Rough notes on the long time behavior of several PDMPs

Florent Malrieu

April 20, 2012

Contents

Chapter 1

Introduction

These notes are about a three hour course I gave during the Journées PDMP 2012 at Marne-La-Vallée. This meeting was perfectly organized by Djalil Chafaï, Bertrand Cloez, Dan Goreac, Marc Hoffmann and Miguel Martinez.

I would want to thank all my coauthors on this topic for the four past years: Jean-Baptiste Bardet, Michel Benaïm, Djalil Chafaï, Alejandra Christen, Joaquin Fontbona, Arnaud Guillin Stéphane Le Borgne, Pierre-André Zitt.

1.1 The setting

Piecewise deterministic Markov processes (PDMPs) were introduced in the literature by Davis [Dav84, Dav93] as a general class of non diffusion stochastic models. PDMPs are a family of Markov processes involving deterministic motion punctuated by random jumps. The motion of the PDMP $(X_t)_{t\geqslant 0}$ depends on three local characteristics, namely, the flow Φ, the jump rate λ and the transition measure Q. Starting from x, the motion of the process follows the flow $\Phi(x,t)$ until the first jump time T_1 which occurs either spontaneously in a Poisson-like fashion with rate $\lambda(x)$ or when the flow $\Phi(x,t)$ hits the boundary of the state space ∂E. In either case, the location of the process at the jump time T_1 is selected by the transition measure $Q(\Phi(x,t),\cdot)$ and the motion restarts from this new point as before. The boundary of the space ∂E can be seen as a region where the jump rate is infinite.

Let us consider the case when $(X_t)_{t\geqslant 0}$ is $\mathbb{R}^d$-valued. Roughly speaking a PDMP $(X_t)_{t\geqslant 0}$ on $E\subset\mathbb{R}^d$ is driven by the generator

$$Lf(x) = \mathcal{X} f(x) + \lambda(x)\int_E (f(y)-f(x))\,Q(x,dy),$$

where $\mathcal{X}$ is a locally Lipschitz continuous vector field on $E\subset\mathbb{R}^d$ determining the flow Φ:

$$\begin{cases}\partial_t\Phi(x,t) = \mathcal{X}\Phi(x,t) & \text{if } t>0,\\ \Phi(x,0)=x.\end{cases}$$

Further details are given in [Dav93] (in particular the domain of L). For any $x\in E$, let us denote by $t_*(x)$ the hitting time of the boundary

$$t_*(x) = \inf\{t>0\ :\ \Phi(x,t)\in\partial E\}\in[0,+\infty].$$

One can associate to $(X_t)_{t\geqslant 0}$ its embedded Markov chain i.e. the process observed at the jump times. Let us define

$$\Lambda(x,t) = \int_0^t \lambda(\Phi(x,s))\,ds, \tag{1.1}$$

for $x \in E$ and $t > 0$. If $X_0 = x$, the density function of $\mathcal{L}(T_1)$ is given by $s \mapsto \lambda(\Phi(x,s))e^{-\Lambda(x,s)}$. In particular, the probability that no jump occurs before time $t_*(x)$ is equal to $e^{-\Lambda(x,t_*(x))}$. At this time $t_*(x)$, X hits ∂E and jumps. As a consequence, the kernel of the embedded chain is given by:

$$\begin{aligned} I(x,A) &= \mathbb{P}_x(X_{T_1} \in A) \\ &= \int_0^{t_*(x)} \lambda(\Phi(x,s))e^{-\Lambda(x,s)}Q(\Phi(x,s),A)\,ds + e^{-\Lambda(x,t_*(x))}Q(\Phi(x,t_*(x)),A). \end{aligned} \tag{1.2}$$

A PDMP has no diffusive part (by definition!) and is (in general) non reversible. For the link with Poisson Point Processes, see for instance [Jac06].

1.2 Motivating examples

1.2.1 The TCP window size process and related examples

This process on $\mathbb{R}_+$ belongs to the subclass of the AIMD (Additive Increase Multiplicative Decrease) processes. Its infinitesimal generator is given by

$$Lf(x) = f'(x) + \lambda(x)\int_0^1 (f(ux) - f(x))\,\nu(du), \tag{1.3}$$

where ν is a probability measure on $[0,1]$ and λ can be constant (easy case, see [LvL08]) or affine (tricky case). It can be viewed as the limit behavior of the congestion of a single channel (see [DGR02, GRZ04] for a rigorous derivation of this limit). [MZ09] gives a generalization of the scaling procedure to interpret various PDMPs as the limit of discrete time Markov chains and in [vLLO09] more general increase and decrease profiles are considered as models for TCP. In the real world (Internet), the AIMD mechanism allows a good compromise between the minimization of network congestion time and the maximization of mean throughput. See also [BDRS02] for a simplified TCP windows size model. See [vLLO09, MZ06, OK08, OKM96, OS07, Ott06, Hes05] for other works dedicated to this process. Generalization to interacting multi-class transmissions are considered in [GR09, GR10].

It is shown in [DGR02] that the invariant measure of the process (1.3) with $\lambda(x) = x$ and ν is the Dirac mass at $\delta \in [0,1)$ is given by

$$x \mapsto \frac{\sqrt{2/\pi}}{\prod_{n=0}^{+\infty}(1-\delta^{2n+1})}\sum_{n=0}^{+\infty}\frac{\delta^{-2n}}{(1-\delta^{-2k})}e^{-\delta^{-2n}x^2/2}$$

see Figure 1.1. This result may follow from the expression of the density function of the invariant measure of the embedded chain and the relation

$$\mathbb{E}(f(W_\infty)) = \frac{1}{\mathbb{E}(T_{V_\infty})}\mathbb{E}\left(\int_0^{T_{V_\infty}} f(V_\infty + s)\,ds\right)$$

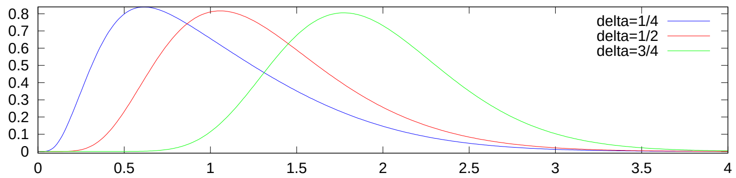

Figure 1.1: Density funtion of the invariant measure associated to the generator (1.3) when $\lambda(x) = x$ and ν is the Dirac mass at δ.

where $\mathcal{L}(W_\infty)$ is the invariant measure associated to (1.3), and where V_∞ is drawn according to the invariant law of the embedded chain and T_v is a jump time for the continuous-time process starting at v.

In [CMP10], a first attemp to get quantitative bounds for the convergence to the invariant measure is proposed. The work [BCG$^+$12] provides good improvement for the estimates in Wasserstein distance and total variation distance.

This process is studied in Chapter 2.

1.2.2 Chemotaxis

Let us briefly describe how bacteria move [ODA88, EO05, EO05]. They alternate two basic behavioral modes: a more or less linear motion, called a run, and a highly erratic motion, called tumbling, the purpose of which is to reorient the cell. During a run the bacteria move at approximately constant speed in the most recently chosen direction. Run times are typically much longer than the time spent tumbling. In practice, the tumbling time is neglected. An appropriate stochastic process for describing the motion of cells is called the *velocity jump process* which is deeply studied in [ODA88]. The velocity belongs to a compact set (the unit sphere for example) and changes by random jumps at random instants of time and the position is deduced by integration of the velocity. The jump rates may depend on the position when the medium is not homogeneous: when bacteria move in a favorable direction *i.e.* either in the direction of foodstuffs or away from harmful substances the run times are increased further. Sometimes, a diffusive approximation is available [ODA88, RS10].

In a one dimensional simple model studied in [FGM10], the particle evolves in $\mathbb{R}$ and its velocity belongs to $\{-1, +1\}$. Its infinitesimal generator is given by:

$$Af(x,v) = v\partial_x f(x,v) + \big(a + (b-a)\mathbb{1}_{\{xv>0\}}\big)\big(f(x,-v) - f(x,v)\big), \tag{1.4}$$

with $0 < a < b$. The dynamics of the process is simple: when X go aways from 0, (resp. go to 0), V flips to $-V$ with rate b (resp. a). This process is reminiscent of the study [HV10] on a generalized telegraph process.

It is shown in [FGM10] that the invariant measure μ of (X, V) driven by (1.4) is the product measure on $\mathbb{R}_+ \times \{-1, +1\}$ given by

$$\mu(dx, dv) = (b-a)e^{-(b-a)x}\, dx \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dv).$$

One can also construct an explicit coupling to get explicit bounds for the convergence to the invariant measure in total variation norm [FGM10].

1.2.3 Reliability

See the works of Dufour and his team.

1.2.4 Neuron activity

The paper [PTW09] establishes limit theorems for a class of stochastic hybrid systems (continuous deterministic dynamic coupled with jump Markov processes) in the fluid limit (small jumps at high frequency), thus extending known results for jump Markov processes. The main results are a functional law of large numbers with exponential convergence speed, a diffusion approximation, and a functional central limit theorem. These results are then applied to neuron models with stochastic ion channels, as the number of channels goes to infinity, estimating the convergence to the deterministic model. In terms of neural coding, the central limit theorems allows to estimate numerically the impact of channel noise both on frequency and spike timing coding.

1.2.5 Chemical reactions

Let E be the set $\{1, 2, \ldots, n\}$, $(\lambda(\cdot, i))_{i \in E}$ be n nonnegative continuous functions on $\mathbb{R}^d$, P be an irreducible stochastic matrix on E and, for any $i \in E$, $F^i(\cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a smooth vector field such that the ordinary differential equation

$$\begin{cases} \dot{x}_t = F^i(x_t) & \text{for } t > 0, \\ x_0 = x \end{cases}$$

has an unique and global solution $t \mapsto \phi_t^i(x)$ on $[0, +\infty)$ for any initial condition $x \in \mathbb{R}^d$. Let us consider the Markov process

$$(Z_t)_{t \geqslant 0} = ((X_t, I_t))_{t \geqslant 0} \text{ on } \mathbb{R}^d \times E$$

defined by its infinitesimal generator L as follows:

$$Lf(x,i) = F^i(x)\nabla_x f(x,i) + \lambda(x,i) \sum_{\tilde{j} \in E} P(i,j)(f(x,j) - f(x,i))$$

for any smooth function $f : \mathbb{R}^d \times E \to \mathbb{R}$.

Chapter 2

The TCP window size process and related examples

The TCP window size process appears in the modeling of the famous Transmission Control Protocol used for data transmission over the Internet. This continuous time Markov process takes its values in $[0,\infty)$, is ergodic and irreversible. The sample paths are piecewise linear deterministic and the whole randomness of the dynamics comes from the jump mechanism. The aim of this chapter is to provide quantitative estimates for the exponential convergence to equilibrium, in terms of the total variation and Wasserstein distances. The main results are established in [CMP10, BCG$^+$12].

2.1 The TCP model with constant jump rate

A baby model of the TCP process it obtained assuming that the jump rate is constant. The infinitesimal generator of the process is given by:

$$Lf(x) = f'(x) + \lambda(f(x/2) - f(x)) \quad (x \geqslant 0).$$

The jump times of this process are the ones of a homogeneous Poisson process with intensity λ. The convergence in Wasserstein distance is obvious.

Lemma 2.1.1 ([PR05, CMP10]). *For any $p \geqslant 1$,*

$$W_p(\delta_x P_t, \delta_y P_t) \leqslant |x-y|e^{-\lambda_p t} \quad \textit{with} \quad \lambda_p = \frac{\lambda(1-2^{-p})}{p}. \tag{2.1}$$

Remark 2.1.2. The case $p=1$ is obtained in [PR05] by PDEs estimates using the following alternative formulation of the Wasserstein distance on $\mathbb{R}$. If the cumulative distribution functions of the two probability measures ν and $\tilde{\nu}$ are F and $\tilde{F}$ then

$$W_1(\nu,\tilde{\nu}) = \int_{\mathbb{R}} |F(x) - \tilde{F}(x)|\,dx.$$

The general case $p \geqslant 1$ is obvious from the probabilistic point of view: choosing the same Poisson process $(N_t)_{t\geqslant 0}$ to drive the two processes provides that the two coordinates jump simultaneously and

$$|X_t - Y_t| = |x-y|2^{-N_t}.$$

As a consequence, since the law of N_t is the Poisson distribution with parameter λt, one has

$$\mathbb{E}_{x,y}(|X_t - Y_t|^p) = |x-y|^p \mathbb{E}(2^{-pN_t}) = |x-y|^p e^{-p\lambda_p t}.$$

This coupling turns out to be sharp. Indeed, one can compute explicitly the moments of X_t (see [LL08, OK08]): for every $n \geqslant 0$, every $x \geqslant 0$, and every $t \geqslant 0$,

$$\mathbb{E}_x((X_t)^n) = \frac{n!}{\prod_{k=1}^n \theta_k} + n! \sum_{m=1}^n \left(\sum_{k=0}^m \frac{x^k}{k!} \prod_{\substack{j=k \\ j \neq m}}^n \frac{1}{\theta_j - \theta_m} \right) e^{-\theta_m t}, \tag{2.2}$$

where $\theta_n = \lambda(1-2^{-n}) = n\lambda_n$ for any $n \geqslant 1$. Obviously, assuming for example that $x > y$,

$$\begin{aligned} W_n(\delta_x P_t, \delta_y P_t)^n &\geqslant \mathbb{E}_x((X_t)^n) - \mathbb{E}_y((Y_t)^n) \\ &\underset{t\to\infty}{\sim} n! \left(\sum_{k=0}^n \frac{x^k - y^k}{k!} \prod_{j=k}^{n-1} \frac{1}{\theta_j - \theta_n} \right) e^{-\theta_n t}. \end{aligned}$$

As a consequence, the rate of convergence in Equation (2.1) is optimal for any $n \geqslant 1$.

