Matrix Liberation Process and A Free Probability Question

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Free Probability

Free probability is a twofold theory. Namely, it is

- a useful toolkit to analyze the natural operator algebra $L(\mathbb{F}_r)$ arising from a free group $\mathbb{F}_r$ and its relatives (including free products of operator algebras);
- a framework to capture and investigate the large N limit of the "empirical distribution" (in what sense ?) of an independent family of RMs.

The key concept of free probability is the so-called free independence, which is the large N limit of "independence" through RMs.

My objective is to obtain a better understanding of free independence through the study of free probability analogs of mutual information. More precisely, we want a correct quantity measuring the "degree" of free independence, and it should be a free probability analog of mutual information. My tools are a matrix-valued stochastic process as well as large deviations techniques following the idea due to Guionnet et al. dealing with indep. Matrix BMs.

Unitary BM

M_N^{sa} = the $N \times N$ selfadjoint matrices ($\cong \mathbb{R}^{N^2}$)
$B(t)$ = an N^2-dimensional standard BM

We regard $B(t)/\sqrt{N}$ as an $N \times N$ matrix-valued stochastic process $H_N(t)$ via $\mathbb{R}^{N^2} \cong M_N^{sa}$ as Euclidean spaces. Then the SDE

$$\mathrm{d}U_N(t) = \sqrt{-1}\,\mathrm{d}H_N(t)U_N(t) - \frac{1}{2}U_N(t)\,\mathrm{d}t, \quad U_N = I_N$$

defines a (unique) BM on the unitary group $\mathrm{U}(N)$.

Known fact.

$U_N(t)$ converges to a Haar distributed unitary RM U_N in distribution as $t \to \infty$.

Matrix liberation process

Given data: $(X_N(0), Y_N(0)) \in (M_N^{sa})^2$.

We call the following pair of matrix-valued stochastic processes

$$t \mapsto (X_N(t), Y_N(t)) := (U_N(t)X_N(0)U_N(t)^*, Y_N(0))$$

the matrix liberation process starting at $(X_N(0), Y_N(0))$.

Facts.

- the spectral information of each of $X_N(t)$ and $Y_N(t)$ is independent of time t.
- the limit $(X_N(\infty), Y_N(\infty))$ in the weak convergence sense as $t \to \infty$ is given by $(U_N X_N(0) U_N{}^*, Y_N(0))$.

SDE for $(X_N(t), Y_N(t))$

$$
\begin{aligned}
\mathrm{d}X_N(t) &= \sqrt{-1}\left[\mathrm{d}H_N(t), X_N(t)\right] - \Big(X_N(t) - \mathrm{tr}_N(X_N(0))\Big)\,\mathrm{d}t, \\
\mathrm{d}Y_N(t) &= 0.
\end{aligned}
$$

If we write $H_N(t) = \sum_{\alpha,\beta=1}^{N} (B_{\alpha\beta}(t)/\sqrt{N})\, C_{\alpha\beta}$ with an orth. basis $C_{\alpha\beta}$ ($1 \le \alpha, \beta \le N$) of M_N^{sa} (w.r.t. HS), then

$$
\begin{aligned}
\mathrm{d}\langle X_N(t), C_{\alpha\beta}\rangle_{HS} = \sum_{\gamma,\delta=1}^{N} &\Big\langle \sqrt{-1}\Big[\frac{1}{\sqrt{N}} C_{\gamma\delta}, X_N(t)\Big], C_{\alpha\beta}\Big\rangle_{HS} \mathrm{d}B_{\gamma\delta}(t) \\
&- \Big\langle\Big(X_N(t) - \mathrm{tr}_N(X_N(0))\Big), C_{\alpha\beta}\Big\rangle_{HS} \mathrm{d}t
\end{aligned}
$$

This expression enables us to use the analysis of Gaussian space like Malliavin calculus.

