# Un modèle PDMP multi-classes pour le contrôle de congestion implémenté par les connexions dans un grand réseau


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**Travail effectué avec**

**Philippe Robert, INRIA Roquencourt**

## Travail effectué avec

**Philippe Robert**, **INRIA Roquencourt**

1. **Interacting multi-class transmissions in large stochastic networks (AAP 2009)**
2. **Self-adaptive congestion control for multi-class intermittent connections in a communication network (QUES 2011)**

**et pour une étude poussée sur les lois invariantes pour 1.,**

**Maaike Verloop**, **CWI Amsterdam**

3. **Stability properties of networks with interacting TCP flows (Proceedings of NET-COOP 2009)**

# 1 Modeling the Internet

**The Internet is a vast, complex, and ill-known structure in perpetual evolution.**

**Undisciplined users**

**access it for varied purposes and have vastly different characteristics and QoS (quality of service) requirements.**

**Connections** (initiated by **users**) have to

**self-adapt**

to the **packet losses** due to the **congestion** they **create**

by **regulating** their **output**

using **algorithms** such as the

**congestion control** part of **TCP**

(**T**ransmission **C**ontrol **P**rotocol).

**It is important to understand better this**

**feedback loop.**

**Highly nonlinear behavior is expected.**

**The situation is a fairly well understood for a**

**single node**

**with asymptotic formulæ for throughput, etc., even if**

**many problems remain open**

**depending on the level of precision and detail sought (delayed information . . . )**

**Much less**

**is understood about the true issue,**

**Stochastic Networks,**

***i.e.*, the coexistence, and hence the interaction,**

- of **varied data flows**

  (file transfers, telephony, video streaming, web browsing, MMORPG, P2P, . . . )

- with **varied characteristics**

  (routes, QoS requirements, rates, . . . )

**each using in its own fashion**

**limited common resources**

- constituted of **varied nodes**

  (links, routers, processors, buffers, . . . )

- with **varied purposes**.

**It is natural to regroup data flows into a**

**reasonable number of classes,**

**according to their characteristics, and we focus on such**

**interacting multi-class network models.**

## One approach:

**Introduced and studied by several distinguished authors,**

***e.g.*,**

**Kelly, Maulloo, and Tan (JORS 1998)**

**Massoulié and Roberts (INFOCOM 1999)**

**Kelly and Williams (AAP 2004)**

**Massoulié (AAP 2007)**

**Kang, Kelly, Lee, and Williams (AAP 2009)**

**Kelly, Massoulié, and Walton (QS 2009).**

**Optimisation problem**: if there are $x_k$ **connections** of class $k \in \{1, \ldots, K\}$, then class $k$ obtains **throughput** $\lambda_k^*$ such that $(\lambda_k^*)$ achieves

$$\max_{\lambda \in \mathscr{C}} \sum_{k=1}^{K} x_k U_k(\lambda_k / x_k)$$

where $U_k$ is some **utility function**, and $\mathscr{C}$ depends on the capacity of the resources.

The **utility function** is used to **summarize** in an **idealized** fashion the **effect** of **TCP** on the **network**.

This is a very **high level macroscopic** approach,

perhaps akin to **Thermodynamics**.

**Our approach:**

**to devise a microscopic model**
**and then derive its macroscopic limit.**

**The idealization of TCP takes place at the micro level.**

**Akin to kinetic models in Statistical Physics.**

# 2 Markovian modeling of user interaction

**We propose**

## a Markovian model for a stochastic network

- constituted of $J \geq 1$ nodes,
- hosting $K \geq 1$ classes of users (or connections),
- with $N_k \geq 1$ users in class $k$ for $1 \leq k \leq K$.

$$N = (N_1, \ldots, N_K) \qquad \text{class size vector},$$

$$|N| = N_1 + \cdots + N_K \qquad \text{total number of users}.$$

**Each user (or connection) alternates between being**

- **active, or on, when it has data to transmit**
- **and inactive, or off, when it has none**

**and thus generates**

**intermittent (intermittent) transmissions.**

## Markov model and Itô-Skorohod equations

This model corresponds to a Markov process

$$W^N(t) = \left( W^N_{n,k}(t), 1 \le n \le N_k, 1 \le k \le K \right), \qquad t \ge 0,$$

where

$$W^N_{n,k}(t) \in \mathbb{R}_+ \cup \{-1\}$$

is the state (output, window) of the $n$-th transmission of class $k$ at time $t$, with $-1$ corresponding to an inactive user.