Nevertheless this estimate for the Wasserstein rate of convergence does not provide on its own any information about the total variation distance between $\delta_x P_t$ and $\delta_y P_t$. It turns out that this rate of convergence is the one of the W_1 distance. This is established in [PR05, Thm 1.1]. Let us provide here an improvement of this result by a probabilistic argument.

Proposition 2.1.3. *For any $x, y \geqslant 0$ and $t \geqslant 0$,*

$$\|\delta_x P_t - \delta_y P_t\|_{\mathrm{TV}} \leqslant \lambda e^{-\lambda t/2}|x-y| + e^{-\lambda t}. \tag{2.3}$$

As a consequence, for any measure ν with a finite first moment and $t \geqslant 0$,

$$\|\nu P_t - \mu\|_{\mathrm{TV}} \leqslant \lambda e^{-\lambda t/2} W_1(\nu, \mu) + e^{-\lambda t}\|\nu - \mu\|_{\mathrm{TV}}. \tag{2.4}$$

Remark 2.1.4. Note that the upper bound obtained in Equation (2.3) is non-null even for $x = y$. This is due to the persistence of a Dirac mass at any time, which implies that taking y arbitrarily close to x for initial conditions does not make the total variation distance arbitrarily small, even for large times.

Proof of Proposition 2.1.3. The coupling is a slight modification of the Wasserstein one. The paths of $(X_s)_{0\leqslant s\leqslant t}$ and $(Y_s)_{0\leqslant s\leqslant t}$ starting respectively from x and y are determined by their jump times $(T_n^X)_{n\geqslant 0}$ and $(T_n^Y)_{n\geqslant 0}$ up to time t. These sequences have the same distribution than the jump times of a Poisson process with intensity λ.

Let $(N_t)_{t\geqslant 0}$ be a Poisson process with intensity λ and $(T_n)_{n\geqslant 0}$ its jump times with the convention $T_0 = 0$. Let us now construct the jump times of X and Y. Both processes make exactly N_t jumps before time t. If $N_t = 0$, then

$$X_s = x + s \quad \text{and} \quad Y_s = y + s \quad \text{for } 0 \leqslant s \leqslant t.$$

Assume now that $N_t \geqslant 1$. The $N_t - 1$ first jump times of X and Y are the ones of $(N_t)_{t\geqslant 0}$:

$$T_k^X = T_k^Y = T_k \quad 0 \leqslant k \leqslant N_t - 1.$$

In other words, the Wasserstein coupling acts until the penultimate jump time T_{N_t-1}. At that time, we have

$$X_{T_{N_t-1}} - Y_{T_{N_t-1}} = \frac{x-y}{2^{N_t-1}}.$$

Then we have to define the last jump time for each process. If they are such that

$$T^X_{N_t} = T^Y_{N_t} + X_{T_{N_t-1}} - Y_{T_{N_t-1}}$$

then the paths of X and Y are equal on the interval $(T^X_{N_t}, t)$ and can be chosen to be equal for any time larger than t.

Recall that conditionally on the event $\{N_t = 1\}$, the law of T_1 is the uniform distribution on $(0,t)$. More generally, if $n \geqslant 2$, conditionally on the set $\{N_t = n\}$, the law of the penultimate jump time T_{n-1} has a density $s \mapsto n(n-1)t^{-n}(t-s)s^{n-2}\mathbb{1}_{(0,t)}(s)$ and conditionally on the event $\{N_t = n, T_{n-1} = s\}$, the law of T_n is uniform on the interval (s,t).

Conditionally on $N_t = n \geqslant 1$ and T_{n-1}, T^X_n and T^Y_n are uniformly distributed on (T_{n-1}, t) and can be chosen such that

$$\begin{aligned}
&\mathbb{P}\left(T^X_n = T^Y_n + \frac{x-y}{2^{n-1}} \,\middle|\, N^X_t = N^Y_t = n,\, T^X_{n-1} = T^Y_{n-1} = T_{n-1}\right)\\
&\quad = \left(1 - \frac{|x-y|}{2^{n-1}(t-T_{n-1})}\right) \vee 0 \geqslant 1 - \frac{|x-y|}{2^{n-1}(t-T_{n-1})}.
\end{aligned}$$

This coupling provides that

$$\begin{aligned}
\|\delta_x P_t - \delta_y P_t\|_{\mathrm{TV}} &\leqslant 1 - \mathbb{E}\left[\left(1 - \frac{|x-y|}{2^{N_t-1}(t-T_{N_t-1})}\right)\mathbb{1}_{\{N_t\geqslant 1\}}\right]\\
&\leqslant e^{-\lambda t} + |x-y|\mathbb{E}\left(\frac{2^{-N_t+1}}{(t-T_{N_t-1})}\mathbb{1}_{\{N_t\geqslant 1\}}\right).
\end{aligned}$$

For any $n \geqslant 2$,

$$\mathbb{E}\left(\frac{1}{t-T_{N_t-1}}\middle| N_t = n\right) = \frac{n(n-1)}{t^n}\int_0^t u^{n-2}\,du = \frac{n}{t}.$$

This equality also holds for $n = 1$. Thus we get that

$$\mathbb{E}\left(\frac{2^{-N_t+1}}{(t-T_{N_t-1})}\mathbb{1}_{\{N_t\geqslant 1\}}\right) = \frac{1}{t}\mathbb{E}\big(N_t 2^{-N_t+1}\big) = \lambda e^{-\lambda t/2},$$

since N_t is distributed according to the Poisson law with parameter λt. This provides the estimate (2.3). The general case (2.4) is a straightforward consequence: if N_t is equal to 0, a coupling in total variation of the initial measures is done, otherwise, we use the coupling above. □

2.2 The TCP process

Let us come back to the "true" TCP process driven by the infinitesimal generator (1.3) with $\nu = \delta_{1/2}$. One of the fine properties of the TCP process is that "it comes back from infinity in finite time". More precisely, one can establish the following result.

Lemma 2.2.1. *For any $p \geqslant 1$ and $t > 0$*

$$M_{p,t} := \sup_{x \geqslant 0} \mathbb{E}_x(X_t^p) \leqslant \left(\sqrt{2p} + \frac{2p}{t}\right)^p.$$

One can get an exponential rate of convergence in Wasserstein distance.

Theorem 2.2.2. *Let us define*

$$M = \frac{\sqrt{2}(3+\sqrt{3})}{8} \sim 0.84 \quad \text{and} \quad \lambda = \sqrt{2}(1-\sqrt{M}) \sim 0.12. \tag{2.5}$$

For any $\tilde{\lambda} < \lambda$, any $p \geqslant 1$ and any $t_0 > 0$, there is a constant $C = C(p, \tilde{\lambda}, t_0)$ such that, for any initial probability measures ν and $\tilde{\nu}$ and any $t \geqslant t_0$,

$$W_p(\nu_t, \tilde{\nu}_t) \leqslant C e^{-(\tilde{\lambda}/p)t}.$$

Sketch of proof. The main idea of the proof is to choose an good coupling of two paths starting from two different points and to study its fine properties. This coupling was introduced in [CMP10]. It is defined by the following generator

$$\mathfrak{L}f(x,y) = (\partial_x + \partial_y)f(x,y) + y\big(f(x/2,y/2) - f(x,y)\big) + (x-y)\big(f(x/2,y) - f(x,y)\big)$$

when $x \geqslant y$ and symmetric expression for $x < y$. We will call the dynamical coupling defined by this generator the Wasserstein coupling of the TCP process (see Figure 2.1 for a graphical illustration of this coupling). This coupling is the only one such that the lower component never jumps alone. Let us give the pathwise interpretation of this coupling. Between two jump times, the two coordinates increase linearly with rate 1. Moreover, two "jump processes" are simultaneously in action:

1. with a rate equal to the minimum of the two coordinates, they jump (*i.e.* they are divided by 2) simultaneously,
2. with a rate equal to the distance between the two coordinates (which is constant between two jump times), the bigger one jumps alone.

At last, one has to overcome the difficulties that come from the fact that the jump rate is equal to 0 at the origin. This can be done using the constant drift that drive the process away from 0. □

Total variation estimates can also be derived from the Wasserstein ones and a coalescent coupling with a unique try (see [BCG+12] for more details).

Theorem 2.2.3. *For any $\tilde{\lambda} < \lambda$ and any $t_0 > 0$, there exists C such that, for any initial probability measures ν and $\tilde{\nu}$ and any $t \geqslant t_0$,*

$$\|\nu_t - \tilde{\nu}_t\|_{\mathrm{TV}} \leqslant C e^{-(2\tilde{\lambda}/3)t},$$

where λ is given by (2.5).

Remark 2.2.4. In both Theorems 2.2.2 and 2.2.3, no assumption is required on the moments nor regularity of the initial measures. In particular they hold uniformly over the Dirac measures. If $\tilde{\nu}$ is chosen to be the invariant measure μ, these theorems provide exponential convergence to equilibrium.

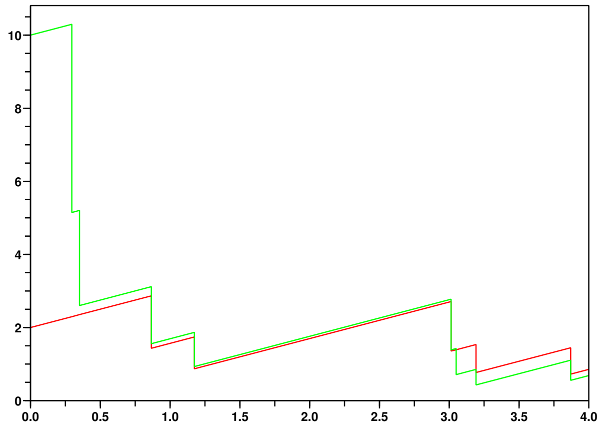

Figure 2.1: Two trajectories following the Wasserstein coupling; the bigger jumping alone can be good, making the distance between both trajectories smaller, or bad.

2.3 A bound via small sets

We describe here briefly the approach of [RR96] (which essentially consists in an kind adaptation of the Meyn-Tweedie method) and compare it with the hybrid Wasserstein/total variation coupling described above. The idea is once more to build a successful coupling between two copies X and Y of the process. In algorithmic terms, the approach is the following:

- wait until X and Y both reach a given set C,
- once they are in C, try to stick them together,
- repeat until the previous step is successful.

The first step is done by independent coupling, until the joint process hits a product set $C \times C$. To get a succesful coupling afterwards, the idea is to find a time $t^\star$, a probability measure ν and an $\alpha > 0$ such that

$$\forall x \in C, \quad \mathcal{L}(X_{t^\star} | X_0 = x) \geqslant \alpha\nu. \tag{2.6}$$

If this holds, once the processes hit C together, we have a probability at least α of coupling them in a time $t^\star$. The set C is called a "small set".

One can weaken the condition and ask for the existence of an α, a $t^\star$ and measures ν_{xy} such that

$$\forall x, y \in C^2, \quad \mathcal{L}(X_t | X_0 = x) \geqslant \alpha\nu_{xy} \text{ and } \mathcal{L}(X_t | X_0 = y) \geqslant \alpha\nu_{xy}. \tag{2.7}$$

In other words, the measure ν in (2.6) may depend on the starting points x and y. In this situation C is called "pseudo-small".

To control the time to come back to $C \times C$, [RR96] proposes an approach via a Lyapunov function. Their result can be stated as follows[1].

[1] In fact, in [RR96], the result is given with A instead of A' in the upper bound. Joaquin Fontbona pointed out to us that Lemma 6 from [RR96] has to be corrected, adding the exponential term $e^{\delta t^\star}$ to the estimate. We thank him for this remark.

Theorem 2.3.1 ([RR96], Theorem 3, Corollary 4 and Theorem 8)**.** *Suppose that there exists a set C, a function $V \geqslant 1$, and positive constants λ, Λ such that*

$$LV \leqslant -\lambda V + \Lambda \mathbb{1}_C. \tag{2.8}$$

Suppose that $\delta = \lambda - \Lambda / \inf_{x \notin C} V > 0$. Suppose additionally that C is pseudo-small, with constant α.

Then for $A = \frac{\Lambda}{\delta} + e^{-\delta t^\star} \sup_{x \in C} V$, $A' = Ae^{\delta t^\star}$, and for any $r < 1/t^\star$,

$$\|\mathcal{L}(X_t) - \mu\|_{\mathrm{TV}} \leqslant (1-\alpha)^{\lfloor rt \rfloor} + e^{-\delta(t-t^\star)} (A')^{\lfloor rt \rfloor - 1} \mathbb{E} V(X_0).$$

If A' is finite, this gives exponential convergence: just choose r small enough so that $(A')^{rt} e^{-\delta t}$ decreases exponentially fast.

Let us try to apply this approach. Choose $V(x) = \frac{1}{x} + e^x$. It is easy to check that (2.8) holds for example with $C = [0.45, 1.85]$, $\Lambda = 3$ and $\lambda = 0.25$. For δ being positive, one has to enlarge C. For example, the interval $C = [0.04, 3.2]$ gives $\inf_{x \notin C} V = 24$ and $\delta = 0.125$. This makes the set C much too large to get a good coupling bound in (2.6): the α obtained is around 2.10^{-19}. For a value of $t = 3.4$, we obtain a value $\alpha \approx 4.10^{-14}$...

Chapter 3

Explosive switched vector fields

This chapter is dedicated to an instructive subclass of switched vector fields presented in Section 1.2.5. The first section present an example for which the computations are explicit. Section 3.2 provides a general result in this framework. This chapter is inspired by [BLBMZ12a].

3.1 An example with phase transition

Let a and b be two positive real numbers and set

$$A_0 = \begin{pmatrix} -1 & ab \\ -a/b & -1 \end{pmatrix} \quad A_1 = \begin{pmatrix} -1 & -a/b \\ ab & -1 \end{pmatrix}$$

and

$$A_{1/2} = \frac{A_1 + A_0}{2} = \begin{pmatrix} -1 & a(b-1/b)/2 \\ a(b-1/b)/2 & -1 \end{pmatrix}.$$

The eigenvalues of A_0 and A_1 are equal to $-1 \pm ia$ whereas the eigenvalues of $A_{1/2}$ are $-1 \pm a(b-1/b)/2$. If $a(b-1/b) > 2$, *i.e.* $b > 1 + \sqrt{1+a^2}$, the matrix $A_{1/2}$ admits a positive and a negative eigenvalue. The associated eigenvectors are $(1,1)$ and $(1,-1)$.