Motivations

(1) We want to unify two approaches to possible mutual information in free probability.
[V1999] Voiculescu, The analogue of entropy and of Fisher's information measure in free probability theory, VI: Liberation and mutual free information, Adv. Math., 149 (1999), 101–166.
[OA level: use only macrostates]

[HMU2009] Hiai, Miyamoto and U., Orbital approach to microstate free entropy, IJM, 20 (2009), 227–273.
Followup: [Biane–Dabrowski2013], [U2014].
[$S = k \log W$ approach: use matricial microstates]

Macrostates = operators vs Microstates = (Random) matrices

(2) We want to investigate the large N limit of "adjoint actions" of (indep.) unitary BMs on matrices.

Framework of free probability

$(\Omega, \mathbb{P}) \rightsquigarrow (L^\infty(\Omega), \tau)$;

$$\tau(X) := \mathbb{E}[X] = \int_\Omega X(\omega)\,\mathbb{P}(d\omega), \quad X \in L^\infty(\Omega).$$

X, Y independent $\Leftrightarrow \tau[(f(X) - \tau(f(X))(g(Y) - \tau(g(Y)))] = 0$

$(L^\infty(\Omega), \tau) \rightsquigarrow$ (vN alg. M, (faithful normal) tracial state τ).

Free independence

$x, y \in M$ freely independent if

$$\tau(w(x, y)) = 0$$

whenever $w(x, y)$ is an alternating words of two kinds of elements $p(x) - \tau(p(x))$ and $q(y) - \tau(q(y))$.

Specht's criterion 1940/Free probability

Theorem. (Specht 1940)

Let $\mathbb{A} = (A_1, \ldots, A_n)$ and $\mathbb{B} = (B_1, \ldots, B_n)$ be n-tuples of $N \times N$ matrices. TFAE:

- A and B are unitarily equivalent, that is, there exists an $N \times N$ unitary matrix U such that $B_i = UA_iU^*$ for all $1 \leq i \leq n$.
- $\mathrm{Tr}_N(A_{i_1}^{\epsilon_1} A_{i_2}^{\epsilon_2} \cdots A_{i_m}^{\epsilon_m}) = \mathrm{Tr}_N(B_{i_1}^{\epsilon_1} B_{i_2}^{\epsilon_2} \cdots B_{i_m}^{\epsilon_m})$ for all possible $(i_1, i_2, \ldots, i_m)$ and $\epsilon_j \in \{\cdot, *\}$.

Conclusion

The noncomm. moments $\mathrm{tr}_N(w(X_N, Y_N))$ with matrices X_N, Y_N form a complete invariant for unitary equivalence.

Free probability deals with the large N limit of those invariants (=distributions) when X_N, Y_N are random.

Noncomm. notion on distributions

X: n-dim. RV $\rightsquigarrow \mu_X \in \mathcal{P}(\mathbb{R}^n) = S(C_0(\mathbb{R}^n))$

$$f \in C_0(\mathbb{R}^n) \mapsto \mu_X(f) = \int_{\mathbb{R}^d} f\, d\mu_X = \mathbb{E}[f(X)].$$

(x, y): a pair of (self-adjoint) RVs in $M \rightsquigarrow$

$$f \mapsto \tau(f(x, y)).$$

What are f ? → noncomm. polyn. in x, y → universal C^*-alg.

$(\mathcal{M}, \tau)$: tracial W^*-probability space, that is,
$\mathcal{M}$ is a vN algebra,
$\tau : \mathcal{M} \to \mathbb{C}$ a f.n.tracial state.
$u(t)$: a unitary operator-valued process in $\mathcal{M}$,
called *a free unitary BM*.

Assume that there exists $(x(0), y(0)) \in (\mathcal{M}^{sa})^2$ such that

$$\mathrm{tr}_N(P(X_N(0), Y_N(0))) \to \tau(P(x(0), y(0))), \quad \forall P$$

and $(x(0), y(0))$ is *freely independent* of $\{u(t), u(t)^*\}$.

Define

$$(x(t), y(t)) := (u(t)x(0)u(t)^*, y(0))$$

for every $t \geq 0$, which should be called the liberation process.

Large N limits – Free probability

Theorem (essentially due to Biane).