We choose to represent it (in law) by the following

**Itô-Skorohod SDE:**

$$
\begin{aligned}
dW^N_{n,k}(t) = \mathbb{1}_{\{W^N_{n,k}(t-)=-1\}} &\int (1+w)\mathbb{1}_{\{0<z<\lambda_k(U^N(t-))\}}\mathscr{A}_{n,k}(dw,dz,dt) \\
+\mathbb{1}_{\{W^N_{n,k}(t-)\geq 0\}} &\Big[ a_k(W^N_{n,k}(t-), U^N(t-))\,dt \\
&- (1-r_k)W^N_{n,k}(t-)\int \mathbb{1}_{\{0<z<b_k(W^N_{n,k}(t-),U^N(t-))\}}\mathscr{N}_{n,k}(dz,dt) \\
&- (1+W^N_{n,k}(t-))\int \mathbb{1}_{\{0<z<\mu_k(W^N_{n,k}(t-),U^N(t-))\}}\mathscr{D}_{n,k}(dz,dt)\Big]
\end{aligned}
$$

with $U^N(t) = (U^N_j(t), 1 \leq j \leq J)$, $\quad U^N_j(t) = \sum_{k=1}^{K} A_{jk} \sum_{n=1}^{N_k} W^N_{n,k}(t)^+$,

where the $\mathscr{A}_{n,k}$ are Poisson point processes with intensity $\alpha_k(dw)dz\,dt$,

and the $\mathscr{N}_{n,k}$ and $\mathscr{D}_{n,k}$ are Poisson point processes with intensity $dz\,dt$.

**The following can be proved using standard arguments.**

**If the functions $a_k$ are Lipschitz**

**and $b_k$, $\lambda_k$, and $\mu_k$ are locally bounded,**

**then there is pathwise existence and uniqueness**

**of solution for the SDE,**

**and the corresponding Markov process is well defined.**

## The mean-field asymptotic regime

**This SDE constitutes**

**a coupled system in very high dimension**

**and an asymptotic study is performed to render it**

**tractable, which reduces**

**the dimension to the number of classes $K$.**

For $1 \le k \le K$, we assume that

$$N_k \to \infty, \qquad \frac{N_k}{|N|} := \frac{N_k}{N_1 + \cdots + N_K} \to p_k\,,$$

and, so as to scale the resource capacities adequately, that a factor $\frac{1}{|N|}$ is introduced inside the coefficients by

$$\text{replacing } U^N \text{ by } \overline{U}^N = \frac{1}{|N|} U^N\,.$$

This yields the following mean-field rescaled SDE:

$$dW^N_{n,k}(t) = \mathbb{1}_{\{W^N_{n,k}(t-)=-1\}} \int (1+w)\mathbb{1}_{\{0<z<\lambda_k(\overline{U}^N(t-))\}} \mathscr{A}_{n,k}(dw,dz,dt)$$

$$+\mathbb{1}_{\{W^N_{n,k}(t-)\geq 0\}} \Big[ a_k(W^N_{n,k}(t-),\overline{U}^N(t-))\, dt$$

$$-(1-r_k)W^N_{n,k}(t-) \int \mathbb{1}_{\{0<z<b_k(W^N_{n,k}(t-),\overline{U}^N(t-))\}} \mathscr{N}_{n,k}(dz,dt)$$

$$-(1+W^N_{n,k}(t-)) \int \mathbb{1}_{\{0<z<\mu_k(W^N_{n,k}(t-),\overline{U}^N(t-))\}} \mathscr{D}_{n,k}(dz,dt) \Big]$$

with $\overline{U}^N(t) = \frac{1}{|N|} U^N(t) = \left(\overline{U}^N_j(t), 1 \leq j \leq J\right)$,

$$\overline{U}^N_j(t) = \frac{1}{|N|} U^N_j(t) = \sum_{k=1}^{K} A_{jk} \frac{N_k}{|N|} \overline{W}^N_k(t), \quad \overline{W}^N_k(t) = \frac{1}{N_k} \sum_{n=1}^{N_k} W^N_{n,k}(t)^+.$$

This system is in multi-class mean-field interaction through

$$\left(\overline{W}_k^N(t), 1 \le k \le K\right) \text{ via } \overline{U}^N(t) = \frac{1}{|N|} U^N(t),$$

the vector of the empirical means of the class outputs

via

the rescaled throughput vector.

For $1 \leq k \leq K$, a natural quantity is the class $k$ empirical measure

$$\Lambda_k^N = \frac{1}{N_k} \sum_{n=1}^{N_k} \delta_{\left(W_{n,k}^N(t), t \geq 0\right)}$$

where $\delta_{(x(t), t \geq 0)}$ denotes the Dirac mass at the sample path $(x(t), t \geq 0)$.