For any $\beta > 0$, Let us define the Markov process (X,I) on $\mathbb{R}^2 \times \{0,1\}$ driven by the generator $\mathcal{L}_\beta$:

$$\mathcal{L}_\beta f(x,i) = \mathcal{L}_C f(x,i) + \beta \mathcal{L}_J f(x,i)$$

where

$$\mathcal{L}_C f(x,i) = A_i \nabla f(x,i) \quad \text{and} \quad \mathcal{L}_J f(x,i) = \frac{1}{2}(f(x,1-i) - f(x,i)).$$

The operator $\mathcal{L}_C$ corresponds to the "continuous" part (the first component x evolves along the flow of the vector field $x \mapsto A_i x$) and $\mathcal{L}_J$ gives the jumps on the second component. If ν is a probability measure on $\mathbb{R}^2 \times \{0,1\}$, we denote by $\mathbb{P}_\nu$ the law of the process (X,I) when the law of (X_0,I_0) is ν.

3.1.1 A polar decomposition

We begin by decomposing the deterministic dynamics. Let A be a matrix on $\mathbb{R}^2$ and $x \in \mathbb{R}^2 \backslash \{0\}$. Consider $(x_t)_{t \geqslant 0}$ the solution of

$$\begin{cases} \dot{x}_t = Ax_t, \\ x_0 = x. \end{cases}$$

First of all, since x is not 0, x_t never reaches 0. Therefore it is possible to define polar coordinates (r_t, θ_t) of x_t. Call e_θ the unit vector $(\cos\theta, \sin\theta)$ and define $u_t = e_{\theta_t}$: x_t may be written $r_t u_t$. Since $r_t^2 = \langle x_t, x_t\rangle$, we have:

$$\begin{aligned} r_t \dot{r}_t &= \langle x_t, A x_t\rangle \\ A(r_t u_t) &= \dot{x}_t = \dot{r}_t u_t + r_t \dot{u}_t. \end{aligned}$$

Therefore:

$$\dot{r}_t = r_t \langle u_t, A u_t\rangle \tag{3.1}$$

$$\dot{u}_t = A u_t - \langle u_t, A u_t\rangle u_t. \tag{3.2}$$

The evolution of u_t on the circle is autonomous. It may be rewritten in terms of θ_t. Since $\dot{u}_t = \dot{\theta}_t e_{\theta_t+\pi/2}$, the scalar product of (3.2) with $e_{\theta_t+\pi/2}$ gives:

$$\begin{aligned} \dot{\theta}_t &= \left\langle A e_{\theta_t}, e_{\theta_t+\pi/2}\right\rangle \\ &= (A_{22} - A_{11}) \sin(\theta_t)\cos(\theta_t) + A_{21}\cos^2(\theta_t) - A_{12}\sin^2(\theta_t). \end{aligned} \tag{3.3}$$

The critical points of this differential equation are related to the eigenvector of A as it is pointed out in the following (easy) lemma.

Lemma 3.1.1. *For any matrix A, the function*

$$d: \ \theta \mapsto d(\theta) = \left\langle A e_\theta, e_{\theta+\pi/2}\right\rangle$$

given by (3.3) *is π-periodic and $d(\theta) = 0$ iff u_θ is an eigenvector of A. As a consequence, the equation $d(\theta) = 0$ admits*

- *four solutions iff A admits two different eigenvalues,*
- *two solutions iff A is a Jordan matrix,*
- *no solution iff the eigenvalues of A are not real.*

Finally, the function d is zero iff $A = \lambda \mathrm{I}_2$.

Proof. From (3.3), it is obvious that d is π-periodic. Moreover, $d(\theta) = 0$ iff Ae_θ is orthogonal to $e_{\theta+\pi/2}$. This happens if and only if e_θ is an eignevector of A. □

3.1.2 The angular process

Now we come back to our stochastic process. Between jumps, the process follows the deterministic dynamics described above, with $A \in \{A_0, A_1\}$. Since the evolution of the angle θ is autonomous for each dynamics, the process (Θ, I) is a Markov process on $\mathbb{R} \times \{0, 1\}$. The evolution of $(R_t)_{t\geqslant 0}$ is determined by the one of the process $((\Theta_t, I_t))_{t\geqslant 0}$, by solving Equation (3.1) between the jumps. If we call $\mathcal{A}(\theta, i) = \langle A_i e_\theta, e_\theta\rangle$, then

$$R_t = R_0 \exp\left(\int_0^t \mathcal{A}(\Theta_s, I_s) ds\right). \tag{3.4}$$

and R_t appears as a multiplicative functional of $((\Theta_s, I_s))_{0\leqslant s\leqslant t}$. Let us define, for $i \in \{0, 1\}$ and $\lambda \in (0, 1)$,

$$\begin{aligned} d_i(\theta) &= \left\langle A_i e_\theta, e_{\theta+\pi/2}\right\rangle, \\ d_\lambda(\theta) &= (1-\lambda) d_0(\theta) + \lambda d_1(\theta). \end{aligned}$$

In our setting,

$$
\begin{aligned}
d_0(\theta) &= -a/b\cos^2(\theta) - ab\sin^2(\theta) < 0 \\
d_1(\theta) &= ab\cos^2(\theta) + a/b\sin^2(\theta) > 0.
\end{aligned}
$$

The generator of the Markov process (Θ, I) is given by:

$$
L_\beta f(\theta, i) = d_i(\theta)\partial_\theta f(\theta, i) + \frac{\beta}{2}(f(\theta, 1-i) - f(\theta, i)),
$$

3.1.3 The invariant measure of the angular process

The Markov process (Θ, I) admits a unique invariant measure and it can be expressed as follows.

Lemma 3.1.2. *The invariant measure μ_β of the angular process is given by*

$$
\mu_\beta(d\theta, i) = \frac{1}{C(\beta)} \frac{1}{|d_i(\theta)|} e^{\beta v(\theta)} \mathbb{1}_{[0,2\pi]}(\theta)\, d\theta,
$$

where

$$
v(\theta) = \begin{cases} \frac{1}{2a}(\arctan(b\tan(\theta)) - \arctan(b^{-1}\tan(\theta))) & \text{if } \theta \neq \pm\frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}
$$

and

$$
C(\beta) = \int_0^{2\pi} \left[\frac{1}{d_1(\theta)} - \frac{1}{d_0(\theta)}\right] e^{\beta v(\theta)}\, d\theta.
$$

Remark 3.1.3. Notice that v belongs to $\mathcal{C}^\infty(\mathbb{T})$ and is π-periodic. Moreover, $v'(\theta) = 0$ if and only if $\theta = \pm\pi/4 + k\pi$. Finally, the function v reaches its maximum at $\pi/4 + k\pi$ and its minimum at $-\pi/4 + k\pi$.

Proof of Lemma 3.1.2. If μ_β is an invariant measure for (Θ, I), then, for any smooth function f on $\mathbb{T} \times \{0, 1\}$, one has

$$
\int_{\mathbb{T}\times\{0,1\}} L_\beta f(\theta, i) d\mu_\beta(\theta, i) = 0.
$$

Let us look for an invariant measure μ_β on $\mathbb{T} \times \{0, 1\}$ that can be written as

$$
\mu_\beta(d\theta, i) = \rho_0(\theta)\mathbb{1}_0(i)\, d\theta + \rho_1(\theta)\mathbb{1}_1(i)\, d\theta,
$$

where ρ_0 and ρ_1 are two smooth and 2π-periodic functions. If f does not depend on the discrete variable $i \in \{0, 1\}$, *i.e.* $f(\theta, i) = f(\theta)$, then

$$
\int_{\mathbb{T}\times\{0,1\}} L_\beta f(\theta) d\mu_\beta(\theta, i) = \int_{\mathbb{T}} \partial_\theta f(\theta)(d_0\rho_0)(\theta)d\theta + \int_{\mathbb{T}} \partial_\theta f(\theta)(d_1\rho_1)(\theta)d\theta,
$$

and an integration by parts leads to

$$
\int_{\mathbb{T}\times\{0,1\}} L_\beta f(\theta) d\mu_\beta(\theta, i) = -\int_{\mathbb{T}} f(\theta)[d_0\rho_0 + d_1\rho_1]'(\theta)d\theta
$$

This ensures that $d_0\rho_0 + d_1\rho_1$ must be constant. Let us assume that one can find ρ_0 and ρ_1 such that $d_0\rho_0 + d_1\rho_1 = 0$. Now, if f is such that $f(\theta, 0) = f(\theta)$ et $f(\theta, 1) = 0$, we get

$$\int_{\mathbb{T}\times\{0,1\}} L_\beta f(\theta, i) d\mu_\beta(\theta, i) = \int_{\mathbb{T}} \left[d_0(\theta)\partial_\theta f(\theta) - \frac{\beta}{2} f(\theta) \right] \rho_0(\theta) d\theta + \int_{\mathbb{T}} \frac{\beta}{2} f(\theta)\rho_1(\theta) d\theta$$

and, after an integration by parts,

$$\int_{\mathbb{T}\times\{0,1\}} L_\beta f(\theta, i) d\mu_\beta(\theta, i) = \int_{\mathbb{T}} f(\theta) \left[-(d_0\rho_0)'(\theta) + \frac{\beta}{2}(\rho_1(\theta) - \rho_0(\theta)) \right] d\theta.$$

Let us define $\phi = d_0\rho_0$. Then $\rho_0 = \frac{\phi}{d_0}$ and $\rho_1 = -\frac{\phi}{d_1}$. The function ϕ is solution of the following ordinary differential equation:

$$\phi' = -\frac{\beta}{2}\left(\frac{1}{d_1} + \frac{1}{d_0}\right)\phi. \tag{3.6}$$

This equation admits a solution on $\mathbb{T}$ (*i.e.* 2π-periodic) since the integral of $\frac{1}{d_1} + \frac{1}{d_0}$ on $[-\pi, \pi]$ is equal to 0. In fact this is already true on $[-\pi/2, \pi/2]$. Since d_0 and d_1 are explicit trigonometric functions, one can find an explicit expression for ϕ. Notice that

$$\begin{aligned} \left[\arctan\left(b^{-1}\tan(\theta)\right)\right]' &= \frac{1}{b}\frac{1+\tan^2(\theta)}{1+\frac{\tan^2(\theta)}{b^2}} = \frac{1}{b\cos^2(\theta) + \frac{1}{b}\sin^2(\theta)} = \frac{a}{d_1(\theta)} \\ [\arctan(b\tan(\theta))]' &= -\frac{a}{d_0(\theta)}. \end{aligned}$$

The differential equation (3.6) becomes $\phi' = \beta v'\phi$ where v is given by (3.5) and its solutions are given by

$$\phi = K\exp(\beta v).$$

This relation provides the expression of ρ_0 and ρ_1 up to the multiplicative constant K. Since we are looking for probability measures, K is such that

$$K\int_{\mathbb{T}}\left(\frac{1}{d_0(\theta)} - \frac{1}{d_1(\theta)}\right)\phi(\theta)d\theta = 1$$

Conversely, it is easy to check that the measure given in Lemma 3.1.2 is invariant for L_β. $\square$

3.1.4 The Lyapunov exponent

Let us recall that we are interested in the function χ given by

$$\chi(\beta) = \int \mathcal{A}(\theta, i)\, d\mu_\beta(\theta, i).$$

Lemma 3.1.4. *The function $\beta \mapsto \chi(\beta)$ is $\mathcal{C}^1$ and monotonous application on $[0, +\infty)$ such that χ' has the sign of $b^2 - 1$ and*

$$\chi(0) = -1, \quad \lim_{\beta\to\infty}\chi(\beta) = \frac{a(b^2-1)}{2b} - 1.$$

Proof. From the definition of A_i and $\mathcal{A}$, we get that, for $i \in \{0,1\}$,

$$\mathcal{A}(\theta, i) = \langle A_i e_\theta, e_\theta \rangle = \frac{a(b^2-1)}{2b} \sin(2\theta) - 1.$$

For sake of simplicity, $\mathcal{A}(\theta)$ stands for $\mathcal{A}(\theta,0) = \mathcal{A}(\theta,1)$. Thus, $\chi(\beta)$ is given by

$$\chi(\beta) = \int_0^{2\pi} \mathcal{A}(\theta) \tilde{\mu}_\beta(d\theta),$$

where

$$\tilde{\mu}_\beta(d\theta) = \frac{1}{C(\beta)} \left(\frac{1}{d_1(\theta)} - \frac{1}{d_0(\theta)} \right) e^{\beta v(\theta)} \mathbb{1}_{[0,2\pi]}\, d\theta.$$

Its derivative is given by

$$\begin{aligned}
\chi'(\beta) &= \int_0^{2\pi} \mathcal{A}(\theta) v(\theta) \tilde{\mu}_\beta(d\theta) - \frac{C'(\beta)}{C(\beta)} \int_0^{2\pi} \mathcal{A}(\theta) \tilde{\mu}_\beta(d\theta) \\
&= \int_0^{2\pi} \mathcal{A}(\theta) v(\theta) \tilde{\mu}_\beta(d\theta) - \int_0^{2\pi} v(\theta) \tilde{\mu}_\beta(d\theta) \int_0^{2\pi} \mathcal{A}(\theta) \tilde{\mu}_\beta(d\theta).
\end{aligned}$$

In other words, one has

$$\begin{aligned}
\chi'(\beta) &= \mathrm{Cov}_{\tilde{\mu}_\beta}(\mathcal{A}(\cdot), v(\cdot)) \\
&= \frac{a(b^2-1)}{2b} \mathrm{Cov}_{\tilde{\mu}_\beta}(\sin(2\cdot), v(\cdot)).
\end{aligned}$$

The mean of $\sin(2\cdot)$ with respect to $\tilde{\mu}_\beta$ is equal to 0. Besides, $\theta \mapsto v(\theta)\sin(2\theta)$ is nonnegative (and non constant) on $\mathbb{T}$. Thus, χ' has the sign of $b^2 - 1$.

