(\mathfrak{M})$ such that for $\mu^{\eta}_{\mathrm{Tail}}$-a.s. ξ

$$\mathbb{P}_{\xi}(\Xi_s \in A) \neq \mathbb{P}_{\xi}(\Xi_t \in A)$$

for some $0 \leq s < t$.

Applications

Let $\mathbf{X}$ be the process in **Ex.1**, that is $\mathbf{X}$ is a solution of

$$dX_t^j = dB_t^j - \frac{1}{2}\nabla\Phi(X_t^j)dt - \frac{1}{2}\sum_{k\in\mathbb{N},k\neq j}\nabla\Psi_{\mathrm{sm}}(X_t^j - X_t^k)dt + \sum_{k\in\Lambda,k\neq j}(X_t^j - X_t^k)dL_t^{jk}, \quad j\in\mathbb{N} \qquad \text{(ISDE2)}$$

Theorem 1.
1) Suppose that Ψ_{sm} satisfies $\sum_{k\neq j}|\nabla^2\Psi_{\mathrm{sm}}(x^j - x^k)| < \infty$ and $\nabla\Phi$ is of Linear growth. Then **(IFC)** holds.
2) Moreover, assume that **(TT)** is satisfied. Then there exists a unique strong solution $(\mathbf{X}, \mathbf{B})$ of (ISDE2) satisfying **(IFC)**, **(μ-AC)** and **(NBJ)**.

Outline of the proof of Theorem 1

The domain of configurations of n balls

$$D_n = \{\mathbf{x} = (x^1, \dots, x^n) \in (\mathbb{R}^d)^n : |x^j - x^k| > r, \quad j \neq k\}$$

Domains of configurations of n balls with moving boundary

$$D_n(\mathbf{X}_t) = \{\mathbf{x} \in D_n : |x^j - X_t^k| > r, 1 \le j \le n,\ n+1 \le k\}$$

$D_n(\mathbf{X}_t)$, $t \ge 0$ do not always satisfy conditions (A) and (B).

The point $x \in \overline{D_n(\mathbf{X}_t)}$ that does not satisfies Conditions (A) or (B) for any $\alpha_0 > 0, \delta_0 > 0$ and $\beta_0 \in [1, \infty)$ is included in

$$\Delta_{t,n} = \{x \in \overline{D_n} : \exists j \text{ s.t. } \sharp\{k : |x^j - x^k| = r \text{ or } |x^j - X_t^k| = r\} \ge 2\}$$

The set of configurations of triple collision.

We introduce the hitting time

$$\tau^m = \inf\{t > 0 : Y_t^m \in \Delta_{t,n}\}.$$

For $t < \tau^m$ solutions of (SDE-m) is pathwaise unique, and

$$(Y_t^{m,1}, \dots, Y_t^{m,m}, X_t^{m+1}, \dots) = \mathbf{X}_t, \quad t < \tau.$$

On the othere hand we see that

$$\mathrm{Cap}_\mu(\Delta) = 0,$$

where Cap_μ is the capacity associated with the Dirichlet form $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ and

$$\Delta = \{\xi = \sum_{j\in\mathbb{N}} \delta_{x^j} \in \mathfrak{X} : \exists k \text{ s.t. } \sharp\{k : |x^j - x^k| = r\} \geq 2\}.$$

Hence, we have $\tau^m = \infty$ a.s.

Logarithmic potentials

Consider the logarithmic potentail:

$$\Psi_{\log}(x) = \beta \log |x|$$

If $\Psi = \Psi_{\log}$, that is the hard core does not exists, there is a quasi-Gibbs state related to the pair potential Ψ. [Osada 2012, 2013]

(i) Sine-β RPF $\mu_{\sin,\beta}$ ($d = 1$, $\beta = 1, 2, 4$) is a $\Psi_{\log}$-quasi-Gibbs state and the log derivative is given by

$$\mathrm{d}(y, \{x^k\}) = \beta \lim_{L\to\infty} \left\{ \sum_{k:|x^k|<L} \frac{1}{y - x^k} \right\}$$

(ii) Ginibre RPF μ_{Gin} ($d = 2$, $\beta = 2$) is a $\Psi_{\log}$-quasi-Gibbs state and the log derivative is given by

$$\mathrm{d}(y, \{x^k\}) = 2 \lim_{L\to\infty} \sum_{k:|y-x^k|<L} \frac{y - x^k}{|y - x^k|^2}$$

Logarithmic potentials 2

The existence and uniqueness of solutions has been shown for the following ISDEs [Osada 2012, Osada-T. arXiv:1412.8674v8]

(i) Dyson model $(d = 1,\ \beta = 1, 2, 4)$

$$dX_t^j = dB_t^j + \lim_{L\to\infty} \left\{ \frac{\beta}{2} \sum_{k\neq j, |X_t^k|<L} \frac{1}{X_t^j - X_t^k} \right\} dt, \quad j \in \mathbb{N}$$

(ii) Ginibre interacting Brownian motions (d=2)

$$dX_t^j = dB_t^j + \lim_{L\to\infty} \sum_{k\neq j,\ |X_t^j - X_t^k|<L} \frac{X_t^j - X_t^k}{|X_t^j - X_t^k|^2} dt, \quad j \in \mathbb{N}$$