Notably

$$\overline{W}_k^N(t) = \langle w^+, \Lambda_k^N(t)(dw) \rangle .$$

# 3 Limit nonlinear Markov process and class interaction

**In the mean-field asymptotic regime,**

**under adequate assumptions for the initial conditions,**

**a propagation of chaos phenomenon is expected:**

- the processes $W^N_{n,k}$ should become independent, and each converge in law to a process $W_k$,
- the empirical measures $\Lambda^N_k$ (random laws on path space) should converge in law (and in probability) to the law of the same process $W_k$,

**where the limit process**

$$W(t) = (W_k(t), 1 \le k \le K), \qquad t \ge 0,$$

**solves the (adequately started) following equation:**

## McKean-Vlasov Itô-Skorohod SDE (Nonlinear Markov process)

$$dW_k(t) = \mathbb{1}_{\{W_k(t-)=-1\}} \int (1+w)\mathbb{1}_{\{0<z<\lambda_k(u_W(t))\}} \mathscr{A}_k(dw,dz,dt)$$
$$+ \mathbb{1}_{\{W_k(t-)\geq 0\}} \Big[ a_k(W_k(t-), u_W(t))\, dt$$
$$- (1-r_k)W_k(t-) \int \mathbb{1}_{\{0<z<b_k(W_k(t-),u_W(t))\}} \mathscr{N}_k(dz,dt)$$
$$- (1+W_k(t-)) \int \mathbb{1}_{\{0<z<\mu_k(W_k(t-),u_W(t))\}} \mathscr{D}_k(dz,dt) \Big]$$

with

$$u_W(t) = (u_{W,j}(t), 1 \leq j \leq J), \quad u_{W,j}(t) = \sum_{k=1}^{K} A_{jk} p_k \mathbb{E}(W_k(t)^+).$$

The **interaction** between coordinates depends on the **mean throughput** vector

$$(u_W(t), t \geq 0)$$

which is a **linear** functional of the **mean class output** vector

$$\mathbb{E}(W(t)^+) = \mathbb{E}(W_k(t)^+, 1 \leq k \leq K) = \langle w^+, \mathscr{L}(W(t)) \rangle .$$

Notably, the infinitesimal generator of the Markov process $(W(t), t \geq 0)$ depends, at time $t$, on the

law $\mathscr{L}(W(t))$ of $W(t)$ itself

and not only on the value of the sample path.

Actually, only on the class marginals of $\mathscr{L}(W(t))$.

**The Kolmogorov equations are**

**nonlinear integro-differential equations**

**and a**

**nonlinear martingale problem**

**can be associated to this process.**

**This is why it is called a**

**Nonlinear Markov process.**

**Using this SDE representation, we have adapted**

**contraction and coupling techniques developed**

**for exchangeable systems by Sznitman (1980's, 1989)**

**to this multi-class setting (technical), and**

**obtained existence and uniqueness**

**for this fixed-point problem,**

**as well as propagation of chaos.**

This was done in G. and Robert (AAP 2009) for

**persistent transmissions**

and has been extended to the general case of

**on-off users with intermittent transmissions**

in G. and Robert (QUES 2011)

**There were many difficulties, *e.g.*,**

- **lack of symmetry of multi-class systems w.r.t. exchangeable ones:**

$$N_1! \cdots N_K! \ll |N|! = (N_1 + \cdots + N_K)!$$

**and rigorous results are rare for multi-class systems,**

- **quadratic behavior of $b_k(w,u) = w\beta_k(u)$,**
- **on-off users introduce discontinuities. Smart choices may simplify computations ($-1$ as cemetery state, ...).**

**The assumptions, notably on initial conditions, reflect this.**

**These must not be too stringent since**

## long-time and stationary behavior

## are essential.

**Gaussian moment assumptions were used.**

# 4 Fixed points and invariant laws for the limit

Recall that solution $(W(t), t \geq 0)$ has in general

an infinitesimal generator at time $t$

depending not only on the state,

but also, through $u_W(t)$, on $\mathbb{E}(W(t)^+)$

and thus on the law $\mathscr{L}(W(t))$ itself.