If $\beta = 0$, one has

$$\begin{aligned}
\chi(0) &= \frac{1}{C(0)} \int_0^{2\pi} \left(\frac{a(b^2-1)}{2b} \sin(2\theta) - 1 \right) \left(\frac{1}{d_1(\theta)} - \frac{1}{d_0(\theta)} \right) d\theta \\
&= -\frac{1}{C(0)} \int_0^{2\pi} \left(\frac{1}{d_1(\theta)} - \frac{1}{d_0(\theta)} \right) d\theta = -1 < 0.
\end{aligned}$$

Finally, as β goes to ∞, the probability measure ν_β converges to a probability measure concentrated on the points $\{\pi/4, 5\pi/4,\}$ where v reaches its maximum. We get

$$\lim_{\beta \to +\infty} \chi(\beta) = \frac{a(b^2-1)}{2b} - 1.$$

This concludes the proof. □

Corollary 3.1.5. *If $b > 1 + \sqrt{1+a^2}$, then there exists $\beta_c \in (0, +\infty)$ such that χ is negative on $(0, \beta_c)$ and positive on $(\beta_c, +\infty)$.*

3.2 The general case

Let $A_0, A_1 \in \mathbb{R}^{2\times 2}$ be two real matrices which admit two eigenvalues with negative real parts: A_0 and A_1 are said to be Hurwitz matrices. In [BBM09], the authors deal with the stability problem for the planar linear switching system $\dot{x}_t = (1-u_t)A_0 x_t + u_t A_1 x_t$, where $u\colon [0,\infty) \to \{0,1\}$ is a measurable function. They provide necessary and sufficient conditions on A_0 and A_1 for the system to be asymptotically stable for arbitrary switching function u. The main hypothesis that ensures the existence of a control u such that the system is not asymptotically stable is the following.

Assumption 3.2.1. *There exists* $\lambda \in (0,1)$ *such that the matrix* $A_\lambda = (1-\lambda)A_0 + \lambda A_1$ *has two real eigenvalues* $-\lambda_- < 0 < \lambda_+$ *with opposite signs. Let us denote by* u_-, u_+ *two associated (real, unit) eigenvectors.*

Under Assumption 3.2.1 the norm of the continuous component X goes to zero if the jumps are rare and to $+\infty$ if the jumps are sufficiently numerous (and $X_0 \neq 0$).

Theorem 3.2.2. *Under Assumption 3.2.1, there exists* $\chi(\beta) \in \mathbb{R}$ *such that, for any initial measure* ν *such that* $\nu(\{0\} \times \{0,1\}) = 0$,

$$\frac{1}{t} \log \|X_t\| \xrightarrow[t\to\infty]{\mathbb{P}_\nu - a.s.} \chi(\beta). \tag{3.7}$$

Moreover, there exist two constants $0 < \beta_1 \leqslant \beta_2 < \infty$ *such that:*

- *if* $\beta < \beta_1$, *then* $\chi(\beta)$ *is negative and* $\|X_t\| \xrightarrow[t\to\infty]{\mathbb{P}_\nu - a.s.} 0$,
- *if* $\beta > \beta_2$, *then* $\chi(\beta)$ *is positive and* $\|X_t\| \xrightarrow[t\to\infty]{\mathbb{P}_\nu - a.s.} \infty$.

Chapter 4

Switched vector fields: quantitative estimates

This chapter is essentially inspired by [BLBMZ12c]. Let E be a finite set, $(\lambda(\cdot, i))_{i\in E}$ be n nonnegative continuous functions on $\mathbb{R}^d$, P be an irreducible stochastic matrix and, for any $i \in E$, $F^i : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a smooth vector field such that the ordinary differential equation

$$\begin{cases} x_t' = F^i(x_t), \quad t > 0; \\ x_0 = x, \end{cases}$$

has a unique and global solution $t \mapsto \varphi_t^i(x)$ on $[0, \infty)$ for any initial condition $x \in \mathbb{R}^d$. Let us consider the Markov process

$$(Z_t)_{t\geqslant 0} = ((X_t, I_t))_{t\geqslant 0} \text{ on } \mathbb{R}^d \times E$$

defined by its extended generator L as follows:

$$Lf(x,i) = \left\langle F^i(x), \nabla_x f(x,i) \right\rangle + \lambda(x,i) \sum_{j\in E} P(i,j)(f(x,j) - f(x,i)) \tag{4.1}$$

for any smooth function $f : \mathbb{R}^d \times E \to \mathbb{R}$ Let us describe the dynamics of this process. Assume that $(X_0, I_0) = (x, i) \in \mathbb{R}^d \times E$. Before the first jump time T_1 of I, the first component X is driven by the vector field F^i and then $X_t = \varphi_t^i(x)$. The time T_1 can be defined by:

$$T_1 = \inf\left\{ t > 0 \, : \, \int_0^t \lambda(X_s, i)\, ds \geqslant E_1 \right\},$$

where E_1 is an exponential random variable with parameter 1. Since the paths of X are deterministic between the jump times of I, the randomness of T_1 comes from the one of E_1 and

$$T_1 = \inf\left\{ t > 0 \, : \, \int_0^t \lambda(\varphi_s^i(x), i)\, ds \geqslant E_1 \right\}.$$

Remark 4.0.3. Notice that $\mathbb{P}_{(x,i)}(T_1 = +\infty) > 0$ if and only if

$$\int_0^{+\infty} \lambda(\varphi_s^i(x), i)\, ds < +\infty.$$

If we assume that $\underline{\lambda} := \inf_{(x,i)} \lambda(x,i) > 0$ then the process I will jump infinitely often. Of course this assumption is not necessary.

At time T_1, the second coordinate I performs a jump with the law $P(i, \cdot)$ and the vector fields that drives the evolution of X is switched...

In order to get explicit rates of convergence, we have to assume that the jump rates are smooth and that each vector field F^i has a unique stable point.

Assumption 4.0.4. *There exist* $0 < \underline{\lambda} \leqslant \overline{\lambda}$ *and* $\kappa > 0$ *such that, for any* $x, \tilde{x} \in \mathbb{R}^d$ *and* $i \in E$,

$$\lambda(x, i) \in [\underline{\lambda}, \overline{\lambda}] \quad \text{and} \quad |\lambda(x, i) - \lambda(\tilde{x}, i)| \leqslant \kappa |x - \tilde{x}|,$$

Assumption 4.0.5. *Assume that there exists* $\alpha > 0$ *such that,*

$$\langle x - \tilde{x}, F^i(x) - F^i(\tilde{x}) \rangle \leqslant -\alpha |x - \tilde{x}|^2, \quad x, \tilde{x} \in \mathbb{R}^d, \ i \in E. \tag{4.2}$$

Assumption 4.0.5 ensures that, for any $i \in E$,

$$\left| \varphi_t^i(x) - \varphi_t^i(\tilde{x}) \right| \leqslant e^{-\alpha t} |x - \tilde{x}|, \quad x, \tilde{x} \in \mathbb{R}^d.$$

As a consequence, the vector fields F^i has exactly one critical point $\sigma(i) \in \mathbb{R}^d$. Moreover it is exponentially stable since, for any $x \in \mathbb{R}^d$,

$$\left| \varphi_t^i(x) - \sigma(i) \right| \leqslant e^{-\alpha t} |x - \sigma(i)|.$$

In particular, X cannot escape from a sufficiently large ball. Let us now state the main result of this chapter

Theorem 4.0.6. *Assume that Assumptions 4.0.4 and 4.0.5 hold and that the supports of* μ_0 *and* $\tilde{\mu}_0$ *are included in the ball* $\bar{B}(0, r)$ *where* r *is given by* (4.3). *Then*

$$W_1(\mu_t, \tilde{\mu}_t) \leqslant 2r(1 + ct) \exp\left(-\frac{\alpha}{1 + \alpha/\gamma} t \right)$$

where

$$\gamma = \frac{(\alpha + 2\underline{\lambda}) - \sqrt{(\alpha + 2\underline{\lambda})^2 - 8p\alpha\underline{\lambda}}}{2} \quad \text{and} \quad c = \frac{\alpha}{\alpha + \gamma} \frac{2ep\alpha\underline{\lambda}}{\sqrt{(\alpha + 2\underline{\lambda})^2 - 8p\alpha\underline{\lambda}}},$$

with $p = e^{-\kappa/\alpha}$ *and* $e = \exp(1)$.

Corollary 4.0.7. *Under Assumptions 4.0.4 and 4.0.5, the process* Z *admits a unique invariant measure* μ *and*

$$W_1(\mu_t, \mu) \leqslant 2r(1 + ct) \exp\left(-\frac{\alpha}{1 + \alpha/\gamma} t \right).$$

As for the TCP process, it is not possible to construct a coupling such that the discrete components I and $\tilde{I}$ jump always together: once I and $\tilde{I}$ are equal, they can go appart with positive probability since the jump rates depend also on X and $\tilde{X}$. Nevertheless, the main idea is the following: if I and $\tilde{I}$ are equal, the distance between X and $\tilde{X}$ decreases exponentially fast and then it should be more and more easier to make the processes I and $\tilde{I}$ jump simultaneously since the jump rates are Lipschitz functions of X.

This section is organized as follows. We firstly we prove Lemma 4.0.8 that ensures that the process X cannot escape from a sufficiently large ball. In particular, the support of the invariant law of X is included in this ball. Then we construct the coupling of two processes (X, I) and $(\tilde{X}, \tilde{I})$ driven by the same infinitesimal generator (4.1) with different initial condition. At last we compare the distance between X and $\tilde{X}$ to an companion process that goes to 0 exponentially fast.

A preliminary estimate

Lemma 4.0.8. *Under Assumtions 4.0.4 and 4.0.5, the process Z cannot escape from the compact set $\bar{B}(0,r)\times E$ where $\bar{B}(0,r)$ is the (closed) ball centered in $0\in\mathbb{R}^d$ with radius r given by*

$$r=\frac{\max_{i\in E}\left|F^i(0)\right|}{\alpha}. \tag{4.3}$$

Moreover, if $|X_0|>r$ then X hits $\bar{B}(0,r)$ exponentially fast.

Proof of Lemma 4.0.8. Setting $\tilde{x}=0$ in (4.2) ensures that, for $\varepsilon\in(0,\alpha)$,

$$\langle f(x,y),x\rangle \leqslant -\alpha|x|^2+\langle f(0,y),x\rangle \leqslant -(\alpha-\varepsilon)|x|^2+C(\varepsilon),$$

if $C(\varepsilon)=\max_{y\in E}|f(0,y)|^2/(4\varepsilon)$. In other words,

$$|X_t|^2-|X_s|^2=\int_s^t 2\langle f(X_u,Y_u),X_u\rangle\,du \leqslant -2(\alpha-\varepsilon)\int_s^t |X_u|^2\,du+2C(\varepsilon).$$

As a consequence,

$$|X_t|^2\leqslant \frac{C(\varepsilon)}{\alpha-\varepsilon}(1-e^{-2(\alpha-\varepsilon)t})+|X_0|^2e^{-2(\alpha-\varepsilon)t}.$$

With $\varepsilon=\alpha/2$, one gets that X cannot escape from the centered closed ball with radius $r=\sqrt{2C(\alpha/2)/\alpha}$. With $\varepsilon=\alpha/4$, one gets that if $|X_0|\geqslant r$, then X will hit $\bar{B}(0,r)$ exponentially fast. □

The coupling

Let us construct a Markov process on $(\mathbb{R}^d\times E)^2$ with marginals driven by (4.1) starting respectively from (x,i) and $(\tilde{x},j)$. This is done *via* its infinitesimal generator which is defined as follows:

- if $i\neq j$

$$\begin{aligned}Af(x,i,\tilde{x},j)=&\left\langle F^i(x),\nabla_x f(x,i,\tilde{x},j)\right\rangle+\left\langle F^j(\tilde{x}),\nabla_{\tilde{x}} f(x,i,\tilde{x},j)\right\rangle\\&+\lambda(x,i)\sum_{i'\in E}P(i,i')(f(x,i',\tilde{x},j)-f(x,i,\tilde{x},j))\\&+\lambda(\tilde{x},j)\sum_{j'\in E}P(j,j')(f(x,y,\tilde{x},j')-f(x,y,\tilde{x},j)).\end{aligned}$$

- if $i=j$ and $\lambda(x,i)\geqslant\lambda(\tilde{x},i)$:

$$\begin{aligned}Af(x,i,\tilde{x},j)=&\left\langle F^i(x),\nabla_x f(x,i,\tilde{x},i)\right\rangle+\left\langle F^i(\tilde{x}),\nabla_{\tilde{x}} f(x,i,\tilde{x},i)\right\rangle\\&+\lambda(\tilde{x},i)\sum_{i'\in E}P(i,i')(f(x,i',\tilde{x},i')-f(x,i,\tilde{x},i))\\&+(\lambda(x,i)-\lambda(\tilde{x},i))\sum_{i'\in E}P(i,i')(f(x,i',\tilde{x},i)-f(x,i,\tilde{x},i)),\end{aligned}$$

- if $i = j$ and $\lambda(x,i) < \lambda(\tilde{x},i)$:

$$
\begin{aligned}
Af(x,i,\tilde{x},j) =& \langle F^i(x), \nabla_x f(x,i,\tilde{x},i)\rangle + \langle F^i(\tilde{x}), \nabla_{\tilde{x}} f(x,i,\tilde{x},i)\rangle \\
&+ \lambda(x,i) \sum_{i'\in E} P(i,i')(f(x,i',\tilde{x},i') - f(x,i,\tilde{x},i)) \\
&+ (\lambda(\tilde{x},i) - \lambda(x,i)) \sum_{i'\in E} P(i,i')(f(x,i,\tilde{x},i') - f(x,i,\tilde{x},i)).
\end{aligned}
$$

Notice that if f depends only on (x,i) or on $(\tilde{x},j)$, then $Af = Lf$. Let us explain how this coupling works. When I and $\tilde{I}$ are different, the two processes (X,I) and $(\tilde{X},\tilde{I})$ evolve independently. If $I = \tilde{I}$ then two jump processes are in competition: a single jump *vs* two simultaneous jumps. The rate of arrival of a single jump is given by $|\lambda(x,i) - \lambda(\tilde{x},i)|$. It is bounded above by $\kappa|x - \tilde{x}|$. The rate of arrival of a simultaneous jump is given by $\lambda(x,i) \wedge \lambda(\tilde{x},i)$.