Logarithmic potentials 3

Suppose that

$$\Psi = \Psi_{\mathrm{hard}} + \Psi_{\mathsf{log}}.$$

We may consider the following problems:

(1) Are there Ψ-quasi Gibbs states having log derivatives

(i) $\beta = 1, 2, 4$, $d = 1$

$$\mathrm{d}(y, \{x^k\}) = \beta \lim_{L \to \infty} \left\{ \sum_{k : |x^k| < L} \frac{1}{y - x^k} \right\}$$

(ii) $\beta = 2$, $d = 2$

$$\mathrm{d}(y, \{x^k\}) = 2 \lim_{L \to \infty} \sum_{k : |y - x^k| < L} \frac{y - x^k}{|y - x^k|^2}$$

(2) Moreover,

(iii) For general $\beta > 0$

(iv) For general dimension

Logarithmic potentials $d = 1, \beta = 1, 2, 4.$

Eigenvalues distribution of $N \times N$ Gaussian ensemble :

$$\check{m}^N_\beta(d\mathbf{x}_N) = \frac{1}{Z} h_N(\mathbf{x}_N)^\beta e^{-\frac{\beta}{4}|\mathbf{x}_N|^2} d\mathbf{x}_N, \quad h_N(\mathbf{x}_N) = \prod_{j<k} |x_j - x_k|$$

($\beta = 1$ GOE, $\beta = 2$ GUE, $\beta = 4$ GSE.)
Under the scaling $y_j = \sqrt{N} x_j$, the distribution of $\{y_j\}_{j=1}^N$ under $\check{m}^N_\beta(d\mathbf{x}_N)$ is given by

$$\check{\mu}^N_{\mathrm{bulk},\beta}(d\mathbf{y}_N) = \frac{1}{Z} h_N(\mathbf{y}_N)^\beta e^{-\frac{\beta}{4N}|\mathbf{y}_N|^2} d\mathbf{y}_N.$$

Sine β RPF $\mu_{\mathrm{sin},\beta}$ is obtained by the limit:

$$\mu^N_{\mathrm{bulk},\beta} \to \mu_{\sin,\beta}, \quad N \to \infty.$$

Logarithmic potentials $d = 1, \beta = 1, 2, 4.$

$\Psi = \Psi_{\log} + \Phi_{\mathrm{hard}}$ (with hard core)

$$\check{\mu}^{N,r}_{\mathsf{bulk},\beta}(d\mathbf{y}_N) = \frac{1}{Z} h_{N,r}(\mathbf{y}_N)^{\beta} e^{-\frac{\beta}{4N}|\mathbf{y}_N|^2} d\mathbf{y}_N,$$

$$h_{N,r}(\mathbf{y}_N) = \prod_{j<k} |x^j - x^k| \mathbf{1}(|x^j - x^k| \geq r)$$

Since $\mathfrak{X}$ is compact there exists a sequence $\{N_\ell\}_{\ell \in \mathbb{N}}$ such that

$$\mu^{N_\ell,r}_{\mathsf{bulk},\beta} \to \mu^{r}_{\mathsf{sin},\beta}, \quad \ell \to \infty.$$

Lemma 1.

There exists $r_0(\beta) > 0$ such that if $r \in (0, r_0)$, $\mu^{r}_{\mathsf{sin},\beta}$, is a quasi-Gibbs state whose log derivative

$$\mathsf{d}(y, \{x^k\}) = \beta \lim_{L \to \infty} \left\{ \sum_{k:|x^k|<L} \frac{1}{y - x^k} \right\}$$

Logarithmic potentials with the hard core

Then we have the following ISDE representation: $(\beta = 1, 2, 4)$

$$dX_t^j = dB_t^j + \lim_{L\to\infty}\left\{\frac{\beta}{2}\sum_{k\neq j, |X_t^k|<L}\frac{1}{X_t^j - X_t^k}\right\}dt + \sum_{k\in\mathbb{N}, k\neq j}(X_t^j - X_t^k)dL_t^{jk}, \quad j\in\mathbb{N}. \tag{ISDE3}$$

Theorem 2.
There exists a unique strong solution $(\mathbf{X}, \mathbf{B})$ of (ISDE3) satisfying **(IFC)**, **(**μ_{Tail}**-AC)** and **(NBJ)**.

Problems

(1) Are there Ψ-quasi Gibbs states having log derivatives

(i) $\beta = 1, 2, 4,\ d = 1$

$$\mathrm{d}(y, \{x^k\}) = \beta \lim_{L\to\infty} \left\{ \sum_{k:|x^k|<L} \frac{1}{y - x^k} \right\}$$

(ii) $\beta = 2,\ d = 2$

$$\mathrm{d}(y, \{x^k\}) = 2 \lim_{L\to\infty} \sum_{k:|y-x^k|<L} \frac{y - x^k}{|y - x^k|^2}$$

(2) Moreover,

(iii) For general $\beta > 0$

(iv) For general dimension

Thank you for your attention