**This nonlinearity is an important complication in studying the behavior of $(W(t), t \geq 0)$, in particular for**

**existence and uniqueness of invariant laws.**

**Note that, starting from an invariant law,**

**$(\mathbb{E}(W(t)^+), t \geq 0)$ and hence $(u_W(t), t \geq 0)$ are constant,**

**and $(W(t), t \geq 0)$ corresponds to a**

## homogeneous Markov process

## in equilibrium.

**Interesting results on the invariant laws were obtained in G. and Robert (AAP 2009, QUES 2011) by reducing the corresponding**

**infinite-dimensional fixed-point problem**

**to a finite-dimensional one.**

**The results for on-off users being still preliminary,**

**we concentrate now on**

**users or connections**

**which emit persistently**

**and thus can be identified with their transmissions.**

**Assume** that, for $1 \le k \le K$,

$$a_k(w,u) = a_k(u), \; b_k(w,u) = w\beta_k(u), \quad w \in \mathbb{R}_+, \, u \in \mathbb{R}_+^J,$$

**where**

$a_k : \mathbb{R}_+^J \longrightarrow \mathbb{R}_+$ is **Lipschitz bounded**

$\beta_k : \mathbb{R}_+^J \longrightarrow \mathbb{R}_+$ is **Lipschitz**.

Then, the invariant laws for the nonlinear SDE are in

**one-to-one relation with**

the solutions of the

**finite-dimensional fixed-point problem**

$$u = (u_j, 1 \le j \le J) \in \mathbb{R}_+^J, \quad u_j = \sum_{k=1}^{K} A_{jk} p_k \psi(r_k) \sqrt{\frac{a_k(u)}{\beta_k(u)}},$$

where

$$\psi(r) = \sqrt{\frac{2}{\pi}} \prod_{n=1}^{\infty} \frac{1 - r^{2n}}{1 - r^{2n-1}}.$$

**Such a solution $u^*$ corresponds to a**

**product-form invariant law with density**

$$\prod_{k=1}^{K} H_{r_k,\rho_k}(w_k), \qquad w = (w_k, 1 \leq k \leq K) \in \mathbb{R}_+^K,$$

**where** $\rho_k = \dfrac{a_k(u^*)}{\beta_k(u^*)}$ **and, for** $x \in \mathbb{R}_+$,

$$H_{r,\rho}(x) = \frac{\sqrt{2\rho/\pi}}{\prod_{n=0}^{\infty}(1 - r^{2n+1})} \sum_{n=0}^{\infty} \frac{r^{-2n}}{\prod_{k=1}^{n}(1 - r^{-2k})} e^{-\rho r^{-2n} x^2/2}.$$

**Note that the decay is appropriate for**

**Gaussian moment conditions.**

**The limit equilibrium throughput for class $k$ users is given by**

$$\lambda_k = \mathbb{E}(\overline{W}_k) = \int_{\mathbb{R}_+} x H_{r_k,\rho_k}(x)\,dx$$
$$= \psi(r_k)\sqrt{\rho_k} = \psi(r_k)\sqrt{\frac{a_k(u^*)}{\beta_k(u^*)}}$$

**with $u^*$ solving a fixed-point problem**

**which can be written as**

$$H(u^*) = 0.$$

**Back to an optimisation problem ?**

**Class $k \in \{1,\ldots,K\}$ receives throughput $\lambda_k^*$, such that $(\lambda_k^*)$ achieves**

$$\max_{\lambda \in \mathscr{C}} \sum_{k=1}^{K} p_k U_k(\lambda_k / p_k)$$

**which can be put (under some conditions) in the form**

$$\nabla G(\lambda^* / p) = 0.$$

**How many solutions for the fixed-point problem ?**

G., Robert, and Verloop (NETCOOP 2009) prove

existence and uniqueness of the invariant law

by contraction and monotonicity methods,

under some assumptions, for topologies such as

- One node, several classes
- Linear networks with cross-traffic
- Trees
- Rings and toruses.

Assume moreover that

$$A_{jk} = \begin{cases} 1 & \text{if node } j \text{ is used by a class } k \text{ user,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_k(u) = \beta_k \left( \sum_{j=1}^{J} A_{jk} u_j \right).$$

(With slight abuse of notations.)

The **fixed-point equation** becomes

$$u_j = \sum_{k=1}^{K} A_{jk} p_k \psi(r_k) \frac{\sqrt{a_k(u)}}{\sqrt{\beta_k\left(\sum_{j=1}^{J} A_{jk} u_j\right)}}, \qquad 1 \le j \le J.$$

We **assume** that $u \to \beta_k(u)$ is **strictly increasing** and **Lipschitz** and that $a_k(u) \equiv a_k$ is **constant**.

## One node, several classes

**There is a unique solution $u = u^*$ for the fixed-point equation**

$$u = \sum_{k=1}^{K} \psi(r_k) p_k \sqrt{\frac{a_k}{\beta_k(u)}},$$

and a unique **invariant law** with density $H_{r,\rho_k}$ and expectation

$$\psi(r_k)\sqrt{\frac{a_k}{\beta_k(u^*)}}$$

yielding the **mean equilibrium output**.