Assume firstly that X_0 and $\tilde{X}_0$ belongs to the ball $\bar{B}(0,r)$ where r is given by (4.3). Let us define D_t as the distance between X_t and $\tilde{X}_t$ for any $t \geqslant 0$. The process $(D_t)_{t\geqslant 0}$ is not Markovian. Nevertheless, as long as $I = \tilde{I}$, D_t decreases with a rate which is greater than α. If it is no longer the case, then D_t can increase. Nevertheless it is still smaller than $d = 2r$. After the coalescent time T_I of two independent independent copies of Y, D decreases once again. If $E = \{0,1\}$, then T_I is equal to the minimum of the jump times of the two independent processes which are both stochastically greater than a random variable of law $\mathcal{E}(\underline{\lambda})$. Thus T_I is (stochastically) smaller than $\mathcal{E}(2\underline{\lambda})$. Then $\mathbb{E}(D_t) \leqslant \mathbb{E}(U_t)$ where the Markov process $(U_t)_{t\geqslant 0}$ on $[0,d] \cup \{d+\varepsilon\}$ is driven by the infinitesimal generator

$$
Gf(x) = \begin{cases} -\alpha x f'(x) + \kappa x (f(d+\varepsilon) - f(x)) & \text{if } x \in [0,d], \\ 2\underline{\lambda}(f(d) - f(d+\varepsilon)) & \text{if } x = d+\varepsilon. \end{cases}
$$

The companion process

Let us consider the Markov process $V = (V_t)_{t\geqslant 0}$ on $[0,1] \cup \{1+\varepsilon\}$ defined by its infinitesimal generator:

$$
Hf(x) = \begin{cases} -\alpha x f'(x) + \kappa x (f(1+\varepsilon) - f(x)) & \text{if } x \in [0,1], \\ b(f(1) - f(1+\varepsilon)) & \text{if } x = 1+\varepsilon. \end{cases}
$$

Theorem 4.0.9. *For any $t \geqslant 0$,*

$$
\mathbb{E}(V_t | V_0 = 1) \leqslant \left(1 + (1+\varepsilon)\left(\frac{p\alpha b e}{\sqrt{(\alpha+b)^2 - 4p\alpha b}}\right)\frac{\alpha t}{\alpha+\gamma}\right) \exp\left(-\frac{1}{1+\alpha/\gamma}\alpha t\right) \tag{4.4}
$$

where

$$
p = e^{-\kappa/\alpha} \quad \textit{and} \quad \gamma = \frac{(\alpha+b) - \sqrt{(\alpha+b)^2 - 4p\alpha b}}{2} = \frac{(\alpha+b) - \sqrt{(\alpha-b)^2 + 4(1-p)\alpha b}}{2}.
$$

Remark 4.0.10. If α goes to ∞, then γ goes to 1 whereas $\gamma \sim p\alpha/b$ if b goes to ∞.

Proof. Starting from $1+\varepsilon$, the process V jumps after a random time with law $\mathcal{E}(b)$ to 1 and then goes to zero exponentially fast until it (possibly) goes back to $1+\varepsilon$. The first jump time T starting from 1 can be constructed as follows: let E be a random variable with law $\mathcal{E}(1)$. Then

$$T \overset{\mathcal{L}}{=} \begin{cases} -\dfrac{1}{\alpha}\log\left(1-\dfrac{\alpha E}{\kappa}\right) & \text{if } E < \dfrac{\kappa}{\alpha}, \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed, conditionally on $\{V_0 = 1\}$,

$$\int_0^t \lambda(V_s)ds = \int_0^t \kappa e^{-\alpha s}\, ds = \frac{\kappa}{\alpha}(1-e^{-\alpha t}).$$

In other words, the cumulative distribution function F_T of T is such that, for any $t \geqslant 0$,

$$1 - F_T(t) = \mathbb{P}(T > t) = \exp\left(-\frac{\kappa}{\alpha}(1-e^{-\alpha t})\right).$$

Let us define $p = e^{-\kappa/\alpha}$. The law of T is the mixture with respective weights p and $1-p$ of a Dirac mass at $+\infty$ and a probability measure on $\mathbb{R}$ with density

$$f\,:\, t \mapsto f(t) = \frac{\kappa}{1-p} e^{-\alpha t} e^{-\frac{\kappa}{\alpha}(1-e^{-\alpha t})} \mathbb{1}_{(0,+\infty)}(t) \tag{4.5}$$

and cumulative density function

$$F\,:\, t \mapsto F(t) = \left(\frac{1-e^{-\frac{\kappa}{\alpha}(1-e^{-\alpha t})}}{1-e^{-\frac{\kappa}{\alpha}}}\right)\mathbb{1}_{(0,+\infty)}(t).$$

Starting at 1, X will return to 1 with probability $1-p$. The Markov property ensures that the number N of returns of X to 1 is a random variable with geometric law with parameter p. The length of a finite loop from 1 to 1 can be written as the sum $S+E$ where the law of S has the density function f given in (4.5), the law of E is the exponential measure with parameter b and S and E are independent.

Remark 4.0.11. In the general case, E is not distributed as an exponential variable but as the coalescent time of a finite Markov chain. Its Laplace transform is finite on a neighbourhood of the origin.

Lemma 4.0.12. *The variable S is stochastically smaller than an exponential random variable with parameter α* i.e. *for any $t \geqslant 0$, $F(t) \geqslant F_\alpha(t)$ where $F_\alpha(t) = (1-e^{-\alpha t})\mathbb{1}_{\{t>0\}}$.*

Proof of Lemma 4.0.12. Recall that $e^{ux} - 1 \leqslant (e^x - 1)u$ for any $x > 0$ and $u \in [0,1]$. Indeed,

$$e^{ux} - 1 = u\sum_{k\geqslant 1} u^{k-1}\frac{x^k}{k!} \leqslant u \sum_{k\geqslant 1}\frac{x^k}{k!} = (e^x-1)u.$$

As a consequence, for any $t \geqslant 0$,

$$1 - F(t) = \frac{e^{\frac{\kappa}{\alpha}e^{-\alpha t}} - 1}{e^{\frac{\kappa}{\alpha}} - 1} \leqslant e^{-\alpha t} = 1 - F_\alpha(t).$$

This ensures the stochastic bound. ☐

As a consequence, the Laplace transform L_S of S with density f is smaller than the one of an exponential variable with parameter α: for any $s < \alpha$,

$$L_S(s) \leqslant \frac{\alpha}{\alpha - s}.$$

If L_e is the Laplace transform of $S + E$, then, for any $s < \alpha \wedge b$, we have

$$L_e(s) \leqslant \frac{\alpha}{\alpha - s} \frac{b}{b - s}.$$

Let us denote by T the last hitting time of 1 *i.e.* the last jump time of X and by L its Laplace transform. Let us introduce $N \sim \mathcal{G}(p)$, $(S_i)_{i \geqslant 1}$ with density f and $(E_i)_{i \geqslant 1}$ with law $\mathcal{E}(b)$. All the random variables are assumed to be independent. Then

$$T \overset{\mathcal{L}}{=} \sum_{i=1}^{N} (S_i + E_i).$$

Classically, for any $s \in \mathbb{R}$ such that $(1-p)L_e(s) < 1$, one has

$$L(s) = \mathbb{E}\big(e^{sT}\big) = \frac{pL_e(s)}{1 - (1-p)L_e(s)} = \frac{p}{1-p}\left(\frac{1}{1 - (1-p)L_e(s)} - 1\right).$$

Let us denote by

$$\gamma = \frac{(\alpha + b) - \sqrt{(\alpha + b)^2 - 4p\alpha b}}{2} \quad \text{and} \quad \tilde{\gamma} = \frac{(\alpha + b) + \sqrt{(\alpha + b)^2 - 4p\alpha b}}{2}$$

the two roots of $X^2 - (\alpha + b)X + p\alpha b = 0$. Notice that $\gamma < \alpha \wedge b < \tilde{\gamma}$. For any $s < \gamma$, one has $(1-p)L_e(s) < 1$ and

$$L(s) \leqslant \frac{p\alpha b}{(\gamma - s)(\tilde{\gamma} - s)} \leqslant \frac{p\alpha b}{\tilde{\gamma} - s} \frac{1}{\gamma - s}. \tag{4.6}$$

Let us now turn to the control of $\mathbb{E}(V_t | V_0 = 1)$. The idea is to discuss wether $T > \beta t$ or not for some $\beta \in (0, 1)$ (and then to choose β as good as possible):

- if $T < \beta t$, then $V_t \leqslant e^{-(1-\beta)\alpha t}$,
- the event $\{T \geqslant \beta t\}$ will have a small probability for large t since T has a finite Laplace transform on a neighbourhood of the origin.

For any $\beta \in (0, 1)$ and $s > 0$,

$$\begin{aligned} \mathbb{E}(V_t | X_0 = 1) &= \mathbb{E}\big(V_t \mathbb{1}_{\{T \leqslant \beta t\}}\big) + \mathbb{E}\big(V_t \mathbb{1}_{\{T > \beta t\}}\big) \\ &\leqslant e^{-(1-\beta)\alpha t} + (1 + \varepsilon)L(s)e^{-s\beta t}. \end{aligned} \tag{4.7}$$

From Equation (4.6), we get that, for any $s < \gamma$, $\log L(s) - \beta ts \leqslant h(s)$ where

$$h(s) = \log\left(\frac{p\alpha b}{\tilde{\gamma} - \gamma}\right) - \log(\gamma - s) - \beta ts.$$

The function h reaches its minimun at $s(t) = \gamma - (\beta t)^{-1}$ and

$$h(s(t)) = \log\left(\frac{p\alpha b}{\tilde{\gamma} - \gamma}\right) + \log(\beta t) + 1 - \gamma\beta t.$$

For $t > 0$ and $\beta \in (0,1)$, choose $s(t) = \gamma - (\beta t)^{-1}$ in (4.7) to get

$$\begin{aligned}\mathbb{E}(V_t) &\leqslant e^{-(1-\beta)\alpha t} + (1+\varepsilon)e^{h(\gamma(t))} \\ &\leqslant e^{-(1-\beta)\alpha t} + (1+\varepsilon)\left(\frac{p\alpha be}{\tilde{\gamma}-\gamma}\right)\beta t e^{-\gamma\beta t}.\end{aligned}$$

At last, one can choose $\beta = \alpha(\alpha+\gamma)^{-1}$ in order to have $(1-\beta)\alpha = \gamma\beta$. This ensures that

$$\mathbb{E}(V_t) \leqslant \left(1 + (1+\varepsilon)\left(\frac{p\alpha be}{\tilde{\gamma}-\gamma}\right)\frac{\alpha t}{\alpha+\gamma}\right)\exp\left(-\frac{\alpha\gamma}{\alpha+\gamma}t\right).$$

Replacing $\tilde{\gamma}-\gamma$ by its expression as a function of α, b and p provides (4.4). $\square$

Chapter 5

Switched vector fields: qualitative estimates

This chapter presents an overview of the papers [BH12] and [BLBMZ12b]. The process under study is a continuous time Markov process ($Z_t = (X_t, Y_t)$) living on $M \times E$ (where M is a subset of $\mathbb{R}^d$ and E is a finite set) whose infinitesimal generator acts on functions

$$g : M \times E \to \mathbb{R},$$
$$(x, i) \mapsto g(x, i),$$

smooth[1] in x, according to the formula

$$Lg(x,i) = \langle F^i(x), \nabla g(x,i) \rangle + \sum_{j \in E} \lambda(x,i,j)(g(x,j) - g(x,i)) \tag{5.1}$$

where

(i) $x \mapsto \lambda(x,i,j)$ is continuous;

(ii) $\lambda(x,i,j) \geqslant 0$ for $i \neq j$ and $\lambda(x,i,i) = 0$;

(iii) For each $x \in M$, the matrix $(\lambda(x,i,j))_{ij}$ is irreducible.

5.1 What can be shown in general ?

The support of the law of the process can be described in term of the solutions set of a differential inclusion induced by the collection $\{F^i : i \in E\}$ (see [BLBMZ12b]).

A natural candidate to support invariant probabilities is the set of the points that can be reached from any other starting point. For all $n \in \mathbb{N}_*$ let $\mathbb{T}_n = E^{n+1} \times \mathbb{R}_+^n$. Given

$$(\mathbf{i}, \mathbf{t}) = ((i_0, \ldots, i_n); (t_1, \ldots, t_n)) \in \mathbb{T}_n$$

and $x \in M$ we let

$$\mathbf{\Phi}_{\mathbf{t}}^{\mathbf{i}}(x) = \Phi_{t_n}^{i_{n-1}} \circ \ldots \circ \Phi_{t_1}^{i_0}(x). \tag{5.2}$$

The *positive* trajectory of x is the set

$$\gamma^+(x) = \{\mathbf{\Phi}_{\mathbf{t}}^{\mathbf{i}}(x) : (\mathbf{i}, \mathbf{t}) \in \bigcup_{n \in \mathbb{N}_*} \mathbb{T}_n\}.$$

[1] meaning that g^i is the restriction to M of a smooth function on $\mathbb{R}^m$

The *accessible set* of (X_t) is the (possibly empty) compact set $\Gamma \subset M$ defined as

$$\Gamma = \bigcap_{x \in M} \overline{\gamma^+(x)}.$$

It can be shown that this *accessible set* Γ is compact, connected, strongly positively invariant and invariant under the differential inclusion induced by $\{F^i : i \in E\}$.