The finite dimensional distribution of $t \mapsto (X_N(t), Y_N(t))$ converges to that of $t \mapsto (x(t), y(t))$, that is,

$$\lim_N \mathbb{E}[\mathrm{tr}_N(P(\{X_N(t), Y_N(t')\}_{t,t'}))] = \tau\big(P(\{x(t), y(t')\}_{t,t'})\big), \quad \forall P.$$

The same proof with more recent results on unitary BMs shows that the above convergence can be strengthened to the almost sure sense:

$$\lim_N \mathrm{tr}_N(P(\{X_N(t), Y_N(t')\}_{t,t'})) = \tau\big(P(\{x(t), y(t')\}_{t,t'})\big), \quad \forall P.$$

(NB: the event of convergence depends on time parameters t_k appearing in P.)

Q. Is there an appropriate LDP for the above convergence ?

Our objects

$$\overset{(X_N(t),Y_N(t)):=}{(U_N(t)X_N(0)U_N(t)^*, Y_N(0))} \quad \overset{t\to\infty}{\longrightarrow} \quad \overset{(X_N(\infty),Y_N(\infty)):=}{(U_N X_N(0) U_N{}^*, Y_N(0))}$$

$$\downarrow_{N\to\infty} \qquad\qquad \downarrow_{N\to\infty}$$

$$(x(t), y(t)) := (u(t)x(0)u(t)^*, y(0)) \quad \overset{t\to\infty}{\longrightarrow} \quad \mathrm{FREE}(x(0), y(0))$$

Vertical limits: almost surely.
Horizontal limits: in distribution.

State space cTS

Assume that $R := \sup_N(\|X_N(0)\|_\infty \vee \|Y_N(0)\|_\infty) < +\infty$.
Consider the universal free product C^*-algebra

$$C := \bigstar_{t\geq 0}(C[-\overset{a(t)}{R}, R] \star C[-\overset{b(t)}{R}, R]),$$

that is, the universal C^*-algebra generated by

$$a(t) = a(t)^*, b(t) = b(t)^* \quad (t \geq 0)$$

with subject to $\|a(t)\|_\infty, \|b(t)\|_\infty \leq R$. Let cTS be the tracial states φ on C such that

$$(t_1, \ldots, t_m) \mapsto \varphi(c_1(t_1) \cdots c_m(t_m)) \quad \text{with } c_i = a \text{ or } b$$

are all continuous, or other words, $a(t), b(t)$ are strongly continuous in t in the GNS repn. associated with φ.

Topology on cTS

The metric $d(\varphi, \psi)$ with $\varphi, \psi \in cTS$ defined to be

$$\sum_{l,m=1}^{\infty} \frac{1}{2^{l+m}} \max_{c_i = a \text{ or } b} \max_{0 \le t_1, \dots, t_l \le m} |(\varphi - \psi)(c_1(t_1) \cdots c_l(t_l))| \wedge 1$$

makes cTS a complete metric space.

$\varphi_N, \varphi_\infty \in cTS$ are defined by

$$\varphi_N(P) := \mathrm{tr}_N(P(\{X_N(t), Y_N(t')\}_{t,t'})),$$
$$\varphi_\infty(P) := \tau(P(\{x(t), y(t')\}_{t,t'}))$$

with $P = P(\{a(t), b(t')\}_{t,t'})$, a polynomial in C.

φ_N are random, but φ_∞ is deterministic.

$d(\varphi_N, \varphi_\infty) \to 0$ corresponds to $(X_N(t), Y_N(t)) \to (x(t), y(t))$ as continuous processes.

Prospective LDP

For any open subset Γ and any closed subset Λ of (cTS, d),

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(\varphi_N \in \Gamma) \geq -\inf_{\psi\in\Gamma} I(\psi),$$
$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(\varphi_N \in \Lambda) \leq -\inf_{\psi\in\Lambda} I(\psi),$$

where $I : cTS \to [0,\infty]$ is a lower semicountinuous function whose level sets $\{I \leq \alpha\}$ are all compact (I is called a good rate function). Moreover, it is preferable that

$$I(\psi) = 0 \quad \Longleftrightarrow \quad \psi = \varphi_\infty,$$

that is, φ_∞ is a unique minimizer for I.