**If** moreover the classes vary only in their RTT's, *i.e.*,

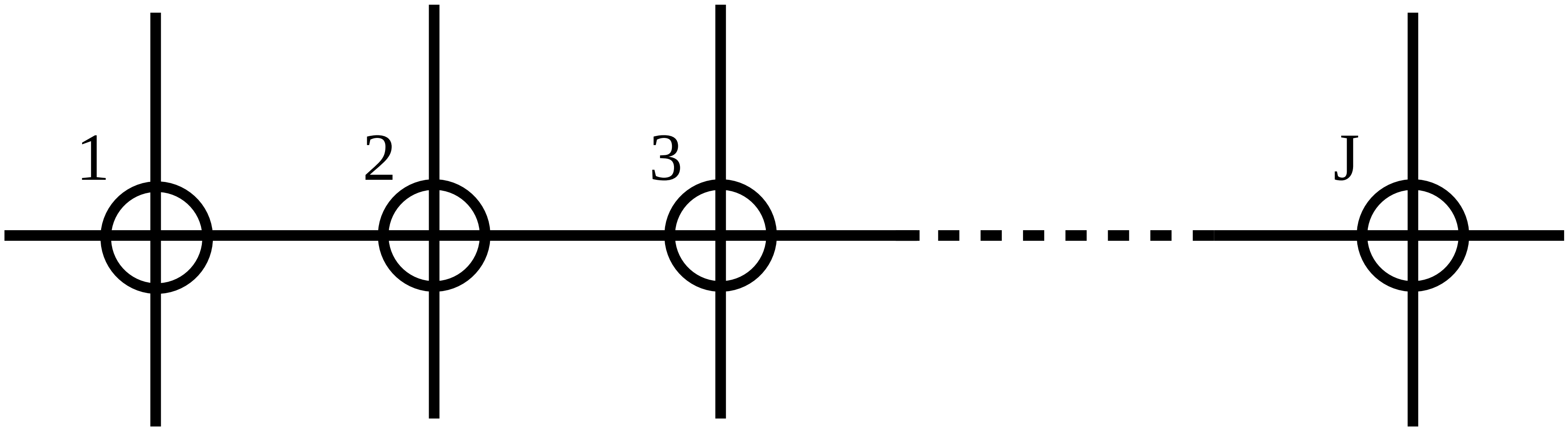
$$\beta_k \equiv \beta\,, \quad r_k \equiv r\,, \qquad 1 \leq k \leq K\,,$$

then the class **outputs** differ **only** by the factors

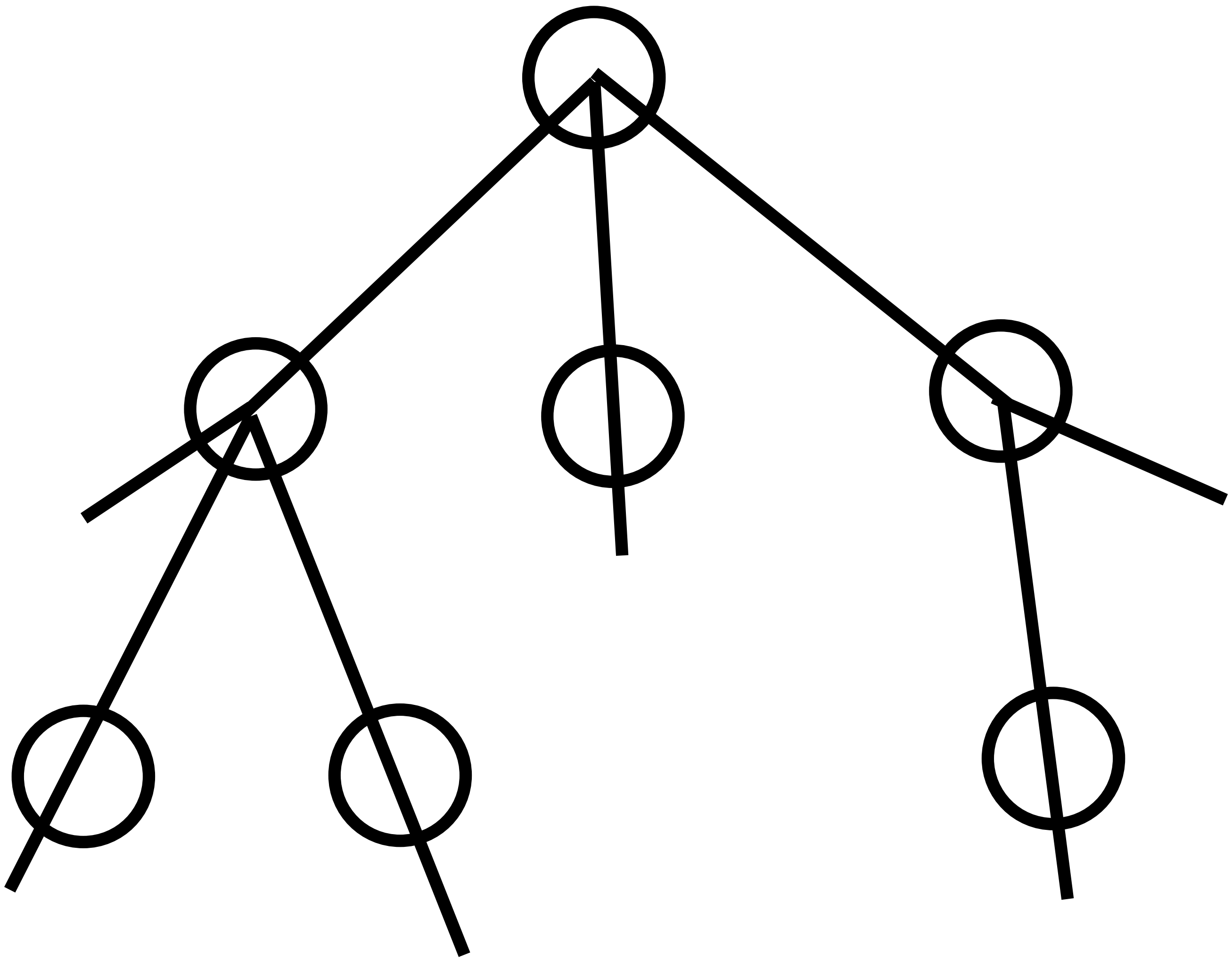
$$\sqrt{a_k} = 1/\sqrt{\mathrm{RTT}_k}\,.$$