Under some Hörmander bracket conditions, the law of jump chain and the process converge exponentially in total variation toward the unique invariant probability of the process.

Let $\mathcal{F}_0$ the collection of vector fields $(F^i : i \in E)$. Let $\mathcal{F}_k = \mathcal{F}_{k-1} \cup \{[F^i, V], V \in \mathcal{F}_{k-1}\}$, and $\mathcal{F}_k(x)$ the vector space (included in $T_x M$) spanned by $\{V(x), V \in \mathcal{F}_k\}$.

Similarly, starting from $\mathcal{G}_0 = \{F^i - F^j, i \neq j\}$, we define $\mathcal{G}_k$ by taking Lie brackets with the F^i, and $\mathcal{G}_k(x)$ the corresponding subspace of $T_x M$.

Definition 5.1.1. *We say that the* weak bracket condition *is satisfied at x if there exists k such that $\mathcal{F}_k(x) = T_x M$. If for some k, $\mathcal{G}_k(x) = T_x M$, we say that the* strong bracket condition *holds.*

Since $\mathcal{G}_k(x)$ is a subspace of $\mathcal{F}_k(x)$, the strong condition implies the weak one. The converse is false, a counter-example is given below in Section 5.2.1.

These two conditions are called A (for the stronger) and B (for the weaker) in [BH12]. The following result is a version of Theorem 2 from [BH12], with an additional uniformity with respect to the initial point and the time t.

Theorem 5.1.2 (Regularity — local form, bracket condition)**.** *If the weak bracket condition holds at x_0, then the process is partly regular at jump times: there exist an integer K', a constant $c > 0$ and non-empty open sets $\mathcal{U}_0$, $\mathcal{V}_0$ such that*

$$\forall x \in \mathcal{U}_0', \forall i, \quad \mathbb{P}_{x,i}\left[X_{T_{K'}} \in \cdot\right] \geqslant c\lambda_{\mathbb{R}^m}(\cdot \cap \mathcal{V}_0). \tag{5.3}$$

If the strong bracket condition holds, the process is partly regular: there is a t_0, two constants $c > 0$ and $\epsilon > 0$, and two non-empty open sets $\mathcal{U}_0$, $\mathcal{V}_0$ such that

$$\forall x \in \mathcal{U}_0, \forall i, \forall t \in [t_0, t_0 + \epsilon], \quad \mathbb{P}_{x,i}[X_t \in \cdot] \geqslant c\lambda_{\mathbb{R}^m}(\cdot \cap \mathcal{V}_0). \tag{5.4}$$

Thanks to this theorem one can easily deduce some exponential convergence to equilibrium with respect to the total variation distance for both the embedded chain and the continuous time process.

Moreover, under the weak bracket condition, the invariant measures of the embedded chain and the continuous time process are unique and absolutely continous.

5.2 Elementary examples

We give here a few examples of systems given by (5.1).

5.2.1 On the torus

Consider the system defined on the torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ by the constant vector fields $F^i = e^i$, where $(e_1, \dots e_n)$ is the standard basis on $\mathbb{R}^n$. Then, as argued in [BH12], the weak bracket condition holds everywhere, and the strong condition does not hold. Therefore the chain $\tilde{Z}$ is ergodic and converges exponentially fast, the empirical means of $\tilde{Z}_n$ and Z_t converge, but the law of Z_t is singular with respect to the invariant measure.

5.2.2 Two planar linear flows

Let A be a 2×2 real matrix whose eigenvalues η_1, η_2 have negative real parts. Set $E = \{0,1\}$ and consider the process defined on $\mathbb{R}^2 \times E$ by

$$F^0(x) = Ax \quad \text{and} \quad F^1(x) = A(x-a)$$

for some $a \in \mathbb{R}^2$. The associated flows are $\Phi_t^0(x) = e^{tA}x$ and $\Phi_t^1(x) = e^{tA}(x-a) + a$.

First note that, by using the Jordan decomposition of A, it is possible to find a scalar product $\langle \cdot \rangle$ on $\mathbb{R}^2$ (depending on A) and some number $0 < \alpha \leqslant \min(-\mathrm{Re}(\eta_1), -\mathrm{Re}(\eta_2))$ such that $\langle Ax, x\rangle \leqslant -\alpha\langle x, x\rangle$. Therefore

$$\langle A(x-a), x\rangle \leqslant -\alpha\langle x, x\rangle - \langle Aa, x\rangle \leqslant \|x\|(-\alpha\|x\| + \|Aa\|).$$

This shows that, for $R > \|Aa\|/\alpha$, the ball $M = \{x \in \mathbb{R}^2, \|x\| \leqslant R\}$ is positively invariant by Φ^0 and Φ^1. Moreover every solution to the differential inclusion induced by $\{F^0, F^1\}$ eventually enters M. In particular $M \times E$ is an absorbing set for the process (Z_t).

Another remark that will prove useful in our analysis is that

$$\det(F^0(x), F^1(x)) = \det(A)\det(a, x),$$

so that

$$\det(F^0(x), F^1(x)) > 0 \;\; (\text{resp.} = 0) \Leftrightarrow \det(a, x) > 0 \;\; (\text{resp.} = 0). \tag{5.5}$$

Case 1: a is an eigenvector

If a is an eigenvector of A the line $\mathbb{R}a$ is invariant by both flows, so that

$$\Gamma = \overline{\gamma^+(0)} = [0, a]$$

and there is one unique invariant probability π (whose support has to be Γ). Indeed, it is easily seen that Γ is an attractor for the set-valued dynamics induced by F^0 and F^1. Therefore the support of every invariant measure equals Γ. If we consider the process restricted to Γ, it becomes one-dimensional and the strong bracket condition holds, proving uniqueness.

Remark 5.2.1. If $X(0) \notin \mathbb{R}a$, X will never reach Γ. As a consequence, the law of X_t and π are singular. In particular, their total variation distance is constant, equal to 1. Note also that the strong bracket condition being satisfied everywhere except on $\mathbb{R}a$, the law of X_t at finite times has a regular part.

Remark 5.2.2. Consider the following example: $A = -Id$, $a = (1,0)$ and $\mathbb{R}a$ is identified to $\mathbb{R}$. If the jump rates are constant and equal to λ, it is easy to check (see [KB04, RMC07]) that the invariant measure μ on $[0,1] \times \{0,1\}$ is given by:

$$\mu = \frac{1}{2}\left(\mu_0 \times \delta_0 + \mu_1 \times \delta_1\right),$$

where μ_0 and μ_1 are Beta laws on $[0,1]$,

$$\mu_0(dx) = C_\lambda x^{\lambda-1}(1-x)^\lambda,$$
$$\mu_1(dx) = C_\lambda x^\lambda(1-x)^{\lambda-1}.$$

In particular, this example shows that the density of the invariant measure (with respect to the Lebesgue measure) may be unbounded: when the jump rate λ is smaller than 1, the densities blow up at 0 and 1.

Case 2: Eigenvalues are reals and a is not an eigenvector

Suppose $\eta_1, \eta_2 < 0$ and that a is not an eigenvector.

Let $\gamma_0 = \{\Phi^0_t(a),\, t \geqslant 0\}$, $\gamma_1 = \{\Phi^1_t(0),\, t \geqslant 0\}$. Note that γ_1 and γ_0 are image of each other by the transformation $T(x) = a - x$. The curve γ_0 (respectively γ_1) crosses the line $\mathbb{R}a$ only at point a (respectively 0). For, otherwise, the trajectory $t \mapsto \Phi^0_t(a)$ would have to cross the line $\mathrm{Ker}(A - \lambda_1 I)$ which is invariant. This makes the curve $\gamma = \gamma_0 \cup \gamma_1$ a simple closed curve in $\mathbb{R}^2$ crossing $\mathbb{R}a$ at 0 and a. By Jordan curve Theorem, $\mathbb{R}^2 \setminus \gamma = B \cup U$ where B is a bounded component and U an unbounded one. We claim that

$$\Gamma = \overline{B}.$$

To prove this claim, observe that thanks to (5.5), F^0 and F^1 both point inward B at every point of γ. This makes $\overline{B}$ positively invariant by Φ^0 and Φ^1. Thus $\Gamma \subset \overline{B}$. Conversely, $\gamma \subset \Gamma$ (because 0 and a are accessibles from everywhere). If $x \in B$ there exists $s > 0$ such that $\Phi^0_{-s}(x) \in \gamma$ (because $\lim_{t \to -\infty} \Phi^0_t(a) = -\infty$) and necessarily $\Phi^0_{-s}(x) \in \gamma_1$. This proves that $x \in \gamma^+(0)$. Finally note that the strong bracket condition is verified in $\Gamma \setminus \mathbb{R}a$ proving uniqueness and absolute continuity of the invariant probability.

Case 3: Eigenvalues are complex conjugates

Suppose now that the eigenvalues have a nonzero imaginary part. By Jordan decomposition, it is easily seen that trajectories of Φ^i converge in spiralling, so that the mappings $\tau^i(x) = \inf\{t > 0 : \Phi^i_t(x) \in \mathbb{R}a\}$ and $h^i(x) = \Phi^i_{\tau^i(x)}$ are well defined and continuous. Let $H : \mathbb{R}a \mapsto \mathbb{R}a$ be the map $h^0 \circ h^1$ restricted to $\mathbb{R}a$. Since two different trajectories of the same flow have empty intersection, the sequence $x_n = H^n(0)$ is decreasing (for the ordering on $\mathbb{R}a$ inherited from $\mathbb{R}$.) Being bounded (recall that M is compact and positively invariant), it converges to $x^* \in \mathbb{R}a$ such that $x^* = H(x^*)$. Let now $\gamma^0 = \{\Phi^1_t(x^*),\, 0 \leqslant t \leqslant \tau^1(x^*)\}, \gamma^1 = \{\Phi^0_t(h^1(x^*)),\, 0 \leqslant t \leqslant \tau^0(h^1(x^*))\}$ and $\gamma = \gamma^0 \cup \gamma^1$. Reasoning as previously shows that Γ is the bounded component of $\mathbb{R}^2 \setminus \gamma$ and that there is a unique invariant and absolutely continuous invariant probability.

We illustrate this situation in Figure 5.1, with

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Remark 5.2.3. Note that if the jump rates are small, the situation is similar to the one described in Remark 5.2.2, the process spends most of its time near the attractive points, and the density is unbounded at these points. Since they are in the interior of Γ, the density is not even continuous in the interior of Γ.

5.3 Knowing the flows is not enough

In this section we study in detail a PDMP on $\mathbb{R}^2$, where the strong bracket condition holds everywhere except on Γ, and where there may be one or more invariant measures, depending on the dynamics of the discrete part of the process.

This model has been suggested by O. Radulescu. The continuous part of the process takes its values on $\mathbb{R}^2$ whereas its discrete part belongs to $\{0, 1\}$. For sake of simplicity we will denote (in a different way than in the beginning of the paper) by $(X_t, Y_t) \in \mathbb{R}^2$ the continuous component. The discrete component $(I_t)_{t \geqslant 0}$ is a continuous time Markov chain

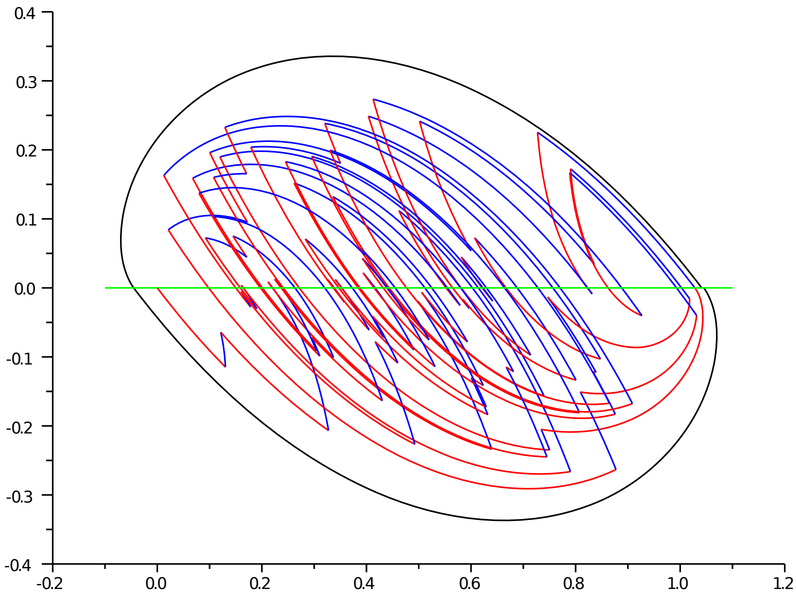

Figure 5.1: Double rotation.

on $E = \{0, 1\}$ with jump rates $(\lambda_i)_{i\in E}$. Let $\alpha > 0$. The two vector fields F^0 and F^1 are given by

$$F^0(x, y) = \begin{pmatrix} -x + \alpha \\ -y + \alpha \end{pmatrix} \quad \text{and} \quad F^1(x, y) = \begin{pmatrix} -x + \dfrac{\alpha}{1 + y^2} \\ -y + \dfrac{\alpha}{1 + x^2} \end{pmatrix}$$

with $(x, y) \in \mathbb{R}^2$. Notice that the quarter plane $(0, +\infty)^2$ is invariant under the action of the vector fields F^0 and F^1. If the support of the initial law of (X, Y) is included in the quarter plane (which is assumed from now on), then it is still the case for the law of (X_t, Y_t) at any time.

5.3.1 General properties of the two vector fields

Obviously, the vector fields F^0 has a unique stable point (α, α), whereas F^1 may admits one or three critical points, according to the value of α.