Rate function

For a given $\psi \in cTS$, construct a new $\psi^s \in cTS$ depending on time s as follows.

Step 1 Construct a tracial W^*-probability space $(\mathcal{M}^\psi, \tau^\psi)$, which includes $a^\psi(t), b^\psi(t)$ of processes whose joint distribution is φ (via $(a(t), b(t)) \mapsto (a^\psi(t), b^\psi(t))$) and a free unitary BM $u(t)$ such that $(a^\psi(\cdot), b^\psi(\cdot))$ and $u(\cdot)$ are freely independent.

Step 2 Consider a new pair $(a^{\psi^s}(t), b^{\psi^s}(t))$:

$$(a^{\psi^s}(t), b^{\psi^s}(t)) = (u((t-s)_+)a^\psi(t \wedge s)u((t-s)_+)^*, b^\psi(t)).$$

Step 3 $(a(t), b(t)) \mapsto (a^{\psi^s}(t), b^{\psi^s}(t))$ gives $\psi^s \in cTS$, that is, ψ^s is the "distribution" of $t \mapsto (a^{\psi^s}(t), b^{\psi^s}(t))$.

ψ^s is the liberation of ψ starting at time s.

Rate function

Let $\mathcal{A}$ be the $*$-subalgebra of C algebraically generated by the $a(t), b(t)$ and $\widetilde{\mathcal{A}}$ be the universal $*$-algebra generated by $\mathcal{A}$ and a "unitary" indeterminate $v(t)$ in addition.

Consider the derivations $\delta_s : \mathcal{A} \to \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}$ determined by

$$\begin{aligned} \delta_s a(t) &= \mathbf{1}_{[0,t]}(s) \\ &\quad \times (a(t)v(t-s) \otimes v(t-s)^* - v(t-s) \otimes v(t-s)^* a(t)), \\ \delta_s b(t) &= 0. \end{aligned}$$

With the mapping $\theta : c \otimes d \mapsto dc$ we define a linear map

$$\mathfrak{D}_s := \theta \circ \delta_s : \mathcal{A} \to \widetilde{\mathcal{A}}.$$

Via $(a(t), b(t), v(t)) \mapsto (a^{\psi^s}(t), b^{\psi^s}(t), u(t))$ we may regard $\widetilde{\mathcal{A}}$ as a $*$-subalgebra of $\mathcal{M}^\psi$.

Rate function

Let $E_s = E_s^{\tau^\psi}$ be the τ^ψ-conditional expectation from $\mathcal{M}^\psi$ onto the W^*-subalgebra generated by all

$$(a^\psi(t), b^\psi(t)) = (a^{\psi^s}(t), b^{\psi^s}(t)), \quad 1 \le t \le s.$$

Our *rate function* $I(\psi)$ of $\psi \in cTS$ is defined to be

$$\sup_{\substack{t \ge 0 \\ P = P^* \in \mathcal{A}}} \left\{ \psi^t(P) - \varphi_\infty(P) - \frac{1}{2} \int_0^t \| E_s(\mathfrak{D}_s P) \|_{\tau^\psi, 2}^2 \, ds \right\}.$$

Proposition.

$I : cTS \to [0, \infty]$ is a good rate function such that

$$I(\psi) = 0 \Longleftrightarrow \psi = \varphi_\infty.$$

Main Result (LD upper bound)

Theorem. [U16]

The LD upper bound holds with the good rate function I above, that is, for any closed $\Lambda \subset cTS$, we have

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(\varphi_N \in \Lambda) \le -\inf_{\psi\in\Lambda} I(\psi).$$

Moreover, $I(\psi) = 0 \iff \psi = \varphi_\infty$.

Corollary.

$$\lim_{N\to\infty} d(\varphi_N, \varphi_\infty) = 0 \quad \text{almost surely.}$$

Namely, the matrix liberation process converges to the corresponding liberation process as continuous processes almost surely.

Proof of Theorem

Use the same strategy as in Biane–Capitaine–Guionnet for self-adjoint matrix BMs.