# Linear network with cross-traffic

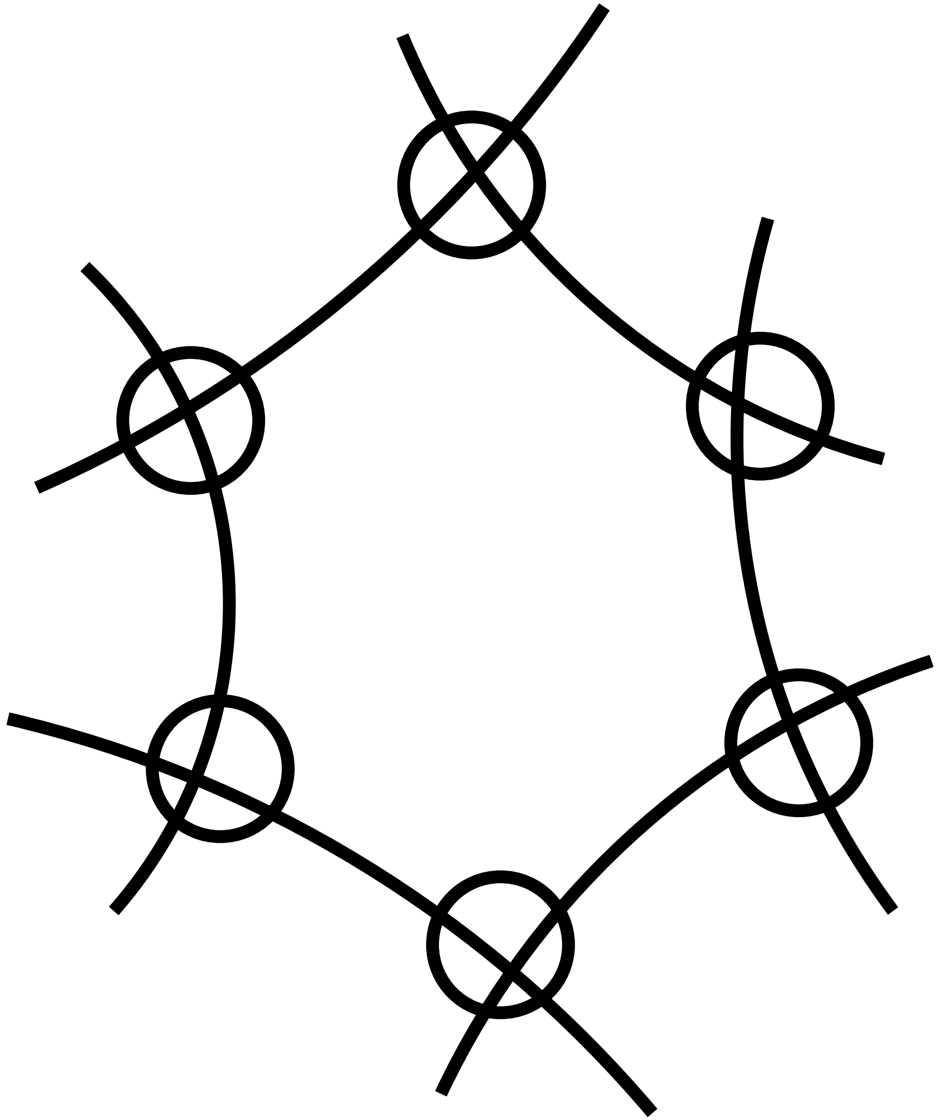
**There are $J$ nodes and $K = J+1$ classes. For $1 \leq j \leq J$, class $j$ transmissions use only node $j$. The transmissions of class $J+1$ use all $J$ nodes.**