Lemma 5.3.1. *Let us define*

$$a = \frac{\alpha + \sqrt{|\alpha^2 - 4|}}{2} \quad \textit{and} \quad b = \left(\frac{\sqrt{4/27 + \alpha^2} + \alpha}{2}\right)^{1/3} - \left(\frac{\sqrt{4/27 + \alpha^2} - \alpha}{2}\right)^{1/3}.$$

Notice that b is positive and is the unique real solution of $b^3 + b = \alpha$. Then

- *if $\alpha \leqslant 2$, then F^1 admits a unique critical point (b, b) and it is stable,*

- *if $\alpha > 2$, then F^1 admits three critical points: (b,b) is unstable whereas (a,a^{-1}) and (a^{-1},a) are stable.*

Proof. If (x,y) is a critical point of F^1 then (x,y) is solution of

$$\begin{cases} x(1+y^2) = \alpha \\ y(1+x^2) = \alpha. \end{cases}$$

As a consequence, x is solution of

$$0 = x^5 - \alpha x^4 + 2x^3 - 2\alpha x^2 + (1+\alpha^2)x - \alpha = (x^2 - \alpha x + 1)(x^3 + x - \alpha).$$

The equation $x^3 + x - \alpha$ admits a unique real solution b. It belongs to $(0,\alpha)$. Obviously, if $\alpha \leqslant 2$, (b,b) is the unique critical point of F^1 whereas, if $\alpha > 2$ and a and a^{-1} are the roots of $x^2 - \alpha x + 1 = 0$, then F^1 admits the three critical points: (b,b), (a,a^{-1}) and (a^{-1},a). Let us have a look to the stability of (b,b). The Jacobian matrix of F^1 at (x,y) is given by

$$\mathrm{Jac}(F^1)(x,y) = \begin{pmatrix} -1 & -\dfrac{2\alpha y}{(1+y^2)^2} \\ -\dfrac{2\alpha x}{(1+x^2)^2} & -1 \end{pmatrix}.$$

Since $1+b^2 = \alpha/b$ one gets that

$$\mathrm{Jac}(F^1)(b,b) = \begin{pmatrix} -1 & -2+\dfrac{2b}{\alpha} \\ -2+\dfrac{2b}{\alpha} & -1 \end{pmatrix}$$

and its eigenvalues are given by

$$\eta_1 = -3 + \frac{2b}{\alpha} = -1 - 2\frac{\alpha - b}{\alpha} \quad \text{and} \quad \eta_2 = 1 - \frac{2b}{\alpha} = \frac{b^3 - b}{\alpha}$$

and are respectively associated to the eigenvectors $(1,1)$ and $(1,-1)$. Since $b < \alpha$, η_1 is smaller than -1. Moreover, η_2 has the same sign than $b-1$ *i.e.* the same sign than $\alpha - 2$. As a conclusion, (b,b) is stable (resp. unstable) if $\alpha < 2$ (resp. $\alpha > 2$).

Assume now that $\alpha > 2$. Then

$$\mathrm{Jac}(F^1)(a,a^{-1}) = \begin{pmatrix} -1 & -\dfrac{2a}{\alpha} \\ -\dfrac{2}{\alpha a} & -1 \end{pmatrix}$$

and its two eigenvalues $-1 \pm 2\alpha^{-1}$ are negative. The critical points (a,a^{-1}) and (a^{-1},a) are stable. □

In the sequel, we assume that $\alpha > 2$. The sets

$$\begin{aligned} D &= \{(x,x) \,:\, x > 0\}, \\ L &= \{(x,y) \,:\, x > 0 \text{ and } 0 < y < x\}, \\ U &= \{(x,y) \,:\, y > 0 \text{ and } 0 < x < y\} \end{aligned}$$

are invariant under the action of the flows F^0 and F^1. Moreover, the set D (and in particular the unique stable point (α,α) of F^0) is included in the stable manifold of the unstable equilibrium (b,b) of F^1.

What happens if (X,Y) starts at a point $(x,y) \in L$? The answer may depend on the parameters λ_0, λ_1, α.

5.3.2 Transience

Lemma 5.3.2. *Assume that* $(X_0, Y_0) \in L$. *Then, for any* $t > 0$,

$$0 \leqslant X_t - Y_t \leqslant (X_0 - Y_0) \exp\left(-\int_0^t \alpha(I_s)\, ds\right),$$

with $\alpha(0) = 1$ *and* $\alpha(1) = 1 - c\alpha < 0$ *with* $c = (3/8)\sqrt{3}$.

Proof. If $I_t = 0$ then

$$\frac{d}{dt}(X_t - Y_t) = -(X_t - Y_t).$$

On the other hand, if $I_t = 1$ then

$$\begin{aligned}\frac{d}{dt}(X_t - Y_t) &= -(X_t - Y_t) + \alpha \frac{X_t^2 - Y_t^2}{(1 + X_t^2)(1 + Y_t^2)} \\ &= -(1 - \alpha h(X_t, Y_t))(X_t - Y_t)\end{aligned}$$

where the function h is defined on $[0, \infty)^2$ by

$$h(x, y) = \frac{x + y}{(1 + x^2)(1 + y^2)}.$$

The unique critical point of h on $[0, \infty)^2$ is $(1/\sqrt{3}, 1/\sqrt{3})$ and h reaches its maximum at this point:

$$c := \sup_{x,y>0} h(x, y) = \frac{3\sqrt{3}}{8}.$$

As a consequence, for any $t \geqslant 0$,

$$\frac{d}{dt}(X_t - Y_t) \leqslant -\alpha(I_t)(X_t - Y_t) \quad \text{where} \quad \begin{cases} \alpha(0) = 1, \\ \alpha(1) = 1 - c\alpha. \end{cases}$$

Integrating this relation concludes the proof. □

Corollary 5.3.3. *Assume that* $(X_0, Y_0) \in L$. *If* $\lambda_1 > \lambda_0(c\alpha - 1)$ *then* (X_t, Y_t) *converges exponentially fast to* D *almost surely. More precisely,*

$$\limsup_{t\to\infty} \frac{1}{t} \log (X_t - Y_t) \leqslant -\frac{\lambda_1 - (c\alpha - 1)\lambda_0}{\lambda_0 + \lambda_1} < 0 \quad a.s. \tag{5.6}$$

In particular, the process (X, Y, I) *admits a unique invariant measure* μ *which support is the set*

$$S = \{(x, x) \, : \, x \in [b, \alpha]\}.$$

Proof. The ergodic theorem for the Markov process $(I_t)_{t\geqslant 0}$ ensures that

$$\frac{1}{t} \int_0^t \alpha(I_s)\, ds \xrightarrow[t\to\infty]{a.s.} \int \alpha(i) d\nu(i)$$

where the invariant measure ν of the process $(I_t)_{t\geqslant 0}$ is the Bernoulli measure with parameter $\lambda_0/(\lambda_0 + \lambda_1)$. The upper bound (5.6) is a straightforward consequence of Lemma 5.3.2. This ensures that the sets L and U are transient. At last, it is quite obvious that the set of recurrent points in D is exactly S. □

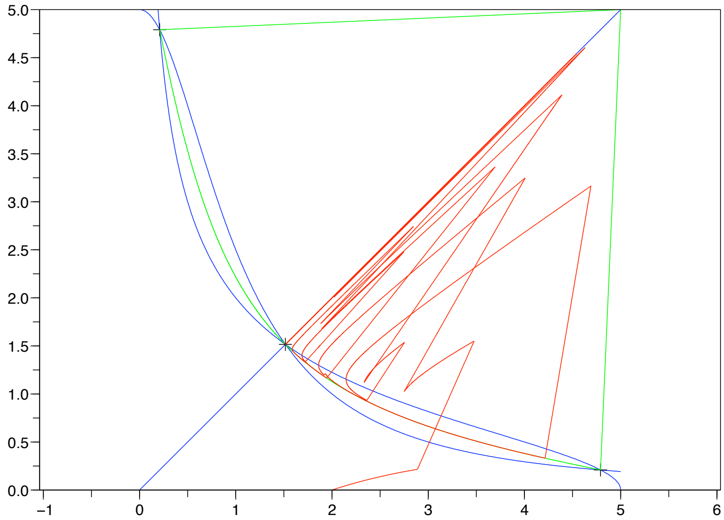

Figure 5.2: Trajectory of (X, Y) (red line) in the plane with $\lambda_0 = 1$, $\lambda_1 = 0.6$ and $\alpha = 5$.

One can also get an estimate for the p^{th} moment of $X_t - Y_t$.

Corollary 5.3.4. *Assume that $(X_0, Y_0) \in L$. Let $p > 0$ such that*

$$\lambda_1 > (\lambda_0 + p)(c\alpha - 1). \tag{5.7}$$

Then there exists two positive constants c_p, μ_p such that

$$\mathbb{E}(|X_t - Y_t|^p) \leqslant c_p \mathbb{E}(|X_0 - Y_0|^p) e^{-\mu_p t}.$$

Proof. Once again, Lemma 5.3.2 ensures that

$$0 \leqslant \mathbb{E}(|X_t - Y_t|^p) \leqslant \mathbb{E}(|X_0 - Y_0|^p) \mathbb{E}\left[\exp\left(-\int_0^t p\alpha(I_s)\, ds\right)\right].$$

According to [BGM10, Prop. 4.1], there exists $c_p \geqslant 1$ such that, for any $t \geqslant 0$,

$$\frac{1}{c_p} e^{-\mu_p t} \leqslant \mathbb{E}\left[\exp\left(-\int_0^t p\alpha(I_s)\, ds\right)\right] \leqslant c_p e^{-\mu_p t}$$

where $\mu_p = -\max\{\operatorname{Re}\eta \,:\, \eta \in \operatorname{Spec}(M_p)\}$ and

$$M_p = \begin{pmatrix} -\lambda_0 - p & \lambda_0 \\ \lambda_1 & -\lambda_1 + p(c\alpha - 1) \end{pmatrix}.$$

The real parts of the eigenvalues of M_p are negative if and only if their sum S is negative and their product P is positive with

$$\begin{aligned} -S &= \lambda_0 + \lambda_1 + p(2 - c\alpha), \\ P &= p(\lambda_1 - (c\alpha - 1)(\lambda_0 + p)). \end{aligned}$$

The sum S is always negative and the positivity of P is given by (5.7). $\square$

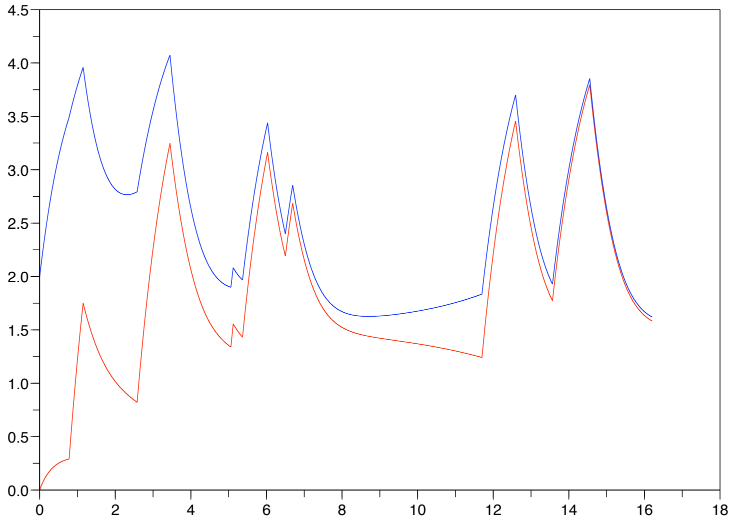

Figure 5.3: Trajectories of X (blue line) and Y (red line) with $\lambda_0 = 1$, $\lambda_1 = 0.6$ and $\alpha = 5$.

5.3.3 Recurrence

In this section, we aim to show that (X, Y, I) may admit several invariant measures if the jump rate λ_0 is large enough. Let us define

$$U_t = \frac{X_t + Y_t}{2} \quad \text{and} \quad V_t = \frac{X_t - Y_t}{2}.$$

Of course (U, V, I) is still a PDMP. If

$$\frac{d}{dt}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = F^1(X_t, Y_t) \quad \text{then} \quad \frac{d}{dt}\begin{pmatrix} U_t \\ V_t \end{pmatrix} = G^1(U_t, V_t),$$

with

$$G^1(u, v) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} F^1(u + v, u - v) = \begin{pmatrix} -u + \dfrac{\alpha(1 + u^2 + v^2)}{(1 + (u + v)^2)(1 + (u - v)^2)} \\ -v + \dfrac{2\alpha uv}{(1 + (u + v)^2)(1 + (u - v)^2)} \end{pmatrix}.$$

Corollary 5.3.3 ensures that, if λ_1/λ_0 is large enough, then V_t goes to 0 exponentially fast.

Let us show that this is no longer true if λ_1/λ_0 is small enough. Let $\varepsilon > 0$. Assume that, with positive probability, $V_t \in (0, \varepsilon)$ for any $t \geqslant 0$. Then, for any time $t \geqslant 0$, $(U_t, V_t) \in [b, \alpha] \times [0, \varepsilon]$. Indeed, one can show that the set $X_t + Y_t \geqslant 2b$ for any $t \geqslant 0$ as soon as it is true at the initial time.