Choose $P = P^* \in \mathcal{A}$, and consider the martingale

$$\begin{aligned} M_N^P(t) &= \mathbb{E}[\mathrm{tr}_N(P(\{X_N(t_1), Y_N(t_2)\}_{t_1,t_2}) \,\big|\, \mathcal{F}_t] \\ &\qquad - \mathbb{E}[\mathrm{tr}_N(P(\{X_N(t_1), Y_N(t_2)\}_{t_1,t_2})] \\ &= \mathbb{E}[\varphi_N(P) \,|\, \mathcal{F}_t] - \mathbb{E}[\varphi_N(P)] \\ &\longrightarrow \varphi^t(P) - \varphi_\infty(P) \quad \text{as } N \to \infty, \end{aligned}$$

where $\mathcal{F}_t$ is the (natural) filtration of σ-subalgebras for the given matrix BM $H_N(t)$.

Apply the Clark–Ocone formula to $M_N^P(t)$. (Need some techniques on SDEs in relation with Malliavin calculus.) Then we can find the integrant for an integral representation of the quadratic variation $\langle M_N^P \rangle(t)$ in terms of P: Find the new cyclic derivation $\mathfrak{D}_s$ in this way.

Proof of Theorem

Lemma.

The exponential function of

$$N^2\Big(M_N^P(t) - \frac{1}{2}\int_0^t \mathrm{d}s$$

$$\Big\|\mathbb{E}[(\mathfrak{D}_s P)(\{X_N(t_1), Y_N(t_2), U_N(t_3+s)U_N(s)^*\}_{t_1,t_2,t_3}) \,\big|\, \mathcal{F}_s]\Big\|_{\mathrm{tr}_N,2}^2\Big)$$

$$\longrightarrow \|E_s(\mathfrak{D}_s P)\|_{\tau^\psi,2}^2$$

defines a martingale $E_N^P(t)$; hence $\mathbb{E}[E_N^P(t)] = \mathbb{E}[E_N^P(0)] = 1$.

This suggests that the formula of our rate function.
We need to compute the large N limit of the red part above, and the keys are: the left increment property for $U_N(t)$, the asymptotic freeness for several indep. GUEs, and **Thierry Lévy's method** for unitary BMs (which plays a key role to give a uniform estimate in time).

Proof of Theorem

For a given $P \in \mathcal{A}$, define

$$(\mathfrak{D}_s P)_N := (\mathfrak{D}_s P)(\{X_N(t_1), Y_N(t_2), U_N(t_3+s)U_N(s)^*\}_{t_1,t_2,t_3}),$$
$$(\mathfrak{D}_s P)^\psi := (\mathfrak{D}_s P)(\{a^{\psi^s}(t_1), b^{\psi^s}(t_2), u(t_3)\}_{t_1,t_2,t_3}).$$

Key proposition.

For any given $P_1, \dots, P_n \in \mathcal{A}$, the $\overline{\lim}_{\varepsilon \searrow 0} \overline{\lim}_{N\to\infty}$ of the supremum over $s \geq 0$ of the essential sup-norm of

$$\mathrm{tr}_N(\mathbb{E}[(\mathfrak{D}_s P_1)_N \,|\, \mathcal{F}_s] \cdots \mathbb{E}[(\mathfrak{D}_s P_n)_N) \,|\, \mathcal{F}_s]) - \tau^\psi(E_s((\mathfrak{D}_s P_1)^\psi) \cdots E_s((\mathfrak{D}_s P_n)^\psi))$$

over the event $(d(\varphi_N, \psi) < \varepsilon)$ becomes 0.

These altogether enable us to prove

$$\frac{1}{N^2}\log \mathbb{P}(d(\varphi_N,\psi) < \varepsilon) \lesssim -\left(\psi^t(P) - \varphi_\infty(P) - \frac{1}{2}\int_0^t \|E_s(\mathfrak{D}_s P)\|_{\tau^\psi,2}^2\, ds\right)$$

as $N \to \infty$ and $\varepsilon \searrow 0$.