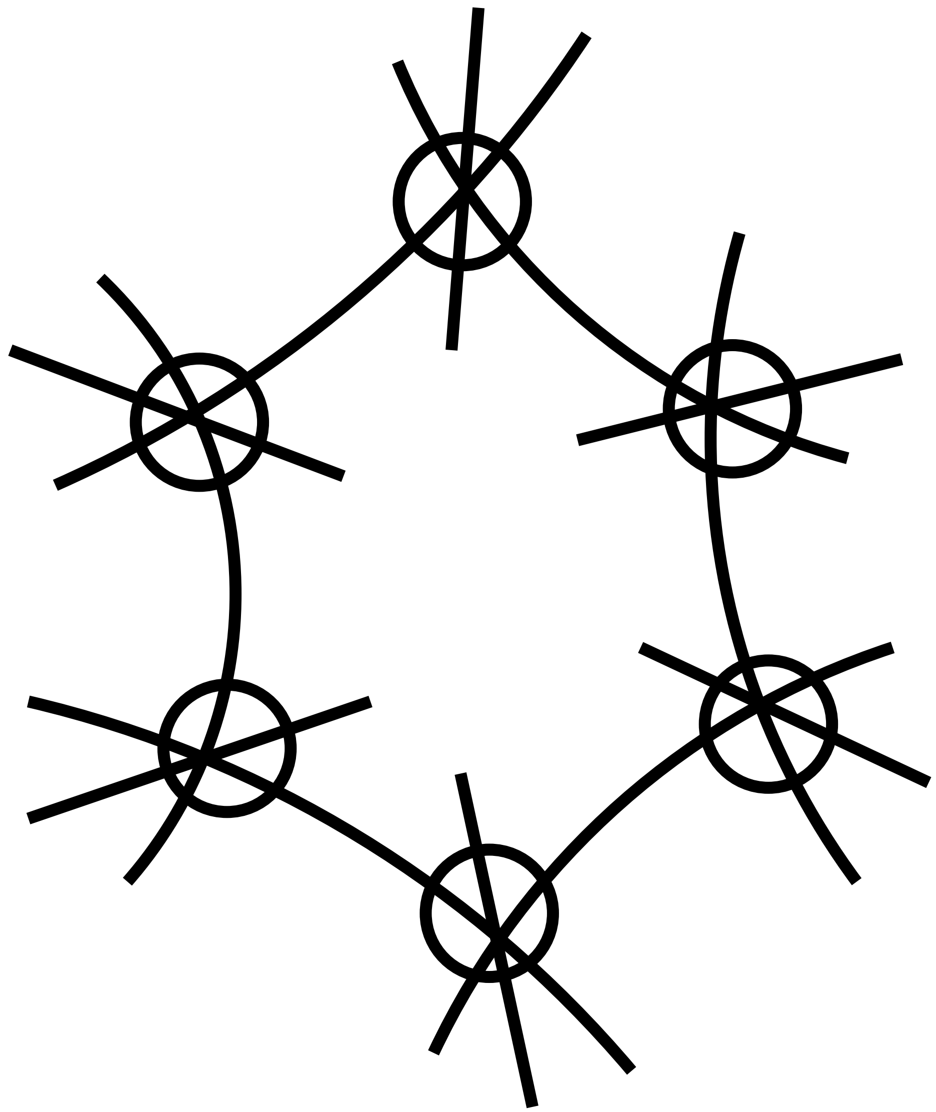
# Trees



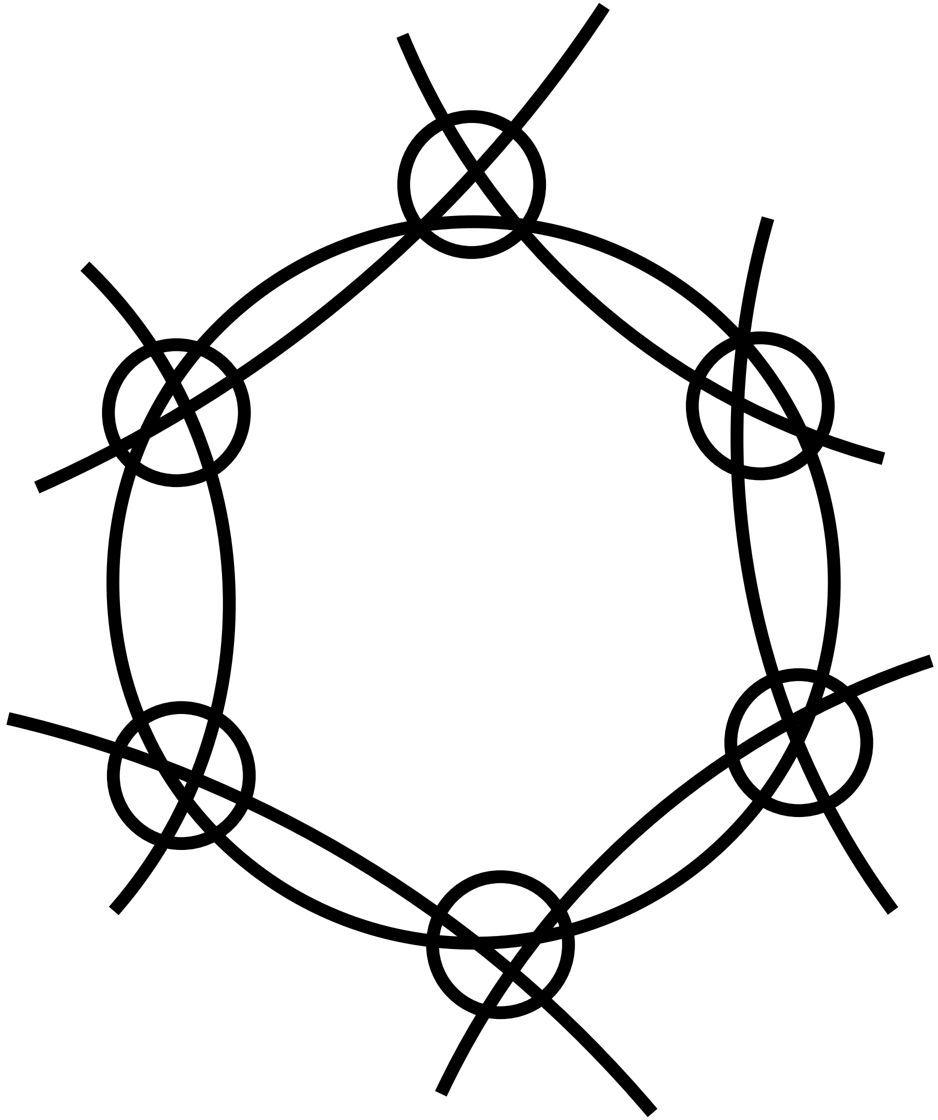
# Ring topology, 1



# Ring topology, 2



## Ring topology, 3



**Is uniqueness always true if**

$$u \longrightarrow \beta_k(u)$$

**is increasing ?**

**There exist examples when**

**not:**

**Raghunathan and Kumar (2007) in a wireless context.**

**Are there**

**meta-stability phenomena**

**in these networks ?**

**Known for**

- **Loss Networks: Gibbens *et al.* (1990), Marbukh (1993),**
- **Wireless Networks: Antunes *et al.* (2008).**

**Are there oscillations ?**

**Such possibilities have been suggested by the literature on the Internet and by simulations:**

**cyclic behavior in congested networks**

***e.g.*, mean-field study for a single node and a single class and delay equations in Baccelli, McDonald, and Reynier (2002)**

# 5 A Conclusion

## Representation of the interaction of flows

- **Using an instantaneous fluid picture yields:**

  **An optimisation problem**

  **Data for Model : Utility function.**

- **Starting from microscopic dynamics yields:**

  **A fixed point equation**

  **Data for Model : Equation coefficients.**

# Perspectives

- Modeling transmissions arriving from the outside world and disappearing after completion. A different scaling.
- More on the invariant laws for on-off users. Transient behavior of on periods needs to be assessed.
- Convergence of invariant laws:

$$\lim_{N\to\infty}\lim_{t\to\infty} = \lim_{t\to\infty}\lim_{N\to\infty} ?$$

  Always difficult, especially with multiple equilibria.
- Relations with the optimization problem approach.

**Merci**

**pour votre attention !**