Lemma 5.3.5. *Assume that $(u, v) \in [b, \alpha] \times [0, \varepsilon]$. Then there exists $u_c \in (b, \alpha)$ and $K, \delta, \gamma, \tilde{\gamma} > 0$ (that do not depend on ε) such that $b_\varepsilon = b + K\varepsilon^2$ and*

$$G_1^1(u, v) \leqslant H_1^1(u, v) \quad \textit{with} \quad H_1^1(u, v) = -\delta(u - b_\varepsilon).$$

and

$$G_2^1(u,v) \geqslant H_2^1(u,v) \quad \textit{with} \quad H_2^1(u,v) = \big((\gamma+\tilde{\gamma})\mathbb{1}_{\{u\leqslant u_c\}} - \tilde{\gamma}\big)v.$$

Proof. Notice firstly that

$$\big|(1+(u+v)^2)(1+(u-v)^2) - (1+u^2)^2\big| \leqslant K\varepsilon^2. \tag{5.8}$$

Thus, using that $u^3+u-\alpha = (u-b)(u^2+bu+\alpha/b)$ we get that

$$\begin{aligned} G_1^1(u,v) &\leqslant -u + \frac{\alpha}{1+u^2} + K\varepsilon^2 \\ &\leqslant -(u-b)\frac{u^2+bu+\alpha/b}{1+u^2} + K\varepsilon^2 \\ &\leqslant -(u-b)\frac{2b^2+\alpha/b}{1+\alpha^2} + K\varepsilon^2. \end{aligned}$$

We get the desired upper bound for G_1^1 with

$$\delta = \frac{2b^2+\alpha/b}{1+\alpha^2} \quad \text{and} \quad b_\varepsilon = b + (K/\delta)\varepsilon^2.$$

Similarly, Equation (5.8) ensures that

$$G_2^1(u,v) \geqslant vk(u) \quad \text{with} \quad k(u) = \frac{2\alpha u}{(1+u^2)^2} - 1 - K\varepsilon^2.$$

Obviously, if ε is small enough, $k(b) > 0$, $k(\alpha) < 0$ and k is decreasing. Thus, if $\tilde{u}$ is the unique zero of k on (b,α), then one can choose

$$u_c = \frac{\tilde{u}+b}{2}, \quad \gamma = k(u_c) \quad \text{and} \quad \tilde{\gamma} = k(\alpha).$$

To get a simpler bound in the sequel we can even set $\tilde{\gamma} = k(\alpha) \vee 1$. □

Finally, define $H_1^0(u,v) = G_1^0(u,v) = -(u-\alpha)$ and $H_2^0(u,v) = G_2^0(u,v) = -v$ and introduce the PDMP $(\tilde{U},\tilde{V},\tilde{I})$ where $\tilde{I} = I$ is the switching process of (U,V,I) and $(\tilde{U},\tilde{V})$ is driven by H^0 and H^1 instead of G^0 and G^1. From Lemma 5.3.5, we get that

$$U_t \leqslant \tilde{U}_t \quad \text{and} \quad \tilde{V}_t \leqslant V_t \quad (t \geqslant 0)$$

assuming that $(\tilde{U}_0,\tilde{V}_0,\tilde{I}_0) = (U_0,V_0,I_0)$. The last step is to study briefly the process $(\tilde{U},\tilde{V},\tilde{I})$. Let us firstly notice that if λ_1/λ_0 is small enough, then $(I_s,\tilde{U}_s)$ spends an arbitrary large amount of time near $(1,b_\varepsilon)$ (and b_ε can be assumed smaller than u_c if ε is small enough). Thus

$$\frac{1}{t}\log\frac{\tilde{V}_t}{\tilde{V}_0} \geqslant \frac{1}{t}\int_0^t ((\gamma+\tilde{\gamma})\mathbb{1}_{\{I_s=1,\tilde{U}_s<u_c\}} - \tilde{\gamma})\,ds$$

since $\tilde{\gamma} \geqslant 1$. The right hand side converges almost surely to a positive limit as soon as λ_1/λ_0 is small enough. This implies that V cannot be bounded by ε forever.

Corollary 5.3.6. *If λ_1/λ_0 is small enough, the process (X,Y,I) admits three ergodic measures.*

Bibliography

[ABC^{+}00] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer, *Sur les inégalités de Sobolev logarithmiques*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 10, Société Mathématique de France, Paris, 2000, With a preface by D. Bakry and M. Ledoux.

[BBM09] M. Balde, U. Boscain, and P. Mason, *A note on stability conditions for planar switched systems*, Internat. J. Control **82** (2009), no. 10, 1882–1888. MR 2567235 (2010i:93122) 3.2

[BCG08] D. Bakry, P. Cattiaux, and A. Guillin, *Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré*, J. Funct. Anal. **254** (2008), no. 3, 727–759, PDF.

[BCG^{+}12] J.B. Bardet, A. Christen, A. Guillin, A. Malrieu, and P.-A. Zitt, *Total variation estimates for the TCP process*, Preprint available on arXiv, 2012. 1.2.1, 2, 2.2

[BDRS02] M. Borkovec, A. Dasgupta, S. Resnick, and G. Samorodnitsky, *A single channel on/off model with TCP-like control*, Stoch. Models **18** (2002), no. 3, 333–367, DOI PDF. 1.2.1

[BGM10] J.-B. Bardet, H. Guérin, and F. Malrieu, *Long time behavior of diffusions with Markov switching*, ALEA Lat. Am. J. Probab. Math. Stat. **7** (2010), 151–170. MR 2653702 (2011k:60263) 5.3.2

[BH12] Y. Bakhtin and T. Hurth, *Invariant densities for dynamical systems with random switching*, preprint available on arXiv, 2012. 5, 5.1, 5.2.1

[BLBMZ12a] M. Benaïm, S. Le Borgne, F. Malrieu, and P.-A. Zitt, *On the stability of planar randomly switched systems*, Preprint available on arXiv, 2012. 3

[BLBMZ12b] ———, *Qualitative properties of certain piecewise deterministic markov processes*, Preprint available on arXiv, 2012. 5, 5.1

[BLBMZ12c] ———, *Quantitative ergodicity for some switched dynamical systems*, Preprint available on arXiv, 2012. 4

[CD00] O. L. V. Costa and F. Dufour, *Invariant probability measures for a class of Feller Markov chains*, Statist. Probab. Lett. **50** (2000), no. 1, 13–21, DOI PDF.

[CD01] ———, *Necessary and sufficient conditions for non-singular invariant probability measures for Feller Markov chains*, Statist. Probab. Lett. **53** (2001), no. 1, 47–57, DOI PDF.

[CD06] ———, *Ergodic properties and ergodic decompositions of continuous-time Markov processes*, J. Appl. Probab. **43** (2006), no. 3, 767–781, DOI PDF.

[CD08] ———, *Stability and ergodicity of piecewise deterministic Markov processes*, SIAM J. Control Optim. **47** (2008), no. 2, 1053–1077, DOI PDF.

[CDPP09] P. Caputo, P. Dai Pra, and G. Posta, *Convex entropy decay via the Bochner-Bakry-,Ä¶mery approach*, Ann. Inst. Henri Poincaré Probab. Stat. **45** (2009), no. 3, 734–753, DOI PDF.

[CGWW09] P. Cattiaux, A. Guillin, F.-Y. Wang, and L. Wu, *Lyapunov conditions for super Poincaré inequalities*, J. Funct. Anal. **256** (2009), no. 6, 1821–1841, DOI PDF.

[CMP10] D. Chafaï, F. Malrieu, and K. Paroux, *On the long time behavior of the TCP window size process*, Stochastic Process. Appl. **120** (2010), no. 8, 1518–1534. MR 2653264 1.2.1, 2, 2.1.1, 2.2

[Cos90] O. L. V. Costa, *Stationary distributions for piecewise-deterministic Markov processes*, J. Appl. Probab. **27** (1990), no. 1, 60–73, PDF.

[Dav84] M. H. A. Davis, *Piecewise-deterministic Markov processes: a general class of nondiffusion stochastic models*, J. Roy. Statist. Soc. Ser. B **46** (1984), no. 3, 353–388, PDF. 1.1

[Dav93] ______, *Markov models and optimization*, Monographs on Statistics and Applied Probability, vol. 49, Chapman & Hall, London, 1993. 1.1

[DC99] F. Dufour and O. L. V. Costa, *Stability of piecewise-deterministic Markov processes*, SIAM J. Control Optim. **37** (1999), no. 5, 1483–1502 (electronic), DOI PDF.

[DGR02] V. Dumas, F. Guillemin, and Ph. Robert, *A Markovian analysis of additive-increase multiplicative-decrease algorithms*, Adv. in Appl. Probab. **34** (2002), no. 1, 85–111, PDF. 1.2.1

[dSDG10] B. de Saporta, F. Dufour, and K. Gonzalez, *Numerical method for optimal stopping of piecewise deterministic markov processes*, Ann. Appl. Probab. **20** (2010), no. 5, 1607–1637, DOI PDF.

[EO05] R. Erban and H. G. Othmer, *From individual to collective behavior in bacterial chemotaxis*, SIAM J. Appl. Math. **65** (2004/05), no. 2, 361–391 (electronic), DOI PDF. 1.2.2

[EO05] ______, *From signal transduction to spatial pattern formation in e. coli: a paradigm for multiscale modeling in biology*, Multiscale Model. Simul. **3** (2005), no. 2, 362–394 (electronic), DOI PDF. 1.2.2

[FGM10] J. Fontbona, H. Guérin, and F. Malrieu, *Quantitative estimates for the long time behavior of a PDMP describing the movement of bacteria*, arXiv, 2010. 1.2.2, 1.2.2

[GR09] C. Graham and Ph. Robert, *Interacting multi-class transmissions in large stochastic networks*, Ann. Appl. Probab. **19** (2009), no. 6, 2334–2361, DOI PDF. 1.2.1

[GR10] ______, *Self-adaptive congestion control for multi-class intermittent connections in a communication network*, arXiv, 2010. 1.2.1

[GRZ04] F. Guillemin, Ph. Robert, and B. Zwart, *AIMD algorithms and exponential functionals*, Ann. Appl. Probab. **14** (2004), no. 1, 90–117, PDF. 1.2.1

[Hes05] J. P. Hespanha, *A model for stochastic hybrid systems with application to communication networks*, Nonlinear Anal. **62** (2005), no. 8, 1353–1383, DOI PDF. 1.2.1

[HM08] M. Hairer and J.C. Mattingly, *Yet another look at Harris' ergodic theorem for Markov chains*, arXiv, 2008.

[HV10] S. Herrmann and P. Vallois, *From persistent random walk to the telegraph noise*, Stoch. Dyn. **10** (2010), no. 2, 161–196, DOI PDF. 1.2.2

[Jac06] M. Jacobsen, *Point process theory and applications*, Probability and its Applications, Birkhäuser Boston Inc., Boston, MA, 2006, Marked point and piecewise deterministic processes. 1.1

[KB04] R. Karmakar and I. Bose, *Graded and binary responses in stochastic gene expression*, Physical Biology **197** (2004), no. 1, 197–214. 5.2.2

[Las04] G. Last, *Ergodicity properties of stress release, repairable system and workload models*, Adv. in Appl. Probab. **36** (2004), no. 2, 471–498, DOI PDF.

[LL08] J. S. H. van Leeuwaarden and A. H. Löpker, *Transient moments of the TCP window size process*, J. Appl. Probab. **45** (2008), no. 1, 163–175. 2.1

[LvL08] A. H. Löpker and J. S. H. van Leeuwaarden, *Transient moments of the TCP window size process*, J. Appl. Probab. **45** (2008), no. 1, 163–175, DOI PDF. 1.2.1

[MT93] S. P. Meyn and R. L. Tweedie, *Markov chains and stochastic stability*, Communications and Control Engineering Series, Springer-Verlag London Ltd., London, 1993.

[MZ06] K. Maulik and B. Zwart, *Tail asymptotics for exponential functionals of Lévy processes*, Stochastic Process. Appl. **116** (2006), no. 2, 156–177, PDF. 1.2.1

[MZ09] ______, *An extension of the square root law of TCP*, Ann. Oper. Res. **170** (2009), 217–232, DOI PDF. 1.2.1

[ODA88] H. G. Othmer, S. R. Dunbar, and W. Alt, *Models of dispersal in biological systems*, J. Math. Biol. **26** (1988), no. 3, 263–298, DOI PDF. 1.2.2

[OK08] T. J. Ott and J. H. B. Kemperman, *Transient behavior of processes in the TCP paradigm*, Probab. Engrg. Inform. Sci. **22** (2008), no. 3, 431–471, PDF. 1.2.1, 2.1

[OKM96] T. J. Ott, J. H. B. Kemperman, and M. Mathis, *The stationary behavior of ideal TCP congestion avoidance*, PDF, 1996. 1.2.1

[OS07] T. J. Ott and J. Swanson, *Asymptotic behavior of a generalized TCP congestion avoidance algorithm*, J. Appl. Probab. **44** (2007), no. 3, 618–635, PDF. 1.2.1

[Ott06] T. J. Ott, *Rate of convergence for the 'square root formula' in the internet transmission control protocol*, Adv. in Appl. Probab. **38** (2006), no. 4, 1132–1154, PDF. 1.2.1

[PR05] B. Perthame and L. Ryzhik, *Exponential decay for the fragmentation or cell-division equation*, J. Differential Equations **210** (2005), no. 1, 155–177. MR 2114128 (2006b:35328) 2.1.1, 2.1.2, 2.1

[PTW09] K. Pakdaman, M Thieullen, and G. Wainrib, *Fluid limit theorems for stochastic hybrid systems with application to neuron models*, arXiv, 2009. 1.2.4

[RMC07] O. Radulescu, A. Muller, and A. Crudu, *Théorèmes limites pour des processus de Markov à sauts. Synthèse des résultats et applications en biologie moléculaire*, Technique et Science Informatiques **26** (2007), no. 3-4, 443–469. 5.2.2

[RR96] G. O. Roberts and J. S. Rosenthal, *Quantitative bounds for convergence rates of continuous time Markov processes*, Electron. J. Probab. **1** (1996), no. 9, approx. 21 pp. (electronic). MR 1423462 (97k:60198) 2.3, 2.3, 1, 2.3.1

[RS10] M. Rousset and G. Samaey, *Individual-based models for bacterial chemotaxis and variance reduced simulation*, HAL, 2010. 1.2.2

[vLLO09] J. S. H. van Leeuwaarden, A. H. Löpker, and T. J. Ott, *TCP and iso-stationary transformations*, Queueing Syst. **63** (2009), no. 1-4, 459–475, DOI PDF. 1.2.1

Florent Malrieu, e-mail: `florent.malrieu(AT)univ-rennes1.fr`

UMR 6625 CNRS Institut de Recherche Mathématique de Rennes (IRMAR)
Université de Rennes 1, Campus de Beaulieu, F-35042 Rennes Cedex, France.