Works in progress toward $(U_N X_N(0) U_N^*, Y_N(0))$

Let C_0 be the universal C^*-algebra generated by $a = a^*$, $b = b^*$ with $\|a\|_\infty, \|b\|_\infty \le R$. Let $\pi_T : C_0 \to C$ be the injective $*$-hom. sending $(a, b) \mapsto (a(T), b(T))$. Let TS be the tracial states on C, and $\pi_T^* : cTS \to TS$ be defined as the dual map, that is,

$$\pi_T^*(\varphi) := \varphi \circ \pi_T, \quad \varphi \in cTS.$$

Fact: $\pi_T^*(\varphi_N) \in TS$ is the distribution of $(X_N(T), Y_N(T))$ with fixed T (the marginal distribution at time T). This is random.

When $T = \infty$, we define $\pi_\infty^*(\varphi_N)$ to be the distribution of $(X_N(\infty), Y_N(\infty)) := (U_N X_N(0) U_N^*, Y_N(0))$ with the unitary RM U_N under Haar prob. on $\mathrm{U}(N)$. This is also random.

Works in progress

The contraction principle implies:

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(\pi_T^*(\varphi_N) \in \Lambda) \leq - \inf_{\sigma\in\Lambda} I_T(\sigma)$$

with

$$I_T(\sigma) := \inf_{\pi_T^*(\varphi)=\sigma} I(\varphi).$$

Q. Does the above LD upper bound still hold at $T = \infty$?

It seems to me that this type of question is usually treated with the concept of 'exponential convergence', but it seems (at least to me) difficult to use the concept in this setting. However, this question itself can be reduced to a question on the "large N and T limit" of the heat kernel on $\mathrm{U}(N)$.

Works in progress

Lemma.

Let $p_t(U)$ be the Heat kernel on $\mathrm{U}(N)$ that is the density of $U_N(t)$ wrt. the Haar prob. Then

$$\lim_{T\to\infty} \liminf_{N\to\infty} \frac{1}{N^2} \log \min_U p_T(U) = \lim_{T\to\infty} \limsup_{N\to\infty} \frac{1}{N^2} \log \max_U p_T(U) = 0.$$

Theorem.

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(\pi^*_\infty(\varphi_N) \in \Lambda) \le -\inf_{\sigma\in\Lambda} J(\sigma),$$

where

$$J(\sigma) := \lim_{\substack{m\to\infty \\ \delta\searrow 0}} \limsup_{T\to\infty} \inf_{\pi^*_T(\tau)\in O_{m,\delta}(\sigma)} I(\sigma)$$

with a standard nbd basis $O_{m,\delta}(\sigma)$ at σ.

Works in progress

Intermediate Questions.

Q1. Does J have a unique minimizer ?
Q2. Find a 'closed formula' of J (the unification problem).

(Q1) is a test for the full LDP for $(U_N X_N(0) U_N{}^*, Y_N)$.
(Q2) is a major question toward the unification between Voiculescu's and our approaches to "mutual information" in free probability.

Proposition.

The rate function J admits a unique minimizer, which is the empirical distribution of the freely independent copies of $x(0)$ and $y(0)$. Therefore, it characterizes free independence.

This also means that the rate function J becomes the "third" candidate for "mutual information" in free probability.

Comments

- The orbital free entropy $\chi_{\mathrm{orb}}(\mathbf{X}, \mathbf{Y})$ of given noncomm. random multi-variables $\mathbf{X}, \mathbf{Y}$ has been established with all the expected properties that the prospective 'free probabilistic mutual information' should possesses. However, its definition still involves two 'drawbacks':
  - Is there a canonical selection of approximating seq. of deterministic matrices (like $X_N(0), Y_N(0)$) to define χ_{orb} ?
  - Can $\limsup_N$ be replaced with $\liminf_N$?

  These drawbacks would be resolved in the affirmative (or more precisely, $\chi_{\mathrm{orb}} = -J$) if the full large deviation principle for the matrix liberation process were established !

- Of course, the main problems are the LD lower bound for the matrix liberation process as well as (Q2). The main issues are all free probabilistic/operator algebraic, though usual stochastic analysis aspects have been already established well.

Thank you for your